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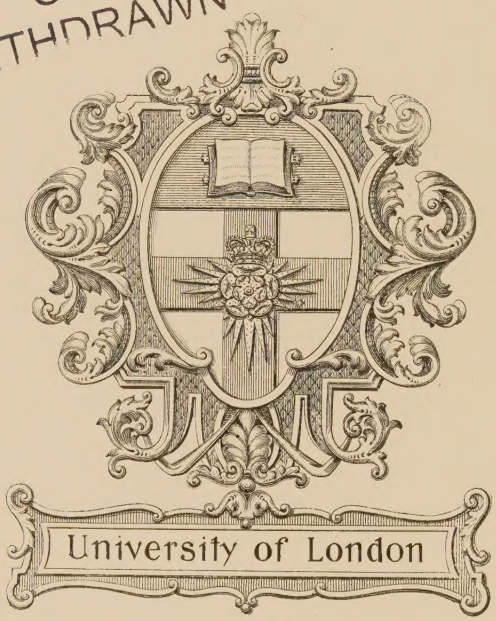

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PAPERS

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ON HOMOLOGIES IN GENERAL SPACES

By B. KAUFMANN.

[Received 13 October, 1936.—Read 12 November, 1936.]

Introduction.

In this paper I prove a general result on intersections of complexes and closed sets of arbitrary dimension, and two theorems on homologies in closed sets.

In a Euclidean n-dimensional space R^n, let $K^p = \Sigma t^i x_i^p$ be a p-dimensional algebraic complex, and let A^q be a q-dimensional closed set, p and q being quite arbitrary positive integers. It is well known that, if $p+q$ is smaller than n, there exists an arbitrarily small transformation of the complex K^p into a complex K_1^p *not meeting* A^q. This result is based on duality theorems, and is of fundamental importance in general topology. It shows that, at least in the case of $p+q<n$, there exists a "general position" of a complex with respect to a set. This is, however, the simplest case of empty intersections.

In the general case of non-empty intersections, *i.e.* when $p+q \geqslant n$, nothing is known, in contrast with the classical theory of intersections of complexes†. The difficulties of the approach to this problem can be seen from the fact that we do not even know which position of a complex with respect to a set is to be considered as *general*. The existence of a general position would imply the possibility of a *finite* simplicial transformation of the complex K^p into a complex K_1^p, such that the intersection $K_1^p A^q$ has a dimension fully determined by the values p, q, and n alone. But such transformations are, in general, impossible, as could easily be demonstrated by means of examples. This shows that, if a solution is possible at all, it must necessarily be by means of infinite simplicial transformations.

† See S. Lefschetz, *Topology* (1930), ch. IV.

Recently I solved this problem in the special case $p+q=n$, and this result enabled me to establish an essentially new type of Phragmén-Brouwer theorem and in particular its inductive generalization†. In Theorem AK (§ I) I now give a complete result for quite arbitrary values of p and q. It shows the existence of an infinite sequence of ϵ_μ modifications of the complex K^p into complexes $K_\mu{}^p$ ($\epsilon_\mu \to 0$ as $\mu \to \infty$) such that the intersections $K_\mu{}^p A^q$ converge to a set of dimension $p+q-n$ at the most. In the dimension of this limit set we can thus observe an analogy with the classical theory of simplicial intersections.

The main result of this paper is the solution of two well-known problems on the internal linking relations in a general space. In Part I, I prove the first hypothesis of Alexandroff's dimensional linking theorem for closed sets of arbitrary dimension (Theorem H_1). I show that any h-dimensional homology ($0 \leqslant h \leqslant r-1$) in a closed r-dimensional set A^r can be destroyed by an at most $(r-h-1)$-dimensional subset of A^r. This result follows from Theorem AK‡.

In Part II, I prove the second part of Alexandroff's dimensional linking theorem. I show that in any r-dimensional closed set A^r there exist homologies $Z^h \sim 0$, of all dimensions $h = 0, 1, 2, \ldots, r-1$, which cannot be destroyed by a subset of dimension less than or equal to $r-h-2$. This result is contained in Theorem H_2. In particular, all homologies of an inductive linking system which can be constructed according to my generalization of the Phragmen-Brouwér theorem possess the required property.

The proof of Theorem H_2 reveals also a close inner connection among the homologies $Z^h \sim 0$ of the inductive linking system for all dimensions $h = 0, 1, 2, \ldots, r-1$. Moreover, any space A^r can be seen to possess a *fibrous structure*; a carrier A^{h+1} of an *arbitrary* homology $Z^h \sim 0$ can be considered as being composed of the carriers A^j of homologies $Z^j \sim 0$ for an arbitrary j smaller than h.

---

† See my papers "On infinitesimal properties of closed sets", *Annals of Math.* (2), 38 (1937), 14–37, and "On the extension of the Pflastersatz", *Proc. Camb. Phil. Soc.*, 32 (1936), 238–247.

‡ The problem of the dimensional linking theorem (*Dimensionstheoretischer Verschlingungssatz*) was much emphasised by Alexandroff [see P. Alexandroff, "Dimensionstheorie", *Math. Annalen*, 106 (1932), 231 and also 238]. It is also connected with the main problem of Alexandroff's theory concerning the relation of Brouwer's and modular dimensions.

I might also mention that the results of this paper lead to a simplification of Alexandroff's *provisional* definition of dimensionally "full valued" (*vollwertige*) sets, which he gave at the end of his fundamental work (p. 238). By Theorem H_1, the first condition of his definition can now be dropped, since it is always satisfied for all sets in Euclidean spaces.

PART I.

Intersection of complexes and sets. Proof of the first hypothesis of Alexandroff's linking theorem.

1. Let A^r be an arbitrary r-dimensional set in R^n. Let $y^p = \Sigma t^i x_i^p$ be a p-dimensional algebraic ϵ complex $(0 \leqslant p \leqslant r)$ in A^r. A sequence $Y^p = y_1^p, y_2^p, \ldots, y_\kappa^p, \ldots$ of algebraic ϵ_κ complexes in A^r, such that $\epsilon_\kappa \to 0$ as $\kappa \to \infty$, is called an *infinite chain* (or complex) in A^r. The set A^r is called a carrier of the chain Y^p. A chain Y^{p_1} is called a *partial* chain of Y^p if each chain y^{p_1} of Y^{p_1} is also in Y^p. A partial chain Y^{p_1} of Y^p which differs from Y^p by a finite number of chains only is apparently equivalent to Y^p; we call Y^{p_1} a *reduced* chain and denote it by Y^p as before (using the same indices $\kappa = 1, 2, \ldots$). If we replace an infinite chain Y^p by a reduced chain (by leaving out a finite number of chains y_κ^p), we speak also of a *reduction* of the chain Y^p.

An *infinite cycle* $Z^p = z_1^p, z_2^p, \ldots, z_\kappa^p, \ldots$ is an infinite chain such that each chain z_κ^p is a cycle (in a prescribed modular or non-modular sense). An infinite cycle Z^p, such that each chain z_κ^p is a cycle mod m_κ (with m_κ depending on κ), is called a cycle with a variable modulus. The definition of infinite chains and cycles can be extended in the usual way mod any closed subset of A^r.

Let A' be an arbitrary subset of A^r and let Z^p be an infinite cycle in A'. Z^p is said to be homologous to 0 ($Z^p \sim 0$ in A') if there exists an infinite chain $y^{p+1} = y_1^{p+1}, y_2^{p+1}, \ldots, y_\kappa^{p+1}, \ldots$ in A' such that, for each κ, $y_\kappa^{p+1} \to z_\kappa^p$. The set A' is called a *carrier of the homology* $Z^p \sim 0$. We call A' an *irreducible carrier* of the homology $Z^p \sim 0$ if no proper subset of A' is a carrier of the same homology. Z^p is *totally* $\not\sim 0$ in A' if no partial cycle of Z^p is homologous to 0 in A'.

Let $Z^p = z_1^p, z_2^p, \ldots, z_\kappa^p, \ldots$ be an arbitrary infinite cycle in A^r such that $Z^p \sim 0$ in A^r and totally $\not\sim 0$ in one of its carriers B. We say a subset A^* of A^r *destroys* the homology $Z^p \sim 0$ in A^r if Z^p is totally $\not\sim 0$ in any compact subset of $A^r - A^*$. In general, we assume A^* to be a closed subset of A^r outside B.

THEOREM *AK. Let A^q be an arbitrary q-dimensional closed set in R^n, and let $K^p = \Sigma t^i x_i^p$ be an arbitrary algebraic complex in R^n such that $p+q \geqslant n$. For any sequence of positive numbers $\epsilon_\mu \to 0$ there exists a sequence of complexes $K_1^p, K_2^p, \ldots, K_\mu^p, \ldots$ such that K_1^p is an ϵ_1-modification*

of K^p, $K_\mu{}^p$ is an ϵ_μ-modification of $K^p_{\mu-1}$, $K_\mu{}^p \underset{\epsilon_\mu}{\sim} K^p_{\mu-1}$†, and

$$\dim \lim_{\mu\to\infty} K_\mu{}^p A^q \leqslant p+q-n.$$

COROLLARY. *Let A^p and A^q be two arbitrary closed sets of dimension p and q $(p+q \geqslant n)$. Then there exists an arbitrarily small ϵ-shifting‡ of the set A^p into a set A^{p*} such that*

$$\dim A^{p*} A^q \leqslant p+q-n.$$

2. Let A^s be an arbitrary s-dimensional closed set in R^n. For any positive δ and each $\nu = 1, 2, \ldots$ we can define a $\delta/3^\nu$-covering of A^s by a finite system of open sets

$$\{U_i^\nu\} = U_1^\nu,\ U_2^\nu,\ \ldots,\ U_i^\nu,\ \ldots,\ U_{\kappa_\nu}^\nu;\ \ d(\bar{U}_i^\nu) < \frac{\delta}{3^\nu},$$

such that the boundary $B_i^{(s-1)\nu}$ of U_i^ν is at most $(s-1)$-dimensional.

We consider the sums

$$\sum_\nu \{U_i^\nu\} = \{U_i^1\} + \{U_i^2\} + \ldots + \{U_i^\nu\} + \ldots$$

and

$$\sum_\nu \{B_i^{(s-1)\nu}\} = \{B_i^{(s-1)1}\} + \{B_i^{(s-1)2}\} + \ldots + \{B_i^{(s-1)\nu}\} + \ldots$$

of all systems $\{U_i^\nu\}$ and $\{B_i^{(s-1)\nu}\}$ for $\nu\to\infty$. The systems $\sum_\nu \{U_i^\nu\}$ and $\sum_\nu \{B_i^{(s-1)\nu}\}$ we form in a quite arbitrary way into sequences

$$\{U_\lambda\} = U_1,\ U_2,\ \ldots,\ U_\lambda,\ \ldots$$

and

$$\{B_\lambda^{(s-1)}\} = B_1^{(s-1)},\ B_1^{(s-1)},\ \ldots,\ B_\lambda^{(s-1)},\ \ldots.$$

The covering of the set A^s by the system $\{U_\lambda\}_{\lambda=1,2,\ldots}$ we call a *complete regular covering* of A^s. Correspondingly we call the dissection of A^s by a system of closed sets $\{B_\lambda^{(s-1)}\}_{\lambda=1,2,\ldots}$ a *complete regular dissection* of A^s. It is clear now what we mean by saying that a given system of closed sets $\{B_\lambda^{(s-1)}\}$ with $\dim B_\lambda^{(s-1)} \leqslant s-1$ defines for $\lambda\to\infty$ a complete regular dissection of a given s-dimensional set. In particular, if P_1 and P_2 are quite arbitrary points of A^s, we can always find an *arbitrarily small* set $B_\lambda^{(s-1)}$ in the system $\{B_\lambda^{(s-1)}\}$ separating P_1 and P_2 in A^s.

---

† Two arbitrary algebraic complexes K^p and $K^{p'}$ are homologous to each other if $K^{p'} - K^p$ is a bounding cycle.

‡ An ϵ-shifting of a set A into a set A^* is a continuous single-valued transformation such that each point P^* of A^* lies within a distance ϵ from its original P in A.

3. Let

$$\{B_\lambda^{(q-1)}\} = B_1^{(q-1)},\ B_2^{(q-1)},\ \ldots,\ B_{\lambda^1}^{(q-1)}\ \ldots;\quad \dim(B_{\lambda^1}^{(q-1)}) \leqslant q-1$$

be a system of closed sets in A^q defining a complete regular dissection of the set A^q (in the above sense). For each $\lambda^1 = 1, 2, \ldots$ let

$$\{B_{\lambda^1\lambda^2}^{(q-2)}\} = B_{\lambda^1 1}^{(q-2)},\ B_{\lambda^1 2}^{(q-2)},\ \ldots,\ B_{\lambda^1\lambda^2}^{(q-2)},\ \ldots;\quad \dim B_{\lambda^1\lambda^2}^{(q-2)} \leqslant q-2$$

be a system of closed sets in $B_{\lambda^1}^{(q-1)}$ defining a complete regular dissection of the set $B_{\lambda^1}^{(q-1)}$. For each $1 \leqslant i \leqslant q$ we assume the sets $B_{\lambda^1\lambda^2\ldots\lambda^{i-1}}^{(q-i+1)}$ to be defined for all combinations $\lambda^1\lambda^2\ldots\lambda^{i-1}$ of indices $\lambda^1, \lambda^2, \ldots, \lambda^{i-1} = 1, 2, \ldots$, and form the system

$$\{B_{\lambda^1\lambda^2\ldots\lambda^i}^{(q-i)}\} = B_{\lambda^1\lambda^2\ldots\lambda^{i-1}1}^{(q-i)},\ B_{\lambda^1\lambda^2\ldots\lambda^{i-1}2}^{(q-i)},\ \ldots,\ B_{\lambda^1\lambda^2\ldots\lambda^{i-1}\lambda^i}^{(q-i)};\quad \dim B_{\lambda^1\lambda^2\ldots\lambda^i}^{(q-i)} \leqslant q-i$$

defining a complete regular dissection of the set $B_{\lambda^1\lambda^2\ldots\lambda^{i-1}}^{(q-i+1)}$. Thus, given an arbitrary pair of points x_1, x_2 of A^q, there exists an at most $(q-1)$-dimensional *arbitrarily small* set $B_\lambda^{(q-1)}$ separating x_1 and x_2 on A^q; for a fixed value $\lambda^1 = \lambda_0{}^1$ and a given pair of points x_1, x_2 on $B_{\lambda_0{}^1}^{(q-1)}$ there exists an at most $(q-2)$-dimensional *arbitrarily small* set $B_{\lambda_0{}^1\lambda^2}^{(q-2)}$ separating x_1 and x_2 on $B_{\lambda_0{}^1}^{(q-1)}$. In general, for fixed values $\lambda^1 = \lambda_0{}^1,\ \lambda^2 = \lambda_0{}^2, \ldots, \lambda^{i-1} = \lambda_0^{i-1}$ and a given pair of points x_1, x_2 on $B_{\lambda_0{}^1\lambda_0{}^2\ldots\lambda_0^{i-1}}^{(q-i+1)}$ there exists an at most $(q-i)$-dimensional set $B_{\lambda_0{}^1\lambda_0{}^2\ldots\lambda_0^{i-1}\lambda^i}^{(q-i)}$ separating x_1 and x_2 on $B_{\lambda_0{}^1\lambda_0{}^2\ldots\lambda_0^{i-1}}^{(q-i+1)}$.

Since $n-p-1 \leqslant q$ (according to the assumption in Theorem AK), the above construction is valid for $i = q+p-n+1$. We consider the system of all at most $(n-p-1)$-dimensional sets

$$B_{\lambda^1\lambda^2\ldots\lambda^{q-n+p+1}}^{(n-p-1)}$$

for all combinations $\lambda^1\lambda^2\ldots\lambda^{q-n+p+1}$ of the indices

$$\lambda^1,\ \lambda^2,\ \ldots,\ \lambda^{q-n+p+1} = 1,\ 2,\ \ldots.$$

It is clear that the complete system of all these sets is enumerable and can be arranged (quite arbitrarily) in a sequence

$$\{B_\mu^{(n-p-1)}\} = B_1^{(n-p-1)},\ B_2^{(n-p-1)},\ \ldots,\ B_\mu^{(n-p-1)},\ \ldots.$$

For a given set $B_{\lambda_0{}^1\lambda_0{}^2\ldots\lambda^{q-n+p}}^{(n-p)}$ there exists always a subsequence of $\{B_\mu^{(n-p-1)}\}$ forming its complete regular dissection.

4. We shall now construct the sequence of p-dimensional complexes $K_\mu{}^p$ satisfying the conditions of Theorem AK. These complexes are

obtained by a process of repeated ϵ-modifications† of the complex K^p with respect to the sets $\{B_\mu^{(n-p-1)}\}$.

We consider the set $K^p B_1^{(n-p-1)}$. Since $B_1^{(n-p-1)}$ is at most $(n-p-1)$-dimensional, it is possible, for an arbitrarily small positive number ϵ_1, to define an ϵ_1-modification of K^p into a complex $K_1{}^p$,

$$K_1{}^p \underset{\epsilon_1}{\sim} K^p,$$

not meeting $B_1^{(n-p-1)}$. Let d_1 be the distance $d(K_1{}^p, B_1^{(n-p-1)})$, and let ϵ_2 be a positive number less than ϵ_1 and less than $\frac{1}{3}d_1$. We note that, in general, the complex K_1^{n-p-1} meets the set $B_2^{(n-p-1)}$. Since $B_2^{(n-p-1)}$ is at most $(n-p-1)$-dimensional, we can define an ϵ_2-modification of the complex $K_1{}^p$ into a complex $K_2{}^p$,

$$K_2{}^p \underset{\epsilon_2}{\sim} K_1{}^p,$$

not meeting $B_2^{(n-p-1)}$. Let d_2 be the distance $d(K_2{}^p, B_2^{(n-p-1)})$. We define a positive number ϵ_3 less than $(1/3^2)d_1$ and $(1/3^2)d_2$, and construct the complex $K_3{}^p$ by an ϵ_3-modification of the complex $K_2{}^p$.

For a given $\epsilon_\mu < \epsilon_{\mu-1}$ and also less than $(1/3^{\mu-1})d_1$, $(1/3^{\mu-1})d_2$, ..., $(1/3^{\mu-1})d_{\mu-1}$, we define an ϵ_μ-modification of the complex $K_{\mu-1}^p$ into a complex $K_\mu{}^p$,

$$K_\mu{}^p \underset{\epsilon_\mu}{\sim} K_{\mu-1}^p,$$

not meeting $B_\mu^{(n-p-1)}$. For a given μ and each κ, $1 \leqslant \kappa \leqslant \mu$, we get the relations

$$d(K_\mu{}^p, B_\kappa^{(n-p-1)}) = d_\kappa \quad (\kappa = \mu)$$

and

$$d(K_\mu{}^p, B_\kappa^{(n-p-1)}) \geqslant d_\kappa - \sum_{i=\kappa}^{\mu-1} \epsilon_{i+1} \geqslant d_\kappa - \sum_{i=\kappa}^{\mu-1} \frac{1}{3^i} d_\kappa \quad (\kappa < \mu).$$

---

† ϵ-modifications are based on duality theorems, and are among the most important simplicial transformations of complexes applied in general topology. If

$$C^{n-p-1} = \Sigma t^i x_i^{n-p-1}$$

is an $(n-p-1)$-dimensional algebraic complex, and A^p a p-dimensional closed set in R^n, then it is always possible to construct a chain $C^{n-p-1*} = \Sigma t^i x_i^{n-p-1}$ *not meeting* A^p, such that $C^{n-p-1*} \underset{\epsilon}{\sim} C^{n-p-1}$ in R^n, the elements y_i of C^{n-p-1*} being themselves complexes in $(1-1)$ correspondence with the simplices of the complex C^{n-p-1}. The complex C^{n-p-1*} can be considered also as a simplicial complex if we dissolve the elements (" blocks ") into simplices. For a definition of ϵ-modifications see P. Alexandroff, *loc. cit.*, § 4. Generalized (" simultaneous ") modifications of a complex with respect to a system of sets of various dimensions are defined in § 3 of my paper " On infinitesimal properties of closed sets " in the *Annals of Math.*, already cited.

It may be noted in particular that an ϵ-modification of an algebraic chain is completely defined by an ϵ-modification of its geometrical skeletons : the new elements (blocks) of the modified complex are always taken with the coefficients of the original chain.

We continue this construction for $\mu \to \infty$ and consider the limit set

$$T = \lim_{\mu \to \infty} \{K_\mu{}^p A^q\}.$$

For any κ we get, from the above inequality,

$$d(T,\ B_\kappa^{(n-p-1)}) \geqslant \tfrac{1}{2} d_\kappa$$

with $d_\kappa = d(K_\kappa{}^p,\ B_\kappa^{(n-p-1)})$. This proves that $T \sum_{\mu=1}^{\infty} B_\mu^{(n-p-1)}$ *is empty.*

5. Assume now that the set T satisfies the relation

$$\dim T \geqslant p+q-n+1.$$

Then there exists a pair of points P_1, P_2 of T and a sufficiently small positive number δ_1 such that *any* closed subset of T with a diameter less than δ_1 separating P_1 and P_2 on T is *at least* $(p+q-n)$-dimensional. Since the system of sets $\{B_{\lambda^1}^{(q-1)}\}$ defines a complete regular dissection of the set A^q, there exists a set $B_{\lambda_0{}^1}^{(q-1)}$ (for a certain $\lambda^1 = \lambda_0{}^1$) with $\delta(B_{\lambda_0{}^1}^{(q-1)}) < \delta_1$, and such that

(i) $B_{\lambda_0{}^1}^{(q-1)} T$ separates P_1 and P_2 on T,

(ii) $\dim (B_{\lambda_0{}^1}^{(q-1)} T) \geqslant p+q-n$.

The set $B_{\lambda_0{}^1}^{(q-1)} T$, being at least $(p+q-n)$-dimensional, contains a pair of points P_1, P_2 such that for a sufficiently small positive number δ_2 *any* subset of $B_{\lambda_0{}^1}^{(q-1)} T$ with a diameter less than δ_2 separating P_1 and P_2 on $B_{\lambda_0{}^1}^{(q-1)} T$ *is at least* $(p+q-n-1)$-dimensional. Since the system of sets $\{B_{\lambda_0{}^1 \lambda^2}^{(q-2)}\}_{\lambda^2 = 1, 2, \ldots}$ defines a complete regular dissection of the set $B_{\lambda_0{}^1}^{(q-1)}$, there exists, for a certain $\lambda^2 = \lambda^2{}_0$, a set $B_{\lambda_0{}^1 \lambda_0{}^2}^{(q-2)}$ with $\delta(B_{\lambda_0{}^1 \lambda_0{}^2}^{(q-2)}) < \delta_2$, such that

(i) $B_{\lambda_0{}^1 \lambda_0{}^2}^{(q-2)} B_{\lambda_0{}^1}^{(q-1)}$ separates P_1 and P_2 on $B_{\lambda_0{}^1}^{(q-1)} T$,

(ii) $\dim (B_{\lambda_0{}^1 \lambda_0{}^2}^{(q-2)} B_{\lambda_0{}^1}^{(q-1)} T) \geqslant p+q-n-1$.

Similarly we define the set $(B_{\lambda_0{}^1 \lambda_0{}^2 \lambda_0{}^3}^{(q-3)} B_{\lambda_0{}^1 \lambda_0{}^2}^{(q-2)} B_{\lambda_0{}^1}^{(q-1)})$. It is obvious that this construction can be repeated $p+q-n+1$ times, and we get, in general, for each $i \leqslant p+q-n+1$, a set $(B_{\lambda_0{}^1 \lambda_0{}^2 \ldots \lambda_0{}^i}^{(q-i)} B_{\lambda_0{}^1 \lambda_0{}^2 \ldots \lambda_0^{i-1}}^{(q-i+1)} \ldots B_{\lambda_0{}^1}^{(q-1)} T)$ such that

(i) $(B_{\lambda_0{}^1 \lambda_0{}^2 \ldots \lambda_0{}^i}^{(q-i)} B_{\lambda_0{}^1 \lambda_0{}^2 \ldots \lambda_0^{i-1}}^{(q-i+1)} \ldots B_{\lambda_0{}^1}^{(q-1)})$

separates a pair of points on

$$(B_{\lambda_0{}^1 \lambda_0{}^2 \ldots \lambda_0^{i-1}}^{(q-i+1)} B_{\lambda_0{}^1 \lambda_0{}^2 \ldots \lambda_0^{i-2}}^{(q-i+2)} \ldots B_{\lambda_0{}^1}^{(q-1)} T),$$

(ii) $\dim (B_{\lambda_0{}^1 \lambda_0{}^2 \ldots \lambda_0{}^i}^{(q-i)} B_{\lambda_0{}^1 \lambda_0{}^2 \ldots \lambda_0^{i-1}}^{(q-i+1)} \ldots B_{\lambda_0{}^1}^{(q-1)} T) \geqslant p+q-n-i+1$.

For $i=p+q-n+1$ we get, in particular, the set

$$\Pi^{p+q-n+1} = B^{(n-p-1)}_{\lambda_0^1 \lambda_0^2 \dots \lambda_0^{p+q-n+1}} B^{(n-p)}_{\lambda_0^1 \lambda_0^2 \dots \lambda_0^{p+q-n}} \dots B^{(q-1)}_{\lambda_0^1} T$$

such that $$\dim \Pi^{p+q-n+1} \geqslant 0.$$

Thus it follows that the set

$$B^{(n-p-1)}_{\lambda_0^1 \lambda_0^2 \dots \lambda_0^{p+q-n+1}} T$$

cannot be empty. But we have shown in §4 that the set $B^{(n-p-1)}_{\lambda_0^1 \lambda_0^2 \dots \lambda_0^{p+q-n+1}}$ is contained in the system $\{B_\mu^{(n-p-1)}\}$ and, by §5, it has an empty product with T. We obtain a contradiction, and we get the relation

$$\dim T \leqslant p+q-n$$

satisfying Theorem AK.

6. We shall now prove the first theorem on the destruction of homologies in a closed set.

THEOREM H_1. *Let F^r be an arbitrary r-dimensional closed set in R^n and let $Z^h = z_1^h, z_2^h, \dots, z_\kappa^h, \dots$ $(0 \leqslant h \leqslant r-1)$ be an arbitrary infinite cycle* (mod m_κ) ~ 0 *in F^r and totally $\nsim 0$ in one of its carriers B. There exists always an at most $(r-h-1)$-dimensional subset $F^{(r-h-1)}$ of F which destroys the homology $Z^h \sim 0$ in F, i.e. Z^h is totally $\nsim 0$ in $F^r - F^{(r-h-1)}$.*

Proof. Let W^n be an arbitrary h-dimensional cube containing F^r in its interior. Since the infinite cycle Z^h is totally $\nsim 0$ in B we can define a sequence of cycles

$$z_1^{n-h-1}, \ z_2^{n-h-1}, \ \dots, \ z_\kappa^{n-h-1}, \ \dots$$

lying inside W^n and outside a certain 3ϵ-neighbourhood $U_{3\epsilon}(B)$ of B, and such that each cycle z_κ^{r-h-1} is linked (mod m_κ) with the corresponding cycle z_κ^h†. Let Σ^n be an arbitrary simplicial subdivision of the cube W^n of a mesh less than or equal to $\frac{1}{2}\epsilon$. After a suitable subdivision each cycle z_κ^{n-h-1} can be homotopically deformed into a cycle $'z_\kappa^{n-h-1}$ in Σ^n lying inside W^n and outside an ϵ-neighbourhood $U_\epsilon(B)$ of B. We denote $'z_\kappa^{n-h-1}$ again by z_κ^{n-h-1}. Since all cycles z_κ^{n-h-1} are subcomplexes of a finite complex Σ^n,

† The existence of the dual cycles z^{n-h-1} outside $U_{3\epsilon}(B)$ follows easily from the well-known duality theorems.

we can represent the sequence $\{z_\kappa^{n-h-1}\}$ as the sum of a finite number of sequences

$$\{z_\kappa^{n-h-1}\} = \{z_{\kappa_1}^{n-h-1}\}_1 + \{z_{\kappa_2}^{n-h-1}\}_2 + \ldots + \{z_{\kappa_i}^{n-h-1}\}_i + \ldots + \{z_{\kappa_m}^{n-h-1}\}_m,$$

$\kappa_1, \kappa_2, \ldots, \kappa_m = 1, 2, \ldots$, such that *all* cycles of each system $\{z_{\kappa_i}^{n-h-1}\}$ are identical with one and the same *geometrical* subcomplex K_i^{n-h-1} of Σ^n.

To each complex K_i^{n-h-1} we now apply the process of repeated ϵ_μ-modifications satisfying the conditions of Theorem AK. For each i we get a sequence of geometrical complexes

$$K_i^{n-h-1},\ K_{i1}^{n-h-1},\ K_{i2}^{n-h-1},\ \ldots,\ K_{i\mu}^{n-h-1},\ \ldots$$

such that K_{i1}^{n-h-1} is an ϵ_1-modification of the complex K_i^{n-h-1} and $K_{i\mu}^{n-h-1}$ is an ϵ_μ-modification of the complex $K_{i\mu-1}^{n-h-1}$ $(\mu \geqslant 2)$. For a sufficiently small ϵ_1, all complexes $K_{i\mu}^{n-h-1}$ will lie outside an ϵ-neighbourhood $U_\epsilon(B)$ of B. For an arbitrary i and a fixed κ_i the sequence of ϵ_μ-modifications $(\mu \to \infty)$ of the complex K_i^{n-h-1} defines also a sequence of ϵ_μ-modifications of the cycle $z_{\kappa_i}^{n-h-1}$ into a sequence of homologous cycles

$$z_{\kappa_i}^{n-h-1} \sim z_{\kappa_i 1}^{n-h-1} \sim z_{\kappa_i 2}^{n-h-1} \sim \ldots z_{\kappa_i \mu}^{n-h-1} \sim \ldots \quad \text{in} \quad W^n - U_\epsilon(B),$$

while for a fixed μ and a fixed i all the cycles $z_{\kappa_i}^{n-h-1}$ (for all κ_i) are geometrically identical with $K_{i\mu}^{n-h-1}$.

For each i we consider now the limit sets

$$T_i = \lim\{K_{i\mu}^{n-h-1}\}.$$

Since each complex $K_{i\mu}^{n-h-1}$ lies outside $U_\epsilon(B)$, we have

$$d(T_i, B) \geqslant \epsilon$$

for each i. The complexes $K_{i\mu}^{n-h-1}$ satisfy for each i the conditions of Theorem AK, and therefore

$$\dim T_i F^r \leqslant (n-h-1)+r-n$$

and

$$\dim \sum_i T_i F^r \leqslant r-h-1.$$

We put $F^{(r-h-1)} = \sum_i T_i$ and consider an arbitrary infinite chain

$$C_1^{h+1},\ C_2^{h+1},\ \ldots,\ C_\kappa^{h+1},\ \ldots,$$

such that, for each κ, $\qquad C_\kappa^{h+1} \to z_\kappa{}^h \pmod{m_\kappa}$.

Each cycle $z_\kappa{}^h$ is linked with the cycle z_κ^{n-h-1}, which is contained in at least one sequence $\{z_{\kappa_i}^{n-h-1}\}_i$. For a certain $\kappa_i = \kappa_i{}^0$, such that $z^{n-h-1} \equiv z_{\kappa_i{}^0}^{n-h-1}$, we

have a sequence of cycles

$$z^{n-h-1}_{\kappa_i^0} \sim z^{n-h-1}_{\kappa_i^0 1} \sim \ldots z^{n-h-1}_{\kappa_i^0 \mu} \sim \ldots$$

linked with $z_\kappa{}^h$. It follows now that

$$C^{h+1}_\kappa z^{n-h-1}_{\kappa_i^0 \mu} \neq 0$$

for each $\mu = 1, 2, \ldots$, and therefore

$$\lim C^{h+1}_\kappa F^{(r-h-1)} \neq 0.$$

Thus we have shown that each carrier of the infinite chain $\{C^{h+1}_\kappa\}$ meets the set $F^{(r-h-1)}$, and therefore $Z^h \frown\!\!\!\!\sim 0$ in $F - F^{(r-h-1)}$.

PART II.

Proof of the second hypothesis of the dimensional linking theorem.

7. The solution of the second problem by Alexandroff is based on a theorem on inductive linkages which I obtained recently and which I called the inductive Phragmen-Brouwer theorem†. This theorem is as follows:

Let $F = B^r$ be an arbitrary r-dimensional closed set in $\bar{U}$ and z^{n-r-1} a cycle (mod 0) $\frown\!\!\!\!\sim 0$ in $U - F$. Then there exist:

(a) a system of closed $(r-j)$-dimensional sets

$$\{B^{r-j}\}_{j=0,1,\ldots,r},$$

such that each set B^{r-j-1} dissects the set B^{r-j} into two proper parts (containing inner points rel B^{r-j}),

$$B^{r-j} = {}^1B^{r-j} + B^{r-j-1} + {}^2B^{r-j};$$

(b) a system of infinite cycles in $\{B^{r-j}\}$,

$$\{Z^{r-j} = z^{r-j}_1, z^{r-j}_2, \ldots, z^{r-j}_\kappa, \ldots\}_{j=0,1,\ldots,r} \pmod{m_\kappa},$$

such that $$z^{r-j}_\kappa = {}^1C^{r-j}_\kappa + {}^2C^{r-j}_\kappa,$$

$${}^1C^{r-j}_\kappa \to z^{r-j-1}_\kappa \text{ in } {}^1B^{r-j}, \quad -{}^2C^{r-j}_\kappa \to z^{r-j-1}_\kappa \text{ in } {}^2B^{r-j};$$

(c) a system of cycles

$$\{z^{n-r+j-1}\}_{j=0,1,\ldots,r},$$

† See my paper "On the extension of the Pflastersatz", *Proc. Camb. Phil. Soc.*, 32 (1936), 238. (The second part of this paper will appear shortly.)

such that

(i) each cycle z^{n-r+j} is dissected into two complexes

$$z^{n-r+j} = {}^1K^{n-r+j} + {}^2K^{n-r+j},$$

$${}^1K^{n-r+j} \to z^{n-r+j-1} \text{ in } U - {}^1B^{r-j}, \quad -{}^2K^{n-r+j} \to z^{n-r+j-1} \text{ in } U - {}^2B^{r-j};$$

(ii) for each j, $z^{n-r+j-1} \sim\!\!/\!\sim 0$ in $U - B^{r-j}$.

The configuration of cycles $\{z^{n-r+j-1}\}$ and the sets and infinite cycles $\{B^{r-j}\}$ and $\{Z^{r-j}\}$ we call an inductive linking configuration with respect to the space F.

LEMMA T. *Let Z^p be the p-dimensional cycle in B^p of the system $\{Z^{r-j}\}_{j=0,1,\ldots,r}$, $0 \leqslant p < r$, and let T be a closed set in F destroying* all *homologies $Z^p \sim 0$ in F, and lying outside a ρ-neighbourhood of B^p. Let Z^s be an s-dimensional cycle of the system $\{Z^{r-j}\}$ in B^s such that $p \leqslant s < r$. Let $Z^s \sim 0$ be an arbitrary homology in F defined by a sequence of complexes $Q_\kappa^{s+1} \to z_\kappa^s$ ($\kappa = 1, 2, \ldots$), and let A_{s+1} be a carrier of this homology in F. Then the set $T_{s+1} = TA_{s+1}$ is at least $(s-p)$-dimensional. In particular, there exists a sufficiently small positive number δ such that the set*

$$t_{s+1,\delta} = T_{s+1} - T_{s+1} V(B^s, \delta),$$

i.e. the subset of T_{s+1} outside a δ-neighbourhood $V(B^s, \delta)$ of B^s is at least $(s-p)$-dimensional.

8. *Proof.* We note first that $Z^p \sim 0$ in B^{p+1}; it follows that T cannot be empty.

For $s = p$ (in particular for $s = p = r-1$) we have $Z^s \equiv Z^p$, and our lemma states merely that the subset of T in an arbitrary carrier A_{p+1} of this homology cannot be empty. This is obvious, since T destroys all homologies $Z^s \sim 0$ in F and thus also any homology in A_{p+1}.

We consider now the general case $s > p$, which we shall prove by induction. For a given s we assume our lemma to be true for $s-1$. Thus, if $Z^{s-1} \sim 0$ is an arbitrary homology in a carrier A_s defined by a sequence of complexes

$$\{Q_\kappa^s \to z_\kappa^{s-1}\}_{\kappa=1,2,\ldots}$$

with respect to the $(s-1)$-dimensional cycle $Z^{s-1} = z_1^{s-1}, z_2^{s-1}, \ldots, z_\kappa^{s-1}, \ldots$ of the system $\{Z^{r-j}\}$, the subset $t_{s,\delta'}$ of T in A_s outside a certain δ'-neighbourhood $V(B^{s-1}, \delta')$ of B^{s-1} is at least $(s-p-1)$-dimensional†.

† The value of δ' depends on the homology $Z^{s-1} \sim 0$ and its carrier A_s.

9. We consider first the homologies $Z^{s-1} \sim 0$ in F. Among these homologies two are defined in the inductive linking system in the components ${}^1B^s$ and ${}^2B^s$ of the dissection

$$B^s = {}^1B^s + B^{s-1} + {}^2B^s.$$

In particular, the homology $Z^{s-1} \sim 0$ in ${}^1B^s$ is defined by the complexes

$$\{{}^1C_\kappa{}^s \to z_\kappa^{s-1}\}_{\kappa=1,2,\ldots} \pmod{m_\kappa}.$$

According to our assumption in §8 there exists a sufficiently small δ' such that the subset of T in ${}^1B^s$ outside a δ'-neighbourhood $V(B^{s-1}, \delta')$ of B^{s-1} is at least $(s-p-1)$-dimensional.

We consider now an arbitrary homology $Z^s \sim 0$ in a subset A_{s+1} of F. Since s is not greater than $r-1$, the existence of such homologies in F is also assured by the inductive linking system. In particular, $Z^s \sim 0$ in the components ${}^1B^{s+1}$ and ${}^2B^{s+1}$ of the dissection

$$B^{s+1} = {}^1B^{s+1} + B^s + {}^2B^{s+1}.$$

It is clear that the subset of T in A_{s+1} cannot be empty; since $A_{s+1} \supset B^s$ we can even assert that the subset of T in A_{s+1} outside a δ'-neighbourhood of B^{s-1} is at least $(s-p-1)$-dimensional. On the other hand, we do not know yet of the existence of points of T in A_{s+1} outside a (suitable) δ-neighbourhood of B^s. We shall show that this set exists and is at least $(s-p)$-dimensional.

We consider the compact space A_{s+1}, and note that A_{s+1} contains the set ${}^1B^s$. We define a covering of the set ${}^1B^s - B^{s-1}$ with respect to the space A_{s+1} by a system of spherical† neighbourhoods $\{V\}$ such that each point P of ${}^1B^s - B^{s-1}$ is contained in a neighbourhood $V(P, \sigma) \in \{V\}$ with a radius $\sigma < \frac{1}{3} d(P, {}^2B^s)$. We denote now by N the sum ΣV of all neighbourhoods $\{V\}$. We have the obvious relations

$$N \supset {}^1B^s - B^{s-1}; \quad \bar{N} \supset {}^1B^s; \quad \bar{N}({}^2B^s - B^{s-1}) = 0;$$

and

$$d(\bar{N}, {}^2\mathfrak{L}^{(r-s)}) \geqq \tfrac{2}{3} d({}^1B^s, {}^2\mathfrak{L}^{(r-s)}) > 0,$$

where

$${}^2\mathfrak{L}^{(r-s)} = z^{r-s}\,{}^2B^s\ddagger.$$

We denote by R the boundary of N with respect to A_{s+1}, *i.e.* the set of points of $\bar{N}(A_{s+1} - N)$.

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† By a spherical neighbourhood of a point P of A_{s+1} we mean the set of all points of A_{s+1} lying within a given distance of P.

‡ The cycle z^{r-s} is defined in § 7.

Let us now assume that our lemma is not true in the case of the homology $Z^s \sim 0$ in A_{s+1}. Then for any arbitrarily small δ the subset of T outside any δ-neighbourhood of B^s is at most $(s-p-1)$-dimensional. Therefore the set TR outside a δ-neighbourhood of B^s (for any δ) must be at most $(s-p-1)$-dimensional. We note also that *a priori* the set R itself, and in particular the set RT outside B^s, could be empty.

Let us assume the set R, and in particular the set $TR-B^s$, to be not empty.

We can now define a covering of the set $TR-B^s$ by an enumerable system of neighbourhoods $\{U\}$ with respect to A_{s+1} satisfying the following conditions:

(i) each point P of $TR-B^s$ has a neighbourhood $U(P, \tau)$ with a diameter $\tau < \frac{1}{3}d(P, B^s)$ and less than $\frac{1}{9}d(\bar{N}, {}^2\mathfrak{E}^{(r-s)})$;

(ii) the boundary $S(U)$ of each neighbourhood $U \epsilon \{U\}$ intersects the set T in an at most $(s-p-2)$-dimensional set.

It is easy to see that condition (ii) can always be satisfied. Each $U \epsilon \{U\}$ is entirely in A_{s+1}, and according to (i) does not meet B^s. On the other hand, the set $TA_{s+1}-B^s$ is, by assumption, at most $(s-p-1)$-dimensional.

We denote now by ΣU the sum of all neighbourhoods in $\{U\}$, and form the open set

$$N^* = N + \Sigma U.$$

Obviously the set N^* satisfies the following conditions:

$$N^*({}^2B^s - B^{s-1}) = 0; \quad d(\bar{N}^*, {}^2\mathfrak{E}^{(r-s)}) > \tfrac{1}{3}d(\bar{N}, {}^2\mathfrak{E}^{(r-s)}).$$

Let R^* be the boundary of the open set N^* with respect to A_{s+1}. According to (ii) the set

$$t_{s+1,\delta}{}^* = TR^* - V(B^s, \delta),$$

for an arbitrary δ, is at most $(s-p-2)$-dimensional.

The definition of the set N^* and of its boundary R^* is valid if

$$TR - B^s \neq 0.$$

If $TR-B^s = 0$, we put $N^* = N$ and also $R^* = R$. There remains still the possibility of R and R^* being empty sets. We shall disprove this later, and note for the present that the open set N^* is defined in any case.

10. We consider now the infinite chain

$$\{Q_\kappa^{s+1} = \Sigma c_\kappa^{\,i} E_{i\kappa}^{s+1}\}_{\kappa=1,2,\ldots}.$$

By leaving out a finite number of complexes Q_κ^{s+1} we reduce the chain $\{Q_\kappa^{s+1}\}$ so that the mesh ϵ_κ of each complex Q_κ^{s+1} becomes smaller than $\frac{1}{3}d(\bar{N}^*, {}^2\mathfrak{L}^{(r-s)})$. We now form, for each κ, the subcomplex $'Q_\kappa^{s+1}$ of all simplices $E_{i\kappa}^{s+1}$ of Q_κ^{s+1} *having at least one vertex in* $\bar{N}^*$. In other words, we omit in the linear form representing Q_κ^{s+1} all simplices in $A_{s+1}-\bar{N}^*$. The complexes $'Q_\kappa^{s+1}$ are defined by linear forms in which simplices with at least one vertex in $\bar{N}^*$ appear with the same coefficients as in Q_κ^{s+1}.

We denote by y_κ^s the boundary of $'Q_\kappa^{s+1}$ for each κ,

$$'Q_\kappa^{s+1} \to y_\kappa^s \pmod{m_\kappa}.$$

Let E_κ^s be an arbitrary s-simplex of $'Q_\kappa^{s+1}$. If E_κ^s is in $\bar{N}^*$ (*i.e.* if all vertices of E_κ^s are in $\bar{N}^*$), the entire star of E_κ^s in Q_κ^{s+1} is also in $'Q_\kappa^{s+1}$. Since $Q_\kappa^{s+1} \to z_\kappa^s \pmod{m_\kappa}$ and ${}^1C_\kappa^s = z_\kappa^s - {}^2C_\kappa^s$ is a complex in ${}^1B^s$, it follows that ${}^1C_\kappa^s$ is a subcomplex of y_κ^s. Hence we can write

$$'Q_\kappa^{s+1} \to {}^1C_\kappa^s + {}^3C_\kappa^s \pmod{m_\kappa},$$

where ${}^3C_\kappa^s$ may possibly vanish. Let us assume that ${}^3C_\kappa^s \not\equiv 0$.

Any s-simplex of ${}^3C_\kappa^s$ not in z_κ^s has at least one vertex in $A_{s+1}-\bar{N}^*$. Any s-simplex of ${}^3C_\kappa^s$ in z_κ^s has at least one vertex in $A_{s+1}-N^*$, since this simplex is in ${}^2B^s$ and therefore not in ${}^1B^s - B^{s-1}$. Thus we see that any s-simplex of ${}^3C_\kappa^s$ has at least one vertex in $A_{s+1}-N^*$. On the other hand, each simplex of ${}^3C_\kappa^s$ is a face of an ϵ_κ simplex with at least one vertex in $\bar{N}^*$. Since $\epsilon_\kappa \to 0$ as $\kappa \to \infty$, it follows that

$$\lim_{\kappa\to\infty} {}^3C_\kappa^s \subset R^*.$$

It is obvious now that the relation ${}^3C_\kappa^s \not\equiv 0$ (for an infinite number of values of κ) involves the relation $R^* \neq 0$. Since, by assumption, ${}^3C_\kappa^s \not\equiv 0$ for almost all values of κ, the distances

$$d(R^*, {}^2\mathfrak{L}^{(r-s)}) > 3\eta > 0$$

and

$$d(R^*, {}^1\mathfrak{L}^{(r-s)}) > 3\eta > 0$$

(where ${}^1\mathfrak{L}^{(r-s)}$ denotes the set $z^{r-s}\,{}^1B^s$, and η is a sufficiently small positive number) can be defined. Let κ_0 be a sufficiently large integer such that, for all $\kappa > \kappa_0$, the complexes ${}^3C_\kappa^s$ are contained in a η-neighbourhood

$V(R^*, \eta)$ of R^*. The complexes ${}^1C_\kappa^s$ are in a general position with respect to the cycle z^{r-s}. By an infinitesimal η_κ-shifting of all vertices of the complexes ${}^3C_\kappa^s$ not in ${}^1C_\kappa^s$ we can transfrom ${}^3C_\kappa^s$ into complexes ${}^3\bar{C}_\kappa^s$ in a general position with respect to z^{r-s}. In so doing, we choose the numbers $\eta_\kappa < \frac{1}{2}\epsilon_\kappa$. The cycles y_κ^s are transformed into new cycles in a general position with respect to z^{r-s}

$$\bar{y}_\kappa^s = {}^1C_\kappa^s + {}^3\bar{C}_\kappa^s$$

such that, for each $\kappa > \kappa_0$ (κ_0 sufficiently large), $\bar{y}_\kappa^s \underset{2\epsilon_\kappa}{\sim} y_\kappa^s$ inside a 3η neighbourhood of $(R^* + {}^1B^s)$. The complexes $3\bar{C}_\kappa^s$ are $2\epsilon_\kappa$ complexes satisfying the following relations:

$$\lim {}^3\bar{C}_\kappa^s \subset R^*,$$

$$d({}^3\bar{C}_\kappa^s, {}^2\mathfrak{E}^{(r-s)}) > 0,$$

$$d({}^3\bar{C}_\kappa^s, {}^1\mathfrak{E}^{(r-s)}) > 0$$

for all $\kappa > \kappa_0$. We consider now the sets

$$z^{r-s}\,{}^3\bar{C}_1^s, \quad z^{r-s}\,{}^3\bar{C}_2^s, \quad \ldots, \quad z^{r-s}\,{}^3\bar{C}_\kappa^s, \quad \ldots$$

and form their limit set

$$L = \lim_{\kappa \to \infty} \{z^{r-s}\,{}^3\bar{C}_\kappa^s\}.$$

Since $z^{r-s}\,B^s = {}^1\mathfrak{E}^{r-s} + {}^2\mathfrak{E}^{r-s}$, we have

$$d(L, B^s) > 0.$$

Since L is contained in R^*, it follows that R^* *is not empty outside* B^s. This result depends on the assumption ${}^3\bar{C}_\kappa^s \not\equiv 0$. We shall now show that ${}^3\bar{C}_\kappa^s \not\equiv 0$ for all values of κ.

The cycles $\bar{y}_\kappa^s$ are in a general position with respect to z^{r-s}, and we can therefore define the following Kronecker indices (mod m_κ) for each κ:

$$\chi(\bar{y}_\kappa^s, z^{r-s}) = \chi({}^1\bar{C}_\kappa^s, z^{r-s}) + \chi({}^3\bar{C}_\kappa^s, z^{r-s}) = 0.$$

According to the properties of the inductive linking system, we have the relation $\chi({}^1C_\kappa^s, z^{r-s}) \neq 0$, and thus we get

$$\chi({}^3\bar{C}_\kappa^s, z^{r-s}) = \chi({}^1C_\kappa^s, z^{r-s}) \neq 0.$$

It follows now that ${}^3\bar{C}_\kappa^s \not\equiv 0$, and therefore ${}^3C_\kappa^s \not\equiv 0$, for each $\kappa = 1, 2, \ldots$.

11. We consider now the infinite cycle

$$Y^s = y_1^s,\ y_2^s,\ \ldots,\ y_\kappa^s,\ \ldots.$$

Since $$y_\kappa^s = {}^1C_\kappa^s + {}^3C_\kappa^s$$

and $${}^1C_\kappa^s \to z_\kappa^{s-1},$$

we have $$-{}^3C_\kappa^s \to z_\kappa^{s-1} \pmod{m_\kappa}.$$

Let $E_1^0, E_2^0, \ldots, E_m^0, \ldots = \{E_m^0\}$ be the system of all vertices in the infinite chain $\{{}^3C_\kappa^s\}$. Since $\lim {}^3C_\kappa^s \subset R^*$, it follows that the (compact) set

$$R^* + \{E_m^0\} = A_s$$

is a carrier of the homology $Z^{s-1} \sim 0$.

According to our lemma for $s-1$, there exists a set

$$t_{s,\delta'} = TA_s - V(B^{s-1}, \delta')$$

for a sufficiently small δ' such that

$$\dim t_{s,\delta'} \geqslant s-p-1 \geqslant 0.$$

We show now that, for a sufficiently small δ',

$$\dim t_{s,\delta'} R^* = \dim t_{s,\delta'}.$$

If $s-1=p$, we may have $\dim t_{s,\delta'} = 0$. Since $d(T, B^p) > 0$, it is sufficient to show in this case that the set TR^* is not empty. Assume that this is not the case. Since $\lim_{\kappa\to\infty} {}^3C_\kappa^s \subset R^*$, it follows that at the most a finite number of chains ${}^3C_\kappa^s$ has vertices in T. By leaving out these chains we obtain a homology $Z^{s-1} \sim 0$ in A_s defined by the reduced chain $\{{}^3C_\kappa^s \to z_\kappa^{s-1}\}$ having no vertices in T, which is impossible, since T destroys all homologies $Z^{s-1} \sim 0$ in F.

We consider now the case $\dim t_{s,\delta'} > 0$. Since $\{E_m^0\}$ is an enumerable set of points, we have $\dim \{E_m^0\} \leqslant 0$. Thus we have proved that, for $s-1 \geqslant p$,

$$\dim (t_{s,\delta'} R^*) = \dim (TA_s) \geqslant s-p-1.$$

But, according to §2, the set $t_{s+1,\delta} R^* = TR^* - V(B^s, \delta)$ is at most $(s-p-2)$-dimensional for an arbitrarily small δ. Since $R^* B^s \subset B^{s-1}$, we can define a sufficiently small δ such that $t_{s+1,\delta} R^* \supset t_{s,\delta'} R^*$. Therefore $t_{s,\delta'} R^*$ could be at most $(s-p-2)$-dimensional, which is impossible. This completes the proof of our lemma.

12. From the proof of Lemma T we can now easily derive a more precise result. We have shown the set

$$t_{s+1,\,\delta} = TA_{s+1} - V(B^s,\,\delta)$$

to be at least $(s-p)$-dimensional. In this result the value of δ still depends on the choice of a particular homology $Z^s \sim 0$ and its carrier A_{s+1}, and could thus become arbitrarily small. We can see, however, that δ can be defined quite independently of the choice of the homology $Z^s \sim 0$. For this purpose we modify the construction of the open set N as follows. We define the covering of the set ${}^1B^s - B^{s-1}$ by a system of spherical neighbourhoods $\{V\}$ such that each point P of ${}^1B^s - B^{s-1}$ is contained in a neighbourhood $V(P,\,\sigma)$ with a radius

$$\sigma = \tfrac{1}{3}d(P,\,B^{s-1}).$$

We then consider the system of *all* carriers $\{A_{s+1}\}$ of the homology $Z^s \sim 0$ in F. Let

$$A^1_{s+1},\;\; A^2_{s+1},\;\; \ldots,\;\; A^\nu_{s+1},\;\; \ldots$$

be an arbitrary sequence of sets in $\{A_{s+1}\}$, and let

$$R^{*1},\;\; R^{*2},\;\; \ldots,\;\; R^{*\nu},\;\; \ldots$$

be the sequence of corresponding sets R^* (§ 9). It follows now from the construction of the sets N^ν that

$$\lim_{\nu\to\infty} [R^{*\nu}(B^s - B^{s-1})] = 0.$$

Let $\{A_s\}$ be the system of *all* carriers of homologies

$$Z^{s-1} \sim 0 \quad (0 < p \leqslant s < \mu).$$

Let us assume that there exists a (*fixed*) number Δ_s such that, for *any* carrier $A_s \epsilon \{A_s\}$, the set

$$t_\Delta(A_s) = TA_s - V(B^{s-1},\;\Delta_{s-1})$$

is at least $(s-p-1)$-dimensional. The existence of the number Δ_p in the case of $s-1=p$ is obvious; in particular, we can take

$$\Delta_p = d(T,\,B^{p-1}).$$

We can now define a positive number $\Delta_{s+1} < \Delta_{s-1}$ such that all sets $\{t_{\Delta_{s-1}}\}$ corresponding to the system $\{A_s\}$ lie outside a neighbourhood $V(B^s, \Delta_s)$.

To each carrier $A_{s+1} \epsilon \{A_{s+1}\}$ of the homology $Z^s \sim 0$ there corresponds a set $t_{\Delta_s}(A_{s+1})$ lying outside the neighbourhood $V(B^s, \Delta_s)$. Proceeding on the lines of the proof of Lemma T, we can now show that the set

$$t_{\Delta_s}(A_{s+1}) = TA_{s+1} - V(B^s, \Delta_s)$$

is at least $(s-p)$-dimensional. Thus we obtain the following extension of Lemma T:

LEMMA T^*. *Let Z^p be the p-dimensional cycle in B, $0 \leqslant p < r$, of the system $\{Z^{r-j}\}_{j=0,1,\ldots,r}$. Let T be a closed set in F destroying* all *homologies $Z^p \sim 0$ in F, and lying outside a ρ-neighbourhood of B^p. Let Z^s be an s-dimensional cycle of the system $\{Z^{r-j}\}$ such that $p \leqslant s < r$, and let $\{A_{s+1}\}$ be the system of* all *carriers of the homology $Z^s \sim 0$ in F. Then there exists a sufficiently small number $\Delta > 0$ such that, for any set $A_{s+1} \epsilon \{A_{s+1}\}$, the set*

$$t_{\Delta_{s+1}} = TA_{s+1} - V(B^s, \Delta)$$

is at least $(s-p)$-dimensional, the number Δ being quite independent of the choice of a particular homology $Z^s \sim 0$ and its carrier.

By putting $s = r-1$ in Lemma T, we obtain at once the following

THEOREM H_2. *Let $\{Z^{r-j}\}_{j=0,1,2,\ldots,r}$ be the infinite cycles of an inductive linking configuration with respect to F. Then, for each $p = 0, 1, 2, \ldots, r-1$, any set T destroying all homologies*

$$Z^p \sim 0 \quad in \quad F$$

is at least $(r-p-1)$-dimensional.

A further close connection between the cycles of a fundamental inductive system is expressed by the following

THEOREM H_2^*. *Let $\{Z^{r-j}\}_{j=0,1,\ldots,r}$ be a fundamental inductive system of cycles in F with the carriers $\{B^{r-j}\}_{j=0,1,\ldots,r}$. Let T_{0,ρ_0} be an arbitrary set destroying* all *homologies $Z^0 \sim 0$ in F, and lying outside a ρ_0 neighbourhood of B^0. Then there exists a sufficiently small number Δ, $\rho_0 > \Delta > 0$, such that the sets*

$$T_{p,\Delta} = T_{0,\rho_0} - V(B^p, \Delta), \quad \dim T_{p,\Delta} \geqslant r-p-1,$$

for each $p = 0, 1, \ldots, r-1$, destroy all homologies

$$Z^p \sim 0 \quad in \quad F.$$

In addition, the sets $T_{p,\Delta}$ *satisfy the condition*

$$T_{0,\rho_0} \supset T_{1,\Delta} \supset T_{2,\Delta} \supset \ldots \supset T_{p,\Delta} \ldots \supset T_{r-1,\Delta}.$$

We choose the number Δ as in Lemma T^*. The relation

$$T_{p,\Delta} \supset T_{p+1,\Delta}$$

follows immediately from the relation $B^p \subset B^{p+1}$. If $Z^p \sim 0$ is an arbitrary homology in a subset A_{p+1} of F, then, by Lemma T^*, the set

$$t_{\Delta_{p+1}} = TA_{p+1} - V(B^p, \Delta)$$

is at least $(r-p-1)$-dimensional, and is certainly not empty. But

$$t_{\Delta_{p+1}} = T_{p,\Delta} A_{p+1},$$

and therefore $T_{p,\Delta} A_{p+1}$ destroys the homology $Z^p \sim 0$ in A_{p+1}. Hence $T_{p,\Delta}$ destroys all homologies $Z^p \sim 0$ in F.

It is obvious that Theorem H_2^* can be stated correspondingly with respect to a set T_{p,ρ_p} for any p.

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BEZIEHUNGEN ZWISCHEN DEN OSZILLATIONSGRENZEN EINER FUNKTION UND IHRER ARITHMETISCHEN MITTEL

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1. In der Arbeit von M. Fekete und C. E. Winn* wird der folgende Fejérsche Satz in etwas veränderter Schreibweise angeführt:

Sei
$$\sigma_n = \frac{1}{n}\sum_{\nu=0}^{n} s_\nu,$$
$$l_s = \operatorname*{osc}_{n=\infty} s_n, \quad l_\sigma = \operatorname*{osc}_{n=\infty} \sigma_n$$
und
$$W(\lambda) = \lim_{n=\infty} \sup \max_{n\leqslant n'\leqslant \lambda n} |s_{n'} - s_n|,$$
so ist
$$l_s \leqslant \frac{\lambda+1}{\lambda-1} l_\sigma + 2W(\lambda). \tag{1}$$

Hier möchte ich zeigen, dass die meisten Resultate der genannten Arbeit* wie auch derjenigen von C. E. Winn† im wesentlichen durch die Verschärfung des letzten Gliedes $2W(\lambda)$ der Fejérschen Ungleichung erhalten werden können, wie dies aus den nachstehenden Sätzen folgt.

SATZ 1. *Sei*
$$\sigma(x) = \frac{1}{x}\int_0^x s(t)\,dt, \tag{2}$$
$$\underline{s} = \lim_{x=\infty} \inf \sup s(x), \quad \overline{\sigma} = \lim_{x=\infty} \sup \inf \sigma(x),$$
$$l_s = \overline{s} - \underline{s}, \quad l_\sigma = \overline{\sigma} - \underline{\sigma},$$
$$-w_1(\lambda) = \lim_{t=\infty} \inf \min_{t\leqslant t'\leqslant \lambda t} \{s(t') - s(t)\} \tag{3}$$

* "On the connection between the limits of oscillation of a sequence and its Cesàro-Riesz means", *Proc. London Math. Soc.* (2), 35 (1932), 488–513.

† "On the oscillation of the means of Cesàro and Riesz of the first order", *Journal London Math. Soc.*, 8 (1933), 27–32.

und $0<\theta<1<\lambda$, *dann ist*

$$(\lambda-1)\bar{s}\leqslant\lambda\bar{\sigma}-\underline{\sigma}+\int_1^\lambda w_1(t)\,dt, \tag{4}$$

$$-(1-\theta)\underline{s}\leqslant\theta\bar{\sigma}-\underline{\sigma}+\int_\theta^1 w_1(1/t)\,dt, \tag{5}$$

also, für $\theta=1/\lambda$,

$$l_s\leqslant\frac{\lambda+1}{\lambda-1}\,l_\sigma+\frac{1}{\lambda-1}\int_1^\lambda\left\{w_1(t)+w_1\left(\frac{\lambda}{t}\right)\right\}dt. \tag{6}$$

Wird ausser (3) *noch*

$$w_2(\lambda)=\lim_{t=\infty}\sup\max_{t\leqslant t'\leqslant\lambda t}\{s(t')-s(t)\} \tag{7}$$

und

$$2w(\lambda)=w_1(\lambda)+w_2(\lambda)$$

gesetzt, so ist

$$(\lambda-\theta)\bar{s}\leqslant\lambda\bar{\sigma}-\theta\underline{\sigma}+\int_\theta^1 w_2(1/t)\,dt+\int_1^\lambda w_1(t)\,dt, \tag{8}$$

$$-(\lambda-\theta)\,\underline{s}\leqslant\theta\bar{\sigma}-\lambda\underline{\sigma}+\int_\theta^1 w_1(1/t)\,dt+\int_1^\lambda w_2(t)\,dt, \tag{9}$$

also

$$l_s\leqslant\frac{\lambda+\theta}{\lambda-\theta}\,l_\sigma+\frac{2}{\lambda-\theta}\left\{\int_\theta^1 w(1/t)\,dt+\int_1^\lambda w(t)\,dt\right\}. \tag{10}$$

Beweis. Aus

$$\begin{aligned}(\lambda-1)\,s(x)&=\lambda\sigma(\lambda x)-\sigma(x)-\int_1^\lambda\{s(tx)-s(x)\}\,dt\\&\leqslant\lambda\sigma(\lambda x)-\sigma(x)-\int_1^\lambda\min_{x\leqslant t'\leqslant tx}\{s(t')-s(x)\}\,dt\\&\leqslant\lambda\bar{\sigma}-\underline{\sigma}+\int_1^\lambda w_1(t)\,dt+o(1)\quad(x\to\infty),\end{aligned}$$

folgt (4). Ebenso folgt aus

$$\begin{aligned}(1-\theta)\,s(x)&=\sigma(x)-\theta\sigma(\theta x)+\int_\theta^1\{s(x)-s(tx)\}\,dt\\&\geqslant\sigma(x)-\theta\sigma(\theta x)+\int_\theta^1\min_{tx\leqslant t'\leqslant x}\{s(t')-s(tx)\}\,dt\\&\geqslant\underline{\sigma}-\theta\bar{\sigma}-\int_\theta^1 w_1(1/t)\,dt+o(1)\quad(x\to\infty),\end{aligned}$$

die Ungleichung (5). In entsprechender Weise können aus

$$(\lambda-\theta)s(x)=\lambda\sigma(\lambda x)-\theta\sigma(\theta x)+\int_\theta^1\{s(x)-s(tx)\}dt-\int_1^\lambda\{s(tx)-s(x)\}dt$$

die Ungleichungen (8) und (9) erhalten werden.

2. Für spezielle Funktionen w können diese Ungleichungen in folgender, verschärfter Form geschrieben werden:

SATZ 2. *Ist im Satze* 1

$$w_1(\lambda)\leqslant H_1\log\lambda, \tag{11}$$

dann ist die schärfste Form der Ungleichungen (4) *und* (5)

$$e^{(\bar s-\bar\sigma)/H_1}-1-\frac{\bar s-\underline\sigma}{H_1}\leqslant 0, \tag{12}$$

und

$$e^{(\underline s-\bar\sigma)/H_1}-1-\frac{\underline s-\underline\sigma}{H_1}\leqslant 0. \tag{13}$$

Ist ausserdem noch

$$w_2(\lambda)\leqslant H_2\log\lambda, \tag{14}$$

so ist die schärfste Form der Ungleichungen (8), (9), *und* (10)

$$H_1(e^{(\bar s-\bar\sigma)/H_1}-1)+H_2(e^{(\underline\sigma-\bar s)/H_2}-1)\leqslant 0, \tag{15}$$

$$H_1(e^{(\underline s-\bar\sigma)/H_1}-1)+H_2(e^{(\bar\sigma-\underline s)/H_2}-1)\leqslant 0, \tag{16}$$

und

$$e^{l_s/(H_1+H_2)}+e^{-l_s/(H_1+H_2)}\leqslant 2e^{l_\bullet/(H_1+H_2)}. \tag{17}$$

Beweis. Wegen (11) folgt aus (4) und (5)

$$0\leqslant(\bar s-\underline\sigma)-\lambda(\bar s-\bar\sigma)+H_1\lambda\log\lambda-H_1(\lambda-1)$$

und

$$0\leqslant\theta(\bar\sigma-\underline s)-(\underline\sigma-\underline s)+H_1\theta\log\theta-H_1(\theta-1),$$

welche Ausdrücke ihren kleinsten Wert für

$$\lambda=e^{(\bar s-\bar\sigma)/H_1}\quad\text{und}\quad\theta=e^{-(\bar\sigma-\underline s)/H_1}$$

annehmen, was die Ungleichungen (12) und (13) liefert.

Ähnlich folgt aus (8), (9), und (10), wegen (11) und (14),

$$0\leqslant\theta(\bar s-\underline\sigma)-\lambda(\bar s-\bar\sigma)+H_2\theta\log\theta-H_2(\theta-1)+H_1\lambda\log\lambda-H_1(\lambda-1),$$

$$0\leqslant\theta(\bar\sigma-\underline s)-\lambda(\underline\sigma-\underline s)+H_1\theta\log\theta-H_1(\theta-1)+H_2\lambda\log\lambda-H_2(\lambda-1)$$

und

$$0\leqslant\theta(l_\sigma+l_s)-\lambda(l_s-l_\sigma)+(H_1+H_2)(\theta\log\theta+\lambda\log\lambda+2-\theta-\lambda),$$

von welchen Ungleichungen die erste die schärfste Form für

$$\theta = e^{-(\bar{s}-\underline{\sigma})/H_2}, \quad \lambda = e^{(\underline{\sigma}-\underline{s})/H_1},$$

die zweite für

$$\theta = e^{-(\bar{\sigma}-\underline{s})/H_1}, \quad \lambda = e^{(\underline{\sigma}-\underline{s})/H_2},$$

und die dritte für

$$\theta = e^{-(l_s+l_\sigma)/(H_1+H_2)}, \quad \lambda = e^{(l_s-l_\sigma)/(H_1+H_2)},$$

annimmt.

3. Der nachstehende Satz von Fekete und Winn ist nun im Satz 2 enthalten:

SATZ 3. *Sei*

$$0 < \lambda_0 < \lambda_1 < \ldots < \lambda_{n-1} < \lambda_n \to \infty \quad (n \to \infty),$$

$$\lambda_{n+1}\sigma_n = \sum_{\nu=0}^{n} (\lambda_{\nu+1}-\lambda_\nu)\, s_\nu,$$

$$\underline{s} = \lim_{n=\infty} \begin{matrix}\sup \\ \inf\end{matrix} s_n, \qquad \underline{\sigma} = \lim_{n=\infty} \begin{matrix}\sup \\ \inf\end{matrix} \sigma_n,$$

und

$$\lambda_n(s_n - s_{n-1}) > -H_1(\lambda_n - \lambda_{n-1}), \tag{18}$$

dann sind die Ungleichungen (12) *und* (13) *erfüllt falls*

$$\lim_{n=\infty} \inf\, (s_n - s_{n-1}) \geqq 0 \tag{19}$$

ist, was insbesondere aus (18) *folgt, wenn* $\lambda_{n+1} \sim \lambda_n$ *ist.*

Ist ausser (18) *noch*

$$\lambda_n(s_n - s_{n-1}) < H_2(\lambda_n - \lambda_{n-1}), \tag{20}$$

so sind die Ungleichungen (15), (16), *und* (17) *erfüllt falls*

$$s_n - s_{n-1} \to 0 \quad (n \to \infty), \tag{21}$$

was insbesondere aus (18) *und* (19) *folgt, wenn* $\lambda_{n+1} \sim \lambda_n$ *ist.*

Wird nämlich in (2)

$$s(x) = s_n = \sum_{\nu=0}^{n} u_\nu \quad \text{für} \quad \lambda_n \leqq x < \lambda_{n+1} \quad (n = 0, 1, 2, \ldots)$$

gesetzt, so ist

$$\sigma(x) = \sum_{\lambda_\nu \leqq x} \left(1 - \frac{\lambda_\nu}{x}\right) u_\nu,$$

was (für $x \to \infty$) dieselben Oszillationsgrenzen wie

$$\sigma_n = \sigma(\lambda_{n+1}) = \frac{1}{\lambda_{n+1}} \sum_{\nu=0}^{n} (\lambda_{\nu+1} - \lambda_\nu) s_\nu, \quad \text{bei} \quad n \to \infty,$$

besitzt, weil $\sigma(x)$ in jedem Intervalle $(\lambda_n, \lambda_{n+1})$, $n = 0, 1, 2, \ldots$, monoton ist.

Ausserdem ist (11), d.h.

$$-w_1(\lambda) = \lim_{n=\infty} \inf \min_{n \leqq \nu \leqq r} \{s_\nu - s_{n-1}\} \geqq -H_1 \log \lambda,$$

mit

$$\lambda_r \leqq \lambda \,.\, \lambda_n < \lambda_{r+1},$$

erfüllt, falls (18) und (19) gilt, weil

$$\begin{aligned} \min_{n \leqq \nu \leqq r} \{s_\nu - s_{n-1}\} &\geqq \min_{n+1 \leqq \nu \leqq r} \sum_{i=n+1}^{\nu} u_i + \tfrac{1}{2}(u_n - |u_n|) \\ &\geqq -H_1 \log \frac{\lambda_r}{\lambda_n} + \tfrac{1}{2}(u_n - |u_n|) \\ &\geqq -H_1 \log \lambda + o(1) \quad (n \to \infty), \end{aligned}$$

ist. Ebenso ist (14) erfüllt falls (20) und (21) gilt, was die Behauptungen liefert.

4. Was die Ungleichung (1) von Fejér betrifft, so gibt (6) ihre schärfste Form falls $\{s(t') - s(t)\}$ einseitig, und (10) falls dieser Ausdruck beiderseits beschränkt ist.

Im Falle wo

$$w_1(\lambda) \leqq H_1 \log \lambda \quad \text{bzw.} \quad w_2(\lambda) \leqq H_2 \log \lambda$$

ist, wird das letzte Glied in (1)

$$2W(\lambda) = 2H \log \lambda, \quad \text{wo} \quad H = \max(H_1, H_2) \text{ ist.}$$

Die Ungleichung (6) liefert aber für dieses Glied

$$H_1 \log \lambda \quad (\text{oder entsprechend } H_2 \log \lambda)$$

und die Ungleichung (10), (indem λ durch $\sqrt{\lambda}$ ersetzt und $\theta = 1/\sqrt{\lambda}$ gesetzt wird), nur

$$H \frac{\sqrt{\lambda}}{1 + \sqrt{\lambda}} \log \lambda$$

wo

$$2H = H_1 + H_2$$

gesetzt ist. In diesem letzten Falle kann aber dieses Glied sogar bis auf

$$\frac{H}{2}\log\lambda$$

herabgedrückt werden, indem in (10) λ durch $2\lambda/(1+\lambda)$ ersetzt und $\theta = 2/(1+\lambda)$ gesetzt wird. In der Tat nimmt dann das letzte Glied von (10) die Form an:

$$\frac{2H}{\lambda-1}\left\{\lambda\log\lambda-(1+\lambda)\log\frac{1+\lambda}{2}\right\}<\frac{H}{2}\log\lambda.$$

Also ist
$$l_s \leqslant \frac{\lambda+1}{\lambda-1}\,l_\sigma+\frac{H}{2}\log\lambda.$$

Wird hier $\log\lambda$ durch $\lambda-1$ vergrössert, und

$$\lambda-1=2\sqrt{(l_\sigma/H)}$$

gesetzt, so wird
$$l_s \leqslant l_\sigma+2\sqrt{(Hl_\sigma)}$$

oder
$$\sqrt{(H+l_s)} \leqslant \sqrt{H}+\sqrt{l_\sigma}$$

was eine schwächere Form der Ungleichung (17) darstellt.

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ON FUNCTIONS BOUNDED AT THE LATTICE POINTS IN AN ANGLE

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1. The problem of integral functions of order less than or equal to two which are bounded at the lattice points $m \pm in$ has been considered by various writers*, in particular by J. M. Whittaker and by Pfluger. These two authors have each applied their methods to a similar problem for functions regular in an angle. Whittaker proved†

THEOREM A. *Let $f(z)$ be a function regular in the sector $S(\alpha, l)$ defined by $|\arg z| \leqslant \alpha < \pi$, $|z| \geqslant l$. Suppose that*

$$\overline{\lim_{r\to\infty}} \frac{\log M(r, \alpha)}{r^2} \leqslant 0, \quad M(r, \alpha) = \max_{-\alpha\leqslant\theta\leqslant\alpha} |f(re^{i\theta})|, \tag{1}$$

$$|f(z)| < A \text{ on the boundary of } S(\alpha, l), \tag{2}$$

(3) $|f(m+in)| < A$ *for all integers m and n such that $m+in$ lies in $S(\alpha, l)$. Then*

$$|f(z)| < A \text{ in } S(\alpha, l). \tag{4}$$

Pfluger‡ has proved a more general result. The chief features of it are that (1) is replaced by

$$\underline{\lim_{r\to\infty}} \frac{\log M(r, \alpha)}{r^2} \leqslant k < \tfrac{1}{2}\pi, \tag{5}$$

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* See Whittaker, **7** and **8**; Pólya, **6**; Ganapathy Iyer, **2**; Pfluger, **4**.

† Whittaker, **8**.

‡ Pfluger, **4**.

and that the A in (3) may be replaced by any positive number A', the conclusion remaining the same.

2. It should be observed that (2) is a very heavy condition. For (1) and (2) imply (4) when $\alpha \leqslant \frac{1}{4}\pi$, and (5) and (2) imply (4) when $\alpha < \frac{1}{4}\pi$, without (3). Also Pfluger's result with A' in (3) shows that the final constant depends on (2), and not on the value of A in (3). Whittaker conjectured that (2) might be replaced by the hypothesis $\alpha > \frac{1}{4}\pi$. It seems to me unlikely that the conclusion could then hold near the boundary of $S(\alpha, l)$, but for sectors $S(\beta, l)$, $\beta < \alpha$, Pfluger's method gives the following result.

THEOREM 1. *Suppose that $f(z)$ is regular in $S(\alpha, l)$, $\alpha \leqslant \pi$, and satisfies* (3) *and* (5). *Suppose that, for some $\delta > 0$,*

$$\gamma = \max_{\alpha-\delta \leqslant |\theta| \leqslant \alpha} \{h(\theta)\} < \tfrac{1}{2}\pi, \quad h(\theta) = \overline{\lim_{r\to\infty}} \frac{\log |f(re^{i\theta})|}{r^2}; \tag{6}$$

and, if $\alpha < \frac{1}{2}\pi$, suppose, in addition, that $\gamma < \frac{1}{2}\pi \sin^2\alpha$. Then

$$|f(re^{i\theta})| = O(1) \tag{7}$$

as $r \to \infty$ uniformly for $|\theta| \leqslant \beta < \alpha$, where $\sin^2(\alpha-\beta) > 2\gamma/\pi$.

In particular, if (1) *and* (3) *hold, then* (7) *holds uniformly for $|\theta| \leqslant \beta$ for every $\beta < \alpha$.*

In this result the value of α has no significance, except in connection with the value of γ when $\gamma > 0$. Since $\gamma < \frac{1}{2}\pi$, it is always possible to find β such that $\sin^2(\alpha-\beta) > 2\gamma/\pi$ when $\alpha \geqslant \frac{1}{2}\pi$. The constant implied in the O depends on the behaviour of the function near the boundary of $S(\alpha, l)$ for small values of $|z|$, as well as on A, γ, and β.

Proof. It will be sufficient to show that if z lies in the square with corners $M \pm \frac{1}{2} + i(N \pm \frac{1}{2})$, where $M+iN$ is any lattice point in $S(\beta, R_0)$ or within a distance 1 of $S(\beta, R_0)$, then

$$|f(z)| < K_1, \tag{8}$$

where K_1 is independent of M and N. For these squares cover the sector $S(\beta, R_0)$. We suppose, of course, that $\alpha-\beta \leqslant \frac{1}{2}\pi$, $\sin^2(\alpha-\beta) > 2\gamma/\pi$, and that R_0 is large.

Let $S'(\beta, R_0)$ denote the region formed by drawing the curve whose shortest distance to the boundary of $S(\beta, R_0)$ is 1 at all points. Let $\sigma(z)$

denote the Weierstrass σ function, and let

$$\phi_{M,N}(z) = \sigma(z-M-iN) \sum_{S(\alpha,l)} \frac{f(m+in)(-1)^{m-M+n-N+(m-M)(n-N)}}{(z-M-iN)\exp\{\frac{1}{2}\pi[(m-M)^2+(n-N)^2]\}},$$

where $M+iN$ is any lattice point in $S'(\beta, R_0)$ and the summation is extended over all lattice points $m+in$ in $S(\alpha, l)$. Then $\phi_{M,N}(m+in) = f(m+in)$, and so

$$\Phi_{M,N}(z) = \frac{f(z)-\phi_{M,N}(z)}{\sigma(z-M-iN)}$$

is regular in $S(\alpha, l)$. Let $r_1, r_2, \ldots r_n \to \infty$ be a sequence such that

$$\lim_{n\to\infty} \frac{\log M(r_n, \alpha)}{{r_n}^2} = k < \tfrac{1}{2}\pi,$$

and let $C(r_n)$ be a contour joining $\arg z = \pm\alpha$, and lying in the region $r_n - 1 \leqslant r \leqslant r_n$. We can certainly choose $C(r_n)$ so that the shortest distance from $C(r_n)$ to any lattice point in $S(\alpha, l)$ is at least $\frac{1}{4}$. Then it follows, as in Pfluger's proof, that

$$|\Phi_{M,N}(z)| < K_2(M, N)\exp(-\delta {r_n}^2) + AK_3, \tag{9}$$

where $0 < \delta < \frac{1}{2}\pi - k$ and K_3 is independent of M and N, on $C(r_n)$. We can also construct a contour $C(\alpha)$ inside $S(\alpha, l)$ at a distance less than 1 from the boundary of $S(\alpha, l)$, and at a distance greater than or equal to $\frac{1}{4}$ from any lattice point. As in Pfluger's theorem, we have

$$\left|\frac{\phi_{M,N}(z)}{\sigma(z-M-iN)}\right| \leqslant A K_4, \tag{10}$$

where K_4 is independent of M and N. It follows from (5) and (6), by the theorems of Phragmén and Lindelöf*, that, for every $\epsilon_1 > 0$,

$$\log|f(re^{i\theta})| < (\gamma+\epsilon_1) r^2$$

for $-\alpha \leqslant \theta \leqslant -\alpha+\delta$, $\alpha-\delta \leqslant \theta \leqslant \alpha$, provided that $r > r_0(\epsilon_1)$. Hence on $C(\alpha)$, for $|z| = r_0$, we have

$$\left|\frac{f(z)}{\sigma(z-M-iN)}\right| < K_5(\epsilon_1)\exp\{(\gamma+\epsilon_1)|z|^2 - (\tfrac{1}{2}\pi-\epsilon_2)|z-M-iN|^2\}$$

* Phragmén and Lindelöf, 5.

for every $\epsilon_2 > 0$, provided that $|z-M-iN| > R_0'(\epsilon_2)$; and for these points z we have

$$|z-M-iN| > |z| \sin(\alpha-\beta)-2,$$

provided that the lattice point $M+iN$ lies in $S'(\beta, R_0)$. Since

$$\gamma < \tfrac{1}{2}\pi \sin^2(\alpha-\beta),$$

we can choose ϵ_1 and ϵ_2 so that

$$\left| \frac{f(z)}{\sigma(z-M-iN)} \right| < K_5 \exp\{-\tfrac{1}{2}\epsilon_2 \sin^2(\alpha-\beta)|z|^2\} < K_6,$$

where K_6 is independent of M and N, on $C(\alpha)$ for $|z| > r_0$, and $M+iN$ in $S'(\beta, R_0)$.

At all points on $C(\alpha)$

$$|z-M-iN| > \{(M^2+N^2)^{\frac{1}{2}}-l\} \sin(\alpha-\beta);$$

and so, using this for $|z| \leqslant r_0$, we have

$$\left| \frac{f(z)}{\sigma(z-M-iN)} \right| < K_6$$

for all lattice points $M+iN$ in $S'(\beta, R_0)$, provided that R_0 is sufficiently large. The right-hand side of (9) tends to AK_3 as $n \to \infty$ for every fixed M and N, and so

$$|\Phi_{M,N}(z)| < K_7, \tag{11}$$

where K_7 is independent of M and N, at all points on $C(r_n)$ and $C(\alpha)$ for $n > n_0$. Since $\Phi_{M,N}(z)$ is regular in the region enclosed by these curves, it follows from the maximum modulus principle that (11) holds inside this region, and in particular in the square with corners $M \pm \frac{1}{2} + i(N \pm \frac{1}{2})$. Now $|\sigma(z-M-iN)| < K_8$ in this square, and so it follows from (10) and (11) that (8) holds in this square, and this, as we have already explained, gives the required result.

3. If (1) holds, the size of α has no bearing on the result, but there are results of a similar kind in which the hypothesis $\alpha > \frac{1}{4}\pi$ suggested by Whittaker is significant. Elsewhere* I have proved the following theorem.

* Cartwright, **1**, Theorem II.

THEOREM B. *Suppose that $f(z)$ is regular in $S(\alpha, l)$, where $\alpha > \frac{1}{4}\pi$, suppose also that*

$$\overline{\lim_{r\to\infty}} \frac{\log\log M(r, \alpha)}{\log r} = \rho \qquad (0 \leqslant \rho \leqslant 2), \tag{12}$$

$$\overline{\lim_{r\to\infty}} \frac{\log|f(re^{i\theta})|}{r^2} = 0 \quad \textit{for} \quad |\theta| \leqslant \alpha. \tag{13}$$

Then, (i) *for every $\delta > 0$,*

$$\overline{\lim_{r\to\infty}} \frac{n(r, \alpha-\delta)}{r^2} = 0,$$

where $n(r, \alpha)$ denotes the number of zeros of $f(z)$ for which $l \leqslant |z| \leqslant r$, $|\arg z| \leqslant \alpha$,

(ii) *for every ϵ, δ, η, $\zeta > 0$, we have*

$$\log|f(re^{i\theta})| > -\epsilon r^2 \tag{14}$$

for $|\theta| \leqslant \alpha-\delta$, $\eta R \leqslant r \leqslant R$, $R > R_0(\epsilon, \delta, \eta, \zeta)$, except perhaps for values of $re^{i\theta}$ which can be enclosed in circles the sum of whose radii is less than ζR.

The only use of (12) is to ensure the uniformity of the limit in (13) with respect to θ, and (12) can be replaced by

$$\overline{\lim_{r\to\infty}} \frac{\log\log M(r, \alpha)}{\log r} \leqslant 2. \tag{15}$$

Since $\alpha > \frac{1}{4}\pi$, it follows, from the theorems of Phragmén and Lindelöf*, that, if (1) holds, then either (13) and (15) hold or $f(z) \equiv 0$. For (15) obviously holds, and

$$h(\theta) = \overline{\lim_{r\to\infty}} \frac{\log|f(re^{i\theta})|}{r^2} \leqslant 0.$$

If $f(z)$ does not vanish identically, any interval in which $h(\theta) < 0$ must be of length $\frac{1}{2}\pi$ at most, and must be adjacent to intervals in which $h(\theta) > 0$ or to the arms of the angle of the theorem. Since $2\alpha > \frac{1}{2}\pi$, $h(\theta)$ cannot be negative, *i.e.* $h(\theta) = 0$ and so (13) holds.

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* Phragmén and Lindelöf, **5.**

Now the *number* of excepted circles in Theorem B depends on the number in Lemma 1 of my paper* which depends on the number in Cartan's lemma. This is at most the number of zeros of the polynomial considered, and in Lemma 1 it is at most $n(R/k')$, where $1 < k' < k$. It follows that the number excepted in Theorem B is at most $n(k'' R, \alpha - \frac{1}{2}\delta)$, where k'' depends only on δ and η. The number of lattice points covered by any circle of diameter greater than 1 is less than four times the square of its diameter, and four times the sum of the squares of the diameters of such circles is at most four times the sum of the radii squared, which is at most $4\zeta^2 R^2$. The number of lattice points covered by circles of diameter less than or equal to 1 is at most four times the number of circles, *i.e.*,

$$4n(k'' R, \alpha - \tfrac{1}{2}\delta) < \epsilon_1 R^2$$

for every $\epsilon_1 > 0$, provided that $R > R_0(\epsilon_1)$. Hence the total number of lattice points excepted from (14) is less than

$$4\zeta^2 R^2 + \epsilon_1 R^2 < \epsilon_2 R^2$$

for every $\epsilon_2 > 0$, $R > R_0(\epsilon_2)$. The total number of lattice points in the region $\eta R \leqslant |z| \leqslant R$, $|\arg z| \leqslant \alpha - \delta$ is approximately $2\alpha(1-\eta) R^2$, and so we may state the result as follows.

THEOREM 2. *Suppose that $f(z)$ is regular in $S(\alpha, l)$, where $\alpha > \frac{1}{4}\pi$, and satisfies* (1). *Then either $f(z) \equiv 0$ or*

$$\log|f(m+in)| > -\epsilon(m^2+n^2) \tag{16}$$

for almost all lattice points $m+in$ in $S(\beta, l)$ for every $\beta < \alpha$.

Mr. Pennycuick† has proved a similar theorem in which (1) is replaced by

$$\lim_{r\to\infty} \frac{\log M(r, \alpha)}{r^2} = 0,$$

and no condition is imposed on α. A consideration of the function e^{-z^2} shows that if $\alpha \leqslant \frac{1}{4}\pi$, (1) is insufficient to obtain (16). Pennycuick's method is based on an interpolation theorem of Whittaker‡.

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* Cartwright, **1**.
† Pennycuick, **3**.
‡ Whittaker, **9**, 57.

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REGULAR SKEW POLYHEDRA IN THREE AND FOUR DIMENSIONS, AND THEIR TOPOLOGICAL ANALOGUES

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The five regular ("Platonic") polyhedra,

$$\{3, 3\}, \quad \{3, 4\}, \quad \{4, 3\}, \quad \{3, 5\} \quad \{5, 3\}^*,$$

have been known and admired for thousands of years. As analogues of these, Kepler drew attention to the three regular plane-fillings,

$$\{4, 4\}, \quad \{3, 6\}, \quad \{6, 3\},$$

which may be regarded as regular polyhedra with infinitely many faces. He also introduced two of the four *star* polyhedra,

$$\{5, \tfrac{5}{2}\}, \quad \{\tfrac{5}{2}, 5\}, \quad \{3, \tfrac{5}{2}\}, \quad \{\tfrac{5}{2}, 3\}\dagger;$$

the remaining two were added by Poinsot, but it was Cauchy‡ who first proved that there are *only* four. These have the disadvantage of possessing "false vertices", *i.e.* points other than vertices, where three or more faces meet.

One day in 1926, J. F. Petrie told me with much excitement that he had discovered two new regular polyhedra; infinite, but free from false vertices. When my incredulity had begun to subside, he described them to me: one consisting of squares, six at each vertex (Fig. i), and one

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* The meaning of these symbols, introduced by Schläfli, is clear from the following example: the *cube* is {4, 3}, because it is bounded by *squares*, *three* at each vertex.

† This extension of the above notation is also due to Schläfli. The four polyhedra are, respectively, the *great dodecahedron*, the *small stellated dodecahedron*, the *great icosahedron*, the *great stellated dodecahedron*.

‡ Cauchy, 1. *Cf.* Coxeter, 1, 203.

consisting of hexagons, four at each vertex (Fig. ii). It was useless to protest that there is *no room* for more than four squares round a vertex. The trick is, to let the faces go up and down in a kind of zig-zag formation, so that the faces that adjoin a given "horizontal" face lie alternately "above" and "below" it. When I understood this, I pointed out a third possibility: hexagons, six at each vertex (Fig. iii).

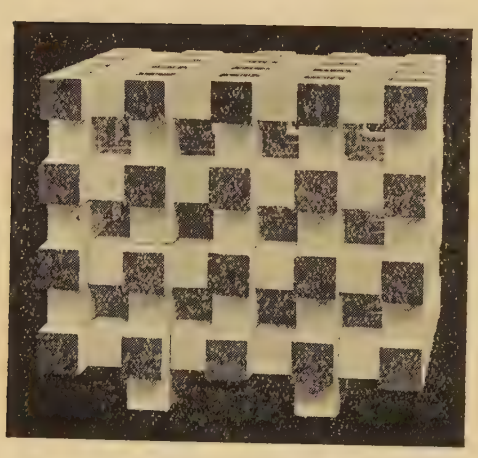


Fig. i: {4, 6|4}.

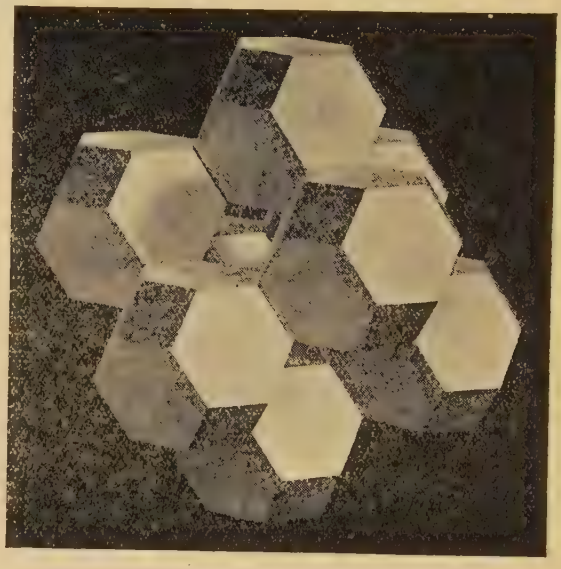


Fig. ii: {6, 4|4}.

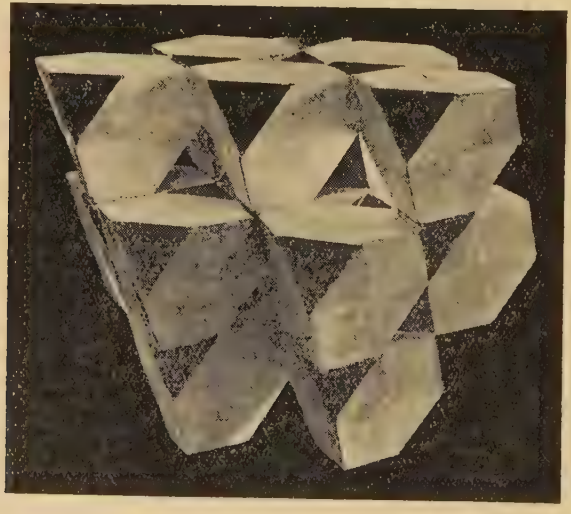


Fig. iii: {6, 6|3}.

It then occurred to us that, although these new polyhedra are infinite, we might find analogous *finite* polyhedra by going into four-dimensional space. Petrie cited one consisting of n^2 squares, four at each vertex. We call such figures *skew polyhedra*, by analogy with "skew polygons". They must not be confused with *polytopes* (which have solid faces). Actually, every regular skew polyhedron is intimately associated with a definite *uniform* polytope*. *E.g.*, Fig. i consists of half the squares of $\{4, 3, 4\}$ or δ_4, the net of cubes; Fig. ii consists of all the hexagons of $t_{1,2}\delta_4$, the net of truncated octahedra; and Fig. iii† consists of all the hexagons of $q\delta_4$, the net of tetrahedra and truncated tetrahedra‡.

The reader naturally asks, What do we mean by saying that these new polyhedra are *regular*? To explain this, we have to go back to first principles.

1. *Definitions.*

We define a *polyhedron* as a connected set of ordinary plane polygons§, such that every side of every polygon belongs also to just one other polygon of the set. We stipulate that any two of the polygons shall have in common either a side (and two vertices), or a single vertex, or nothing at all (thus ruling out the Kepler-Poinsot polyhedra). We define a *symmetry* (or "symmetry operation") of any figure as a congruent transformation of the figure into itself (*i.e.*, a combination of translations, rotations, and reflections). A completely irregular figure has no symmetry save identity.

A polyhedron is said to be *regular* if it possesses two particular symmetries: one which cyclically permutes the vertices of any face c, and one which cyclically permutes the faces that meet at a vertex C, C being a vertex of c. It follows that these two symmetries, say R and S, generate a group which is transitive on the vertices, on the edges, and on the faces. Moreover, the faces are regular polygons (in the most elementary sense). This clearly agrees with our preconceived notion of regularity.

Perhaps this will seem clearer if we give the analogous definition for a regular *polygon*. A polygon (which may be skew) is said to be regular if it possesses a symmetry which cyclically permutes the vertices (and therefore also the sides) of the polygon. Regular polygons can be classified

* Coxeter, **4**.

† For this picture some apology is required. The lines which divide each hexagon into six triangles are to be ignored, since they arise merely from the manner in which this particular model was built up.

‡ Andreini, **1**, 106 (Fig. 14), 107 (Fig. 15).

§ In § 5, we relax this definition, and consider a connected set of *topological* polygons.

according to the nature of this symmetry. The cases that arise in ordinary space are as follows:

Symmetry.	*Polygon.*
Reflection	Digon, $\{2\}$, with two vertices and two coincident sides.
Rotation (through $2\pi/n$)	Ordinary n-gon, which we denote by $\{n\}$.
Rotary reflection (involving rotation through π/n)	Finite skew $2n$-gon.
Translation	Apeirogon, $\{\infty\}$ (*i.e.*, an infinite straight line broken into equal segments).
Glide	Plane zig-zag (dividing the plane into two equal parts).
Screw	Helical polygon.

Since the "square" of a rotary reflection is a pure rotation, the vertices of a regular *finite* skew polygon lie alternately on two circles which reflect into one another in a plane parallel to the planes of the circles (and therefore half-way between them). In fact, the sides of such a polygon are the lateral edges of an *antiprism* (*e.g.*, Fig. iv). Hence the number of sides must be *even*. (If there are only four sides, these are edges of a tetragonal bisphenoid, which can be regarded as a *digonal* antiprism.)

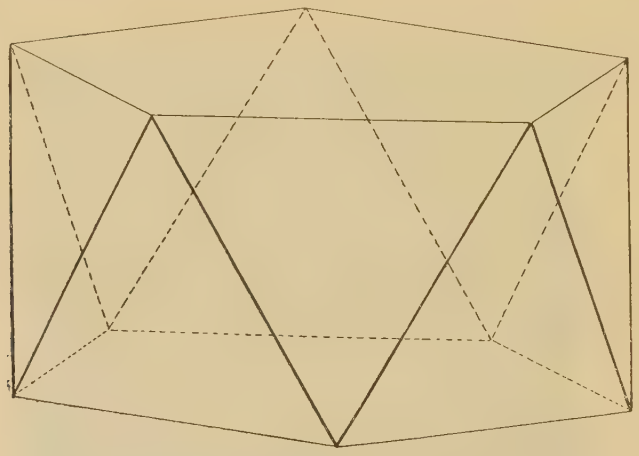


Fig. iv: The skew decagon in a pentagonal antiprism.

The line joining two alternate vertices of an ordinary regular n-gon (of unit side) is of length $2\cos(\pi/n)$, and is called the *vertex figure* of the $\{n\}$.

For a regular polyhedron (of unit edge), consider those vertices which are joined by edges to any one vertex. These can be regarded as the vertices of a finite polygon whose sides are vertex figures of faces of the polyhedron. This polygon is called the *vertex figure* of the polyhedron. It is a *regular* polygon, since its sides are cyclically permuted by the special symmetry S.

If a regular polyhedron is four-dimensional and finite, its vertices are equidistant from their centroid, which we call the *centre* of the polyhedron. Hence those vertices which are joined by edges to any one vertex lie on the intersection of two hyper-spheres, *i.e.* on an ordinary sphere. Thus in this case, as well as when the polyhedron (finite or infinite) is three-dimensional, the vertex figure lies in a three-space. By taking as vertex figure a plane polygon, we obtain an ordinary regular solid or plane-filling; but the vertex figure may just as well be a skew polygon, whose vertices lie alternately in two parallel planes.

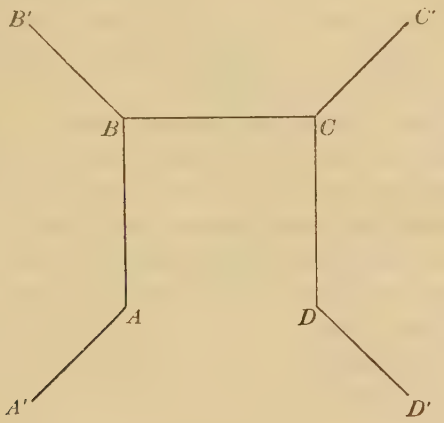


Fig. v.

Let us first suppose the polyhedron to lie in ordinary space. It divides space into two parts, one "inside" and one "outside" the polyhedron. When the vertex figure is a skew polygon, the inside and outside are alike (*i.e.* congruent) in the neighbourhood of one vertex, and are therefore alike altogether. (In fact, R and S each interchange the inside and outside.) Hence the polyhedron must be infinite. On account of the zig-zag nature of the skew polygon, the faces that surround a given "horizontal" face are alternately "above" and "below" it. Hence R, like S, is a rotary reflection.

The situation when the polyhedron is finite in four dimensions is very similar. For, after projecting the polyhedron on to its circumscribed hyper-sphere, we have again a surface lying in a manifold of three dimen-

sions, and dividing that manifold into two equal parts. R and S are still rotary reflections, when regarded as operating in the spherical three-space.

Let A′ABB′ ..., B′BCC′ ..., C′CDD′ ... (Fig. v) be three faces of any regular polyhedron. We may suppose that R cyclically permutes the edges B′B, BC, CC′, ... which belong to the second of these faces, while S cyclically permutes the edges CD, CC′, CB, ... which meet at C. Then the product RS reverses the edge BC, while the "quotient" RS^{-1} cyclically permutes the edges AB, BC, CD, ... Hence ABCD ... is a regular polygon (but may be skew). The lines AA′, BB′, CC′, DD′, ... are clearly concurrent, say at O. (When the faces are squares, so that these edges are parallel, we say that O is at infinity.) If the polyhedron is in three dimensions, the planes ABB′, DCC′ are equally inclined to the plane B′BCC′, on the same side of it; therefore ABCD ... is a plane polygon. If the polyhedron is finite in four dimensions, the symmetry RS^{-1}, being the product of two rotary reflections, is a *positive* transformation. Since it leaves the centre invariant, this must be either a simple rotation or a double rotation. Since it also leaves the point O invariant, it can only be a simple rotation. Hence ABCD ... is again a plane polygon.

In Figs. i, ii, iii, this polygon appears as a hole (triangular in Fig. iii, square in Figs. i and ii), and it is convenient to adopt the term *hole* in other cases, too. The hole may be described as a path along edges, such that at the end of each edge we leave two faces on (say) the left. In other words, at any vertex of the path, the edges to be selected are not adjacent but alternate. When (as above) the hole is a plane polygon, we may determine it from its vertex figure, which is the join of two alternate vertices of the vertex figure of the polyhedron. In the case of the ordinary polyhedron $\{l, m\}$, whose vertex figure is an $\{m\}$ of side $2\cos(\pi/l)$, it follows that the "hole"* is an $\{n\}$, where

$$2\cos(\pi/n) = 2\cos(\pi/l) \,.\, 2\cos(\pi/m),$$

i.e.

$$\cos(\pi/n) = 2\cos(\pi/l)\cos(\pi/m). \tag{1.1}$$

We shall use the symbol $\{l, m \mid n\}$ to denote any regular polyhedron which is uniquely determined by

$$\begin{cases} l, \text{ the number of vertices or sides of a face,} \\ m, \text{ the number of edges or faces at a vertex,} \\ n, \text{ the number of vertices or sides of a hole.} \end{cases}$$

* Here the term "hole" is not entirely appropriate. But triangular "pits" are clearly visible in the star polyhedron $\{5, \frac{5}{2}\}$ (Fig. xiii).

We shall see that every regular polyhedron in three dimensions, or finite in four, is so determined. From this point of view,

$$\{n, 3\} \text{ is } \{n, 3|n\},$$
$$\{3, n\} \text{ is } \{3, n|n\},$$
$$\{4, 4\} \text{ is } \{4, 4|\infty\}.$$

2. *The trigonometrical criterion for* $\{l, m|n\}$.

Consider first the general infinite regular polyhedron in three dimensions. Let λ, λ' denote the planes that perpendicularly bisect two adjacent edges, BC, CC', belonging to the face B'BCC'... (Fig. v), and let μ, μ' denote the planes that bisect the dihedral angles at those same edges. These four planes bound a tetrahedron, whose angles we proceed to calculate.

The planes λ, μ, and likewise λ', μ', are obviously perpendicular. The planes λ, λ' are both perpendicular to the plane BCC', and meet it in the perpendicular bisectors of two adjacent sides of the $\{l\}$ B'BCC'...; so the angle between them is $2\pi/l$. The bisecting planes of the dihedral angles at the edges through C all pass through the axis of the rotary reflection S; since there are m such edges, the angle between two consecutive bisecting planes, such as μ, μ', is $2\pi/m$. The planes λ, μ' are both perpendicular to the plane BCD, and meet it in the bisectors of the side BC and angle BCD of the $\{n\}$ ABCD...; so the angle between them is π/n. Collecting results, we see that the six dihedral angles of the tetrahedron $\lambda\lambda'\mu\mu'$ are

$$(2.1) \qquad \begin{cases} (\lambda\mu) = (\lambda'\mu') = \frac{1}{2}\pi, \\ (\lambda\lambda') = 2\pi/l, \\ (\mu\mu') = 2\pi/m, \\ (\lambda\mu') = (\lambda'\mu) = \pi/n. \end{cases}$$

We have now reduced the problem of enumerating the possible regular polyhedra to that of enumerating the possible tetrahedra of a special type. For each tetrahedron, we have *two* polyhedra,

$$\{l, m|n\} \quad \text{and} \quad \{m, l|n\},$$

which may be called *reciprocal*, since the vertices of each are the centres of the faces of the other. Thus $\{4, 6|4\}$ and $\{6, 4|4\}$ (Figs. i and ii) are reciprocal, while $\{6, 6|3\}$ (Fig. iii) is self-reciprocal.

Now, just as the three angles of a plane triangle are not independent, but add up to π, so also the six dihedral angles of a tetrahedron are not

independent, but satisfy the relation

$$(2.2)\quad \Delta \equiv \begin{vmatrix} 1 & -\cos(a\,b) & -\cos(a\,c) & -\cos(a\,d) \\ -\cos(b\,a) & 1 & -\cos(b\,c) & -\cos(b\,d) \\ -\cos(c\,a) & -\cos(c\,b) & 1 & -\cos(c\,d) \\ -\cos(d\,a) & -\cos(d\,b) & -\cos(d\,c) & 1 \end{vmatrix} = 0,$$

where a, b, c, d are the four bounding planes. In the case under consideration,

$$\Delta = \left(4\cos^2\frac{\pi}{l}\cos^2\frac{\pi}{m} - \cos^2\frac{\pi}{n}\right)\left(4\sin^2\frac{\pi}{l}\sin^2\frac{\pi}{m} - \cos^2\frac{\pi}{n}\right).$$

After discarding two definitely positive factors, we are left with the condition

$$(2.3)\quad \left(2\cos\frac{\pi}{l}\cos\frac{\pi}{m} - \cos\frac{\pi}{n}\right)\left(2\sin\frac{\pi}{l}\sin\frac{\pi}{m} - \cos\frac{\pi}{n}\right) = 0.$$

By (1.1), the vanishing of the first factor gives the ordinary polyhedron $\{l, m\}$. Hence, to find all the possible skew polyhedra (in three dimensions), we merely have to solve in integers the equation

$$(2.4)\quad 2\sin(\pi/l)\sin(\pi/m) = \cos(\pi/n).$$

Apart from the plane-fillings

$$\{3, 6\,|\,6\},\quad \{6, 3\,|\,6\},\quad \{4, 4\,|\,\infty\},$$

which are already covered by (1.1)*, the only solutions are:

$$\{4, 6\,|\,4\},\quad \{6, 4\,|\,4\},\quad \{6, 6\,|\,3\}.$$

These infinite polyhedra may be called "complements" of the cube, octahedron, and tetrahedron, respectively, since each solution $\{l, m\,|\,n\}$ of (2.4) corresponds to a solution $\{l', m'\,|\,n\}$ of (1.1), where

$$1/l + 1/l' = 1/m + 1/m' = \tfrac{1}{2}.$$

We shall see later that "complementary" polyhedra have the same dihedral angles.

For each value of n, the relation between l and m may be presented graphically by taking l and m as Cartesian coordinates in a plane. (2.3) is then the equation for a couple of curves, which are shown in Fig. vi when $n = 3$, Fig. vii when $n = 4$. The finite polyhedra $\{l, m\}$ appear as O's on the curve

$$\cos(\pi/l)\cos(\pi/m) = \tfrac{1}{2}\cos(\pi/n),$$

* For the plane-fillings we have $1/l + 1/m = \frac{1}{2}$, so they can be regarded indifferently as ordinary or skew polyhedra.

while the infinite polyhedra $\{l, m \mid n\}$ appear as dots on the curve

$$\sin(\pi/l) \sin(\pi/m) = \tfrac{1}{2} \cos(\pi/n).$$

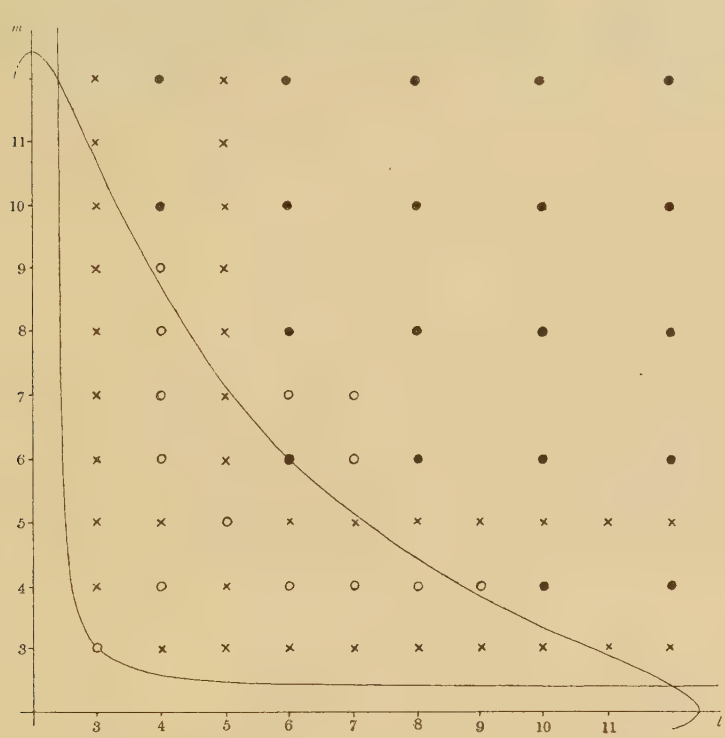


Fig. vi: Graphical enumeration of polyhedra $\{l, m \mid 3\}$.

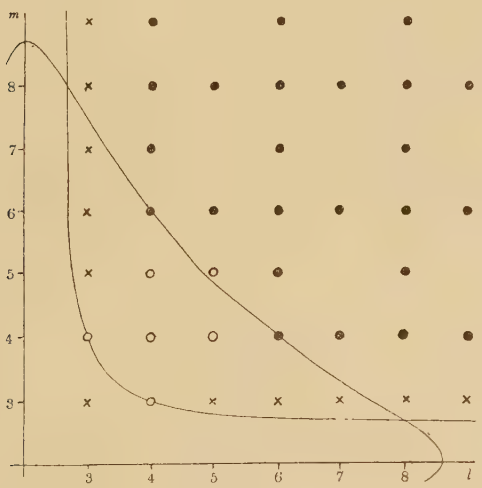


Fig. vii: Graphical enumeration of polyhedra $\{l, m \mid 4\}$.

In the case of the general finite regular polyhedron in four dimensions, the planes $\lambda, \lambda', \mu, \mu'$ have to be replaced by three-spaces through the centre, but the angles between them are still given by (2. 1). They can be regarded as bounding a spherical tetrahedron on the circumscribed hyper-sphere of the polyhedron. As before, the two polyhedra $\{l, m \mid n\}$ and $\{m, l \mid n\}$ are derivable from the same tetrahedron, and are *reciprocal* in the sense that the vertices, edges, and faces of one correspond to the faces, edges, and vertices of the other. The equation (2.2) has to be replaced by the inequality*

$$\Delta > 0,$$

which reduces to

$$\left(2 \cos \frac{\pi}{l} \cos \frac{\pi}{m} - \cos \frac{\pi}{n}\right)\left(2 \sin \frac{\pi}{l} \sin \frac{\pi}{m} - \cos \frac{\pi}{n}\right) > 0.$$

It is easily seen that the only significant case is when these two factors are positive†. *E.g.*, in Figs. vi and vii, the significant lattice points are those which lie inside the region enclosed by the two curves.

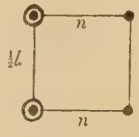
Since we restrict consideration to finite polyhedra in four dimensions, R and S (as we have seen) are rotary reflections; hence their periods, l and m, must be *even*. Thus the only admissible solutions are:

$$\{4, 4 \mid n\},$$

$$\{4, 6 \mid 3\}, \quad \{6, 4 \mid 3\},$$

$$\{4, 8 \mid 3\}, \quad \{8, 4 \mid 3\}.$$

3. *Derivation of the polyhedra from uniform polytopes.*

Clearly, the planes or three-spaces $\lambda, \lambda', \mu, \mu'$ are primes of symmetry of the polyhedron, and the reflections in them generate a sub-group of the complete symmetry-group of the polyhedron. (R and S generate a different subgroup.) The vertex C (Fig. v) is the mid-point of the edge $\mu\mu'$ of the tetrahedron $\lambda\lambda'\mu\mu'$, and lies on the bisector of the opposite dihedral angle $(\lambda\lambda')$. The reflections in λ, λ' transform C into B, C′, respectively, while the reflection in μ' transforms B into D. In fact, Wythoff's construction‡ leads to the uniform polytope 

whose vertices, edges,

* Coxeter, 2, 137.

† When the second factor is negative (and the first positive), the space is Minkowskian instead of Euclidean.

‡ Coxeter, 4, 329.

$\{l\}$'s, and $\{n\}$'s are the vertices, edges, faces, and holes of $\{l, m \mid n\}$. Moreover, the vertex figure of the polytope is an antiprism, whose lateral edges form the vertex figure of the polyhedron.

When $l = m = 4$, the polytope is the double n-gonal prism, $[\{n\}, \{n\}]$ or $\{n\} \times \{n\}$, bounded by $2n$ n-gonal prisms. The corresponding polyhedron,

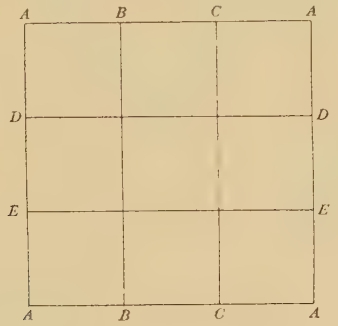


Fig. viii: "Net" of $\{4, 4 \mid 3\}$.

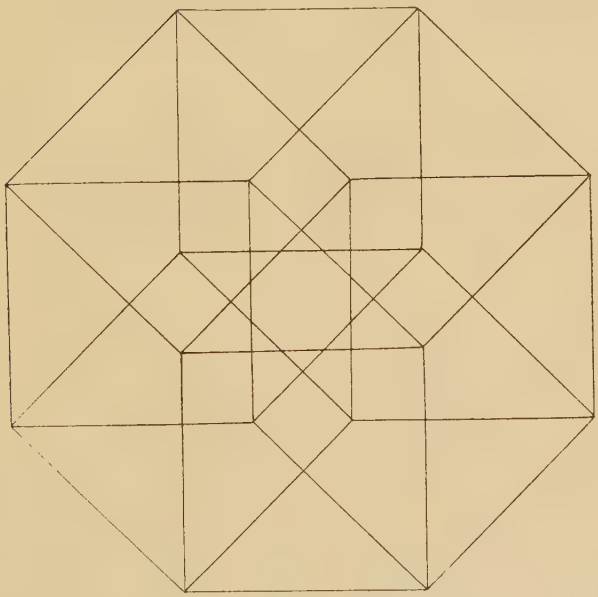


Fig. ix: Orthogonal projection of $\{4, 4 \mid 4\}$.

$\{4, 4 \mid n\}$, can be unfolded into a "net" which consists of n^2 squares forming a large square. The opposite sides of this large square are to be identified (or brought into coincidence) in the manner appropriate to the formation of a torus. (See, for instance, Fig. viii, where $n = 3$. The "squared-

paper" plane-filling, $\{4, 4\}$, can be regarded as the limiting form as n tends to infinity.) When $n = 4$, the double prism becomes the tessaract or hyper-cube, γ_4, and the faces of $\{4, 4 \mid 4\}$ are sixteen of the twenty-four squares of this polytope. The vertex figure of γ_4 is a regular tetrahedron (of edge $\sqrt{2}$), which can be regarded as a digonal antiprism in three ways. Therefore one γ_4 leads to three $\{4, 4 \mid 4\}$'s. Fig. ix shows the familiar octagonal projection of the tessaract (in which the eight bounding cubes are all plainly visible). In this projection, eight of the twenty-four squares are undistorted, while the remaining sixteen are foreshortened into rhombs. These sixteen squares are the faces of one $\{4, 4 \mid 4\}$.

When $l > 4$, the faces of $\{l, 4 \mid n\}$ are simply all the $\{l\}$'s of the polytope* $t_{1,2}\{n, \frac{1}{2}l, n\}$. In the finite cases, this polytope can be described as the common content of two equal reciprocal regular polytopes $\{n, \frac{1}{2}l, n\}$. In particular, $t_{1,2}\{3, 3, 3\}$ or $t_{1,2}\alpha_4$ can also be described as a central section of the five-dimensional measure-polytope γ_5, analogous to the hexagonal section of the cube (γ_3). The general measure-polytope may be defined as the totality of points $(x_1, x_2, \ldots)$ for which $|x_i| \leqslant 1$; we take its section by the prime $\Sigma x_i = 0$. Thus the vertices of $t_{1,2}\alpha_4$, and so also of $\{6, 4 \mid 3\}$, are the thirty points

$$(1, 1, 0, -1, -1)$$

(permuted). (See Fig. x.) The polytope is bounded by ten truncated tetrahedra; the polyhedron separates these into two sets of five.

The regular 24-cell, $\{3, 4, 3\}$, may be defined as the totality of points for which

$$|x_i \pm x_j| \leqslant \sqrt{2}\dagger \quad (i, j = 1, 2, 3, 4; \ i < j).$$

Its reciprocal, with vertices $\pm(1, 1, 0, 0)$ (permuted), is then defined by

$$|x_i| \leqslant 1, \quad |x_1 \pm x_2 \pm x_3 \pm x_4| \leqslant 2.$$

Thus the vertices of $t_{1,2}\{3, 4, 3\}$, and so also of $\{8, 4 \mid 3\}$, are the 288 points

$$\pm(1, \sqrt{2}-1, \sqrt{2}-1, 3-2\sqrt{2}), \quad \pm(2\sqrt{2}-2, 2-\sqrt{2}, 2-\sqrt{2}, 0).$$

The polytope is bounded by forty-eight truncated cubes‡; the polyhedron separates these into two sets of twenty-four.

---

* Coxeter, **4**, 331.

† Or any other positive constant.

‡ Threlfall and Seifert, 1, 63.

When $l = 4$ and $m > 4$, the polytope is $t_{0,3}\{n, \frac{1}{2}m, n\}$, which can be derived from the regular polytope $\{n, \frac{1}{2}m, n\}$ by uniformly shrinking all the bounding $\{n, \frac{1}{2}m\}$'s, and inserting n-gonal prisms between adjacent pairs of them*. The shrinkage is adjusted so that the lateral faces of the

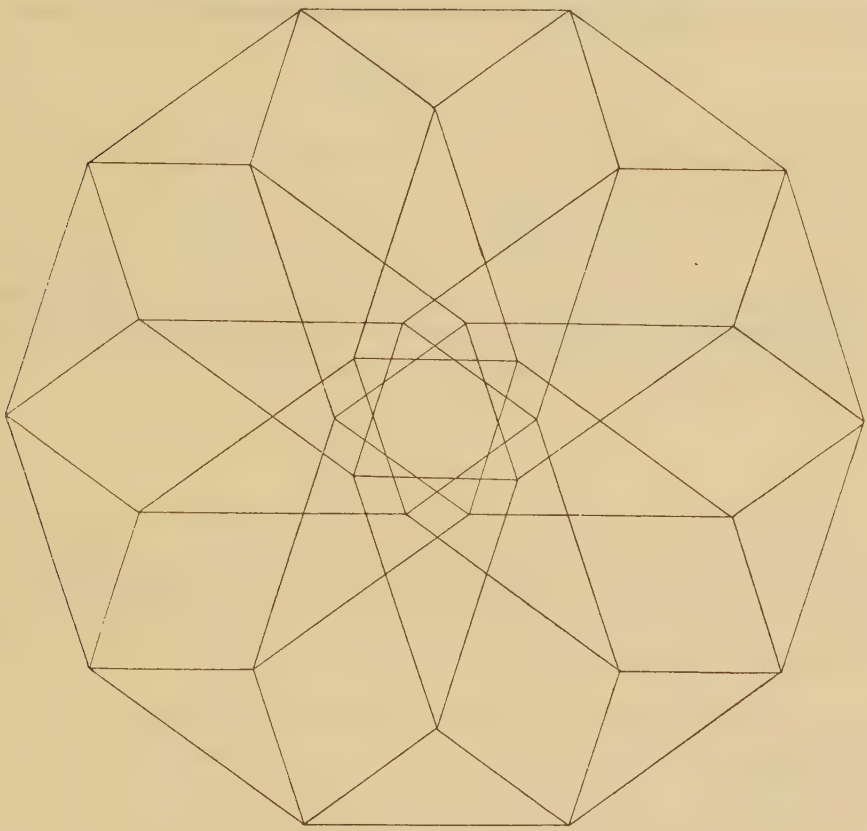


Fig. x: Orthogonal projection of $\{6, 4|3\}$.

prisms are squares; these squares are then the faces of $\{4, m|n\}$. It follows that the number of faces of $\{4, m|n\}$ is n times the number of edges (or plane faces) of $\{n, \frac{1}{2}m, n\}$. The vertices of $t_{0,3}\alpha_4$, as also of $\{4, 6|3\}$, are the twenty points

$$(1, 0, 0, 0, -1).$$

* Apart from a change of scale, this is equivalent to Mrs. Stott's "expansion" e_3. (Stott, 1, 9.)

The vertices of $t_{0,3}\{3, 4, 3\}$, as also of $\{4, 8|3\}$, are the 144 points

$$\pm(\sqrt{2}, 2-\sqrt{2}, 0, 0), \quad \pm(1, 1, \sqrt{2}-1, \sqrt{2}-1).$$

Any two adjacent faces of $\{l, m|n\}$ belong to a bounding $t_{0,1}\{\frac{1}{2}l, n\}$* of the corresponding polytope. The dihedral angle between them is therefore equal to the dihedral angle of the ordinary polyhedron† $\{\frac{1}{2}l, n\}$, namely

$$2 \arcsin \frac{\cos(\pi/n)}{\sin(2\pi/l)}.$$

In the three-dimensional cases, (2.3) shows that this is equal to

$$2 \arcsin \frac{\cos(\pi/m)}{\sin(\pi/l)} \quad \text{or} \quad 2 \arcsin \frac{\sin(\pi/m)}{\cos(\pi/l)}$$

according as the polyhedron $\{l, m|n\}$ is ordinary or skew. This explains the fact that the infinite skew polyhedra have the same dihedral angles as the ordinary polyhedra that are their "complements".

4. *The symmetry-groups.*

It is well known that the complete symmetry-group of the ordinary regular polyhedron $\{l, m\}$ is representable on a concentric sphere in such a way that the fundamental region is a spherical triangle of angles π/l, π/m, $\pi/2$. The group is generated by the reflections in the sides of this triangle, and has the abstract definition

$$R_1^2 = R_2^2 = R_3^2 = (R_1 R_2)^l = (R_2 R_3)^m = (R_1 R_3)^2 = 1.$$

The rotations

$$R = R_1 R_2, \quad S = R_2 R_3$$

generate a sub-group $\mathfrak{G}$, of order g, whose abstract definition is

$$R^l = S^m = (RS)^2 = 1. \tag{4.1}$$

This consists of all *rotations* that are symmetries of the polyhedron. The complete group is derivable from it by adjoining a single reflection, and so is of order $2g$. The area of the fundamental region is $(1/l+1/m-\frac{1}{2})\pi$. Since $2g$ such regions exactly cover the sphere, we have

$$2/g = 1/l+1/m-\tfrac{1}{2}. \tag{4.2}$$

---

* For $l = 4$, an n-gonal prism.

† For $l = 4$, of the n-gon itself.

Instead of representing the group on a sphere, we can just as well represent it on the surface of the polyhedron itself. The fundamental region is now a plane right-angled triangle $O_1 O_2 O_3$, where

O_3 is the centre of a face of the polyhedron,

O_2 is the mid-point of a side of this face, and

O_1 is one end of this side.

R_1, R_2, R_3 are reflections in the planes joining the centre of the polyhedron to the sides $O_2 O_3$, $O_3 O_1$, $O_1 O_2$ of this triangle. Every face can be divided into $2l$ such triangles, and then each edge of the polyhedron belongs to four of them, and each vertex to $2m$. It is convenient to imagine the triangles to be coloured alternately white and black, so that there are l of either colour in each face, two of either colour at each edge, m of either colour at each vertex, and g of either colour altogether. Hence, if the polyhedron has f faces, e edges, and v vertices,

$$lf = 2e = mv = g,$$

i.e.

$$f = g/l, \quad e = g/2, \quad v = g/m. \tag{4.3}$$

Thus (4.2) is equivalent to Euler's theorem

$$f+v-e = 2. \tag{4.4}$$

The faces of a regular *skew* polyhedron can be divided into right-angled triangles in the same manner. Any such triangle is still a fundamental region for the complete symmetry group, but the generating operations which transform the first triangle into its neighbours need no longer be reflections. Actually, R_1 and R_3 still are reflections (in the planes or three-spaces that we called λ and μ), but R_2 is the rotation through π about the hypotenuse $O_1 O_3$ (or about the plane joining this line to the centre). Hence the sub-group generated by R and S no longer consists solely of rotations. However, since it transforms white triangles into white and black into black, we shall call it the *intrinsic rotation-group* of $\{l, m \mid n\}$, and still denote it by $\mathfrak{G}$, and its order by g. Both groups now require an extra relation in their abstract definitions, on account of the multiple connectivity of the surface. The formulae (4.3) still hold, but (4.4) has to be replaced by

$$f+v-e = 2-2p,$$

where p is the genus; consequently (4.2) becomes

$$p = \tfrac{1}{2}g(\tfrac{1}{2} - 1/l - 1/m) + 1. \tag{4.5}$$

This enables us to find p when g is known, but we have no simple expression for g in terms of l, m, n.

Actually, we can find g in any given case by considering the group generated by the reflections in the planes or three-spaces λ, λ', μ, μ'. When we denote these reflections by L, L', M, M', (2.1) shows that this group is defined by

$$L^2 = L'^2 = M^2 = M'^2 = (LM)^2 = (L'M')^2 = (LL')^{\frac{1}{2}l} = (MM')^{\frac{1}{2}m} = (LM')^n = (L'M)^n = 1. \tag{4.6}*$$

Its fundamental region includes two of our right-angled triangles, namely one white and one black, with a common hypotenuse. Hence it is a subgroup of index 2 in the complete group of the polyhedron, so that its order is g†. The rotation R_2, which interchanges the white and black triangles, is a symmetry of the tetrahedron $\lambda\lambda'\mu\mu'$, and transforms L, L', M, M' into L', L, M', M, respectively. Thus we have

$$L = R_1, \quad L' = R_2 R_1 R_2, \quad M = R_3, \quad M' = R_2 R_3 R_2.$$

The complete group is derivable from (4.6) by adjoining the involutory operation R_2. By direct substitution, its abstract definition is thus

$$R_1^2 = R_2^2 = R_3^2 = (R_1 R_2)^l = (R_2 R_3)^m = (R_1 R_3)^2 = (R_1 R_2 R_3 R_2)^n = 1. \tag{4.7}$$

Finally, the intrinsic rotation-group $\mathfrak{G}$ can be derived from this (by writing $R_1 R_2 = R$, $R_2 R_3 = S$) in the form

$$R^l = S^m = (RS)^2 = (RS^{-1})^n = 1. \tag{4.8}$$

The fact that these relations suffice to define $\mathfrak{G}$ may be verified by adjoining the involutory operation R_2 (which transforms R and S into their inverses), and so reconstructing (4.7).

In the case of $\{4, 4 \mid n\}$, the group generated by reflections is $[n, 2, n]$ or $[n] \times [n]$, the direct product of two dihedral groups of order $2n$. Hence

---

* Coxeter, 3.

† Therefore g is infinite when $2 \sin(\pi/l) \sin(\pi/m) \leqslant \cos(\pi/n)$ and l, m are even.

$\mathfrak{G}$, defined by*

$$R^4 = S^4 = (RS)^2 = (RS^{-1})^n = 1,$$

is likewise of order $4n^2$. It has a simple representation as a permutation-group of degree $2n$, namely

$$R = (1\ 2\ 3\ 4)(5\ 6\ 7\ 8)(\ldots\ 2n), \quad S = (1\ 2)(3\ 4\ 5\ 6)(\ldots\ 2n);$$

e.g., when $n = 3$†,

$$R = (1\ 2\ 3\ 4)(5\ 6), \qquad S = (1\ 2)(3\ 4\ 5\ 6),$$

and when $n = 4$‡,

$$R = (1\ 2\ 3\ 4)(5\ 6\ 7\ 8), \qquad S = (1\ 2)(3\ 4\ 5\ 6)(7\ 8).$$

In the case of $\{4, 6 \mid 3\}$ or $\{6, 4 \mid 3\}$, the group generated by reflections is [3, 3, 3], the symmetric group of order 120. Hence $\mathfrak{G}$, defined by

$$R^4 = S^6 = (RS)^2 = (RS^{-1})^3 = 1,$$

is likewise of order 120; and by considering the permutations (1 4 2 5), (1 5) (2 3 4), we see that it is again the symmetric group. Geometrically, [3, 3, 3] permutes the five vertices of the regular simplex α_4. $\mathfrak{G}$ differs from it in having each *odd* permutation combined with the central inversion.

In the case of $\{4, 8 \mid 3\}$ or $\{8, 4 \mid 3\}$, the group generated by reflections is [3, 4, 3], of order 1152§. Hence $\mathfrak{G}$, defined by

$$R^4 = S^8 = (RS)^2 = (RS^{-1})^3 = 1,$$

is likewise of order 1152 (although it is a quite different group). The operation

$$(R^2 S^2)^6 = (R_1 . R_2 R_1 R_2 . R_2 R_3 R_2 . R_3)^6 = (LL' M' M)^6$$

is the central inversion; and, by considering the permutations

$$(1\ 8)\ (2\ 7\ 3\ 6)\ (4\ 5), \quad (1\ 6\ 3\ 7\ 4\ 5\ 2\ 8),$$

---

* Burnside, **1**, 419 (III; $a = n$, $b = 0$).

† Miller, **1**, 368 (Degree Six, Order 36, No. 2). The relation $(s_1^2 s_2^2)^3 = 1$ is superfluous.

‡ Burns, **1**, 208 (Order 64, No. 4). The relation $(s_1 s_2^2)^4 = 1$ is superfluous.

§ Goursat, **1**, 87.

we can identify the central quotient group

$$R^4 = S^8 = (RS)^2 = (RS^{-1})^3 = (R^2 S^2)^6 = 1$$

with Dr. Burns's group of order 576, No. 3*.

5. *Topological extension of the theory: the general polyhedron* $\{l, m \mid n\}$.

Every regular polyhedron can be interpreted topologically as a *regular map* in the sense of Brahana†, the faces of the polyhedron being the countries of the map. If the faces are l-gons, m at each vertex, the map may be built up, face by face, by lettering the vertices, and calling the faces ABC..., ABE..., and so on. *E.g.*, when $l = 4$ and $m = 3$, we have the "topological cube"

ABCD, ABEF, ADGF, BCHE, CDGH, EFGH.

If this abstract construction is carried on in such a way that new letters are introduced whenever possible (*i.e.*, using only the number of sides of the face and the number of faces at a vertex), then the map is automatically regular, and forms a simply-connected surface, finite or infinite according as

$$1/l + 1/m > \text{or} \leqslant \tfrac{1}{2}.$$

It can be metrically realized as a partition of the sphere, or of the Euclidean or hyperbolic plane, into regular l-gons; hence no confusion can arise by using the symbol $\{l, m\}$.

Any other regular map of l-gons, m at each vertex, can be derived from $\{l, m\}$ by *identifying* certain edges. The intrinsic rotation-group $\mathfrak{G}$, of order $g = lf = 2e = mv$, may be defined topologically as a permutation group on the edges of the map; namely, as generated by R, which cyclically permutes the edges of one face, and S, which cyclically permutes the edges at one vertex‡. It is clearly a factor group of the rotation-group of $\{l, m\}$, whose abstract definition is (4.1). The identification of a pair of edges of $\{l, m\}$ can at once be interpreted as an extra relation between R and S, and will necessitate the identification of every other pair that are similarly related.

We may, for instance, identify two edges belonging to a chain of edges which leaves, at each vertex, two faces on the left (and $m-2$ on the right).

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* Burns, **1**, 210. The central quotient group of [3, 4, 3], on the other hand, is No. 2.

† Brahana, **1**, 269.

‡ Our generators R, S differ trivially from Brahana's S, T; in fact, our R, S are his S, $S^{-1}T$, while his S, T are our R, RS. Also our l, f are his k, n.

It is easily seen (Fig. xi*) that the operation RS^{-1} takes us one step along such a chain. Thus the relation

$$(RS^{-1})^n = 1$$

corresponds to the identification of edges which differ by n steps.

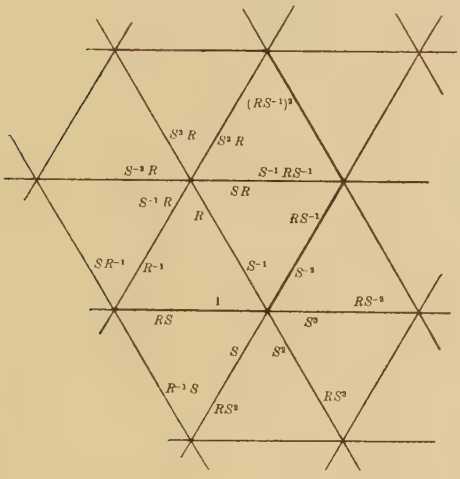

Fig. xi: $R^3 = S^6 = (RS)^2 = 1$.

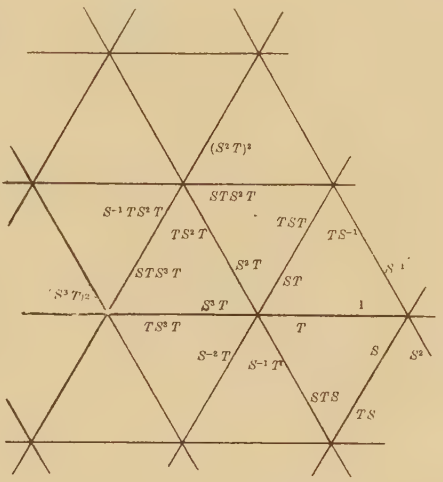

Fig. xii: $S^6 = T^2 = (ST)^3 = 1$.

We now have an interpretation for the symbol $\{l, m \mid n\}$ regardless of the parity of l and m. § 4 shows that this agrees with our previous interpretation in the case when l and m are both even.

In a family of abstract groups with various periods assigned to certain fixed combinations of generators, it usually happens that the greater periods correspond to greater order†. Accordingly, we should expect the criterion

$$2 \sin(\pi/l) \sin(\pi/m) > \cos(\pi/n) \tag{5.1}$$

(which precisely determines the finiteness of the polyhedron $\{l, m \mid n\}$

* In the special case when $l = 3$, (4.1) implies $(RS^{-1})^m = 1$. But when l and m are both greater than 3, the period of RS^{-1} is infinite. This follows from (1.1), by considering the Minkowskian polyhedron $\{l, m\}$; for, if $l \geqslant 4$ and $m \geqslant 4$, then $\cos(\pi/n) \geqslant 1$, and π/n is a hyperbolic angle.

† This is merely a working rule, and not a demonstrable theorem, as the following example shows: although $\{6, 6 \mid 3\}$ is infinite, we shall see later that $\{6, 7 \mid 3\}$ and $\{7, 7 \mid 3\}$ are finite.

when l and m are even) to retain some significance when the parity of l and m is unrestricted. Actually, we find that it fails in two respects.

First, for certain values of l, m, n, satisfying the criterion (as well as for certain values violating it), the group *collapses*, in the sense that the relations (4.8) imply $R = S = 1$. *E.g.*, there is no polyhedron $\{4, 5 \mid 3\}$. Secondly, for certain values of l, m, n, violating the criterion, the group remains finite. In fact, the order of the group, considered as a function of l, m, n, has a tendency to take, when l or m is odd, a smaller value than that which we should expect by interpolation from the values when l and m are even. (Since the value when l and m are even is determined by the volume of a spherical tetrahedron, this interpolation could be given a precise meaning in terms of Schläfli functions.) However, we can say this: *whenever the group is infinite, the criterion is violated, i.e.*, whenever the criterion is satisfied, the group either is finite or collapses. In Figs. vi and vii, we mark the known finite polyhedra as O's, the known infinite polyhedra as dots, and the known cases of collapse as crosses. We observe that the O's are all inside or "just outside" the region of validity of the criteria. (Unmarked lattice points correspond to groups which have not yet been investigated.)

When l or $m = 3$, we must have $n = m$ or l (respectively); any other value of n causes collapse (although, if the assigned value of n is a multiple of its proper value, the collapse will be merely partial).

When $m > 3$, and l and n are both even (or when $l > 3$, and m and n are both even), *the criterion* (5.1) *holds perfectly*; *e.g.*, $\{4, 7 \mid 4\}$ and $\{5, 6 \mid 4\}$ are infinite. For (4.8) can be written in the form

$$S^m = T^2 = (ST)^l = (S^2 T)^n = 1, \tag{5.2}$$

which has been thoroughly investigated elsewhere* in the case when l and n are even.

By putting S^2 in place of S in (5.2), we see that *$\{l, 5 \mid n\}$ and $\{n, 5 \mid l\}$ have the same group*. In particular, $\{5, 5 \mid 3\}$ has the same group as $\{3, 5 \mid 5\}$ or $\{3, 5\}$, namely the icosahedral group. $\{5, 5 \mid 3\}$ can, in fact, be realized metrically as the *great dodecahedron*† $\{5, \frac{5}{2}\}$ (Fig. xiii). All other polyhedra $\{n, 5 \mid 3\}$, $\{5, n \mid 3\}$ are impossible, since they would have the same group as $\{3, 5 \mid n\}$.

* Coxeter, **5**, 284.

† *Cf.* Brahana and Coble, **1**, 14 (Fig. VII), The other three Kepler-Poinsot polyhedra give nothing fresh. In fact, $\{\frac{5}{2}, 5\}$, $\{\frac{5}{2}, 3\}$, $\{3, \frac{5}{2}\}$, are topologically identical with $\{5, \frac{5}{2}\}$, $\{5, 3\}$, $\{3, 5\}$, respectively.

Apart from

$$\{3, m \mid n\}, \quad \{m, 3 \mid n\} \quad (n \neq m),$$

$$\{n, 5 \mid 3\}, \quad \{5, n \mid 3\} \quad (n \neq 5),$$

there are no known cases of collapse.

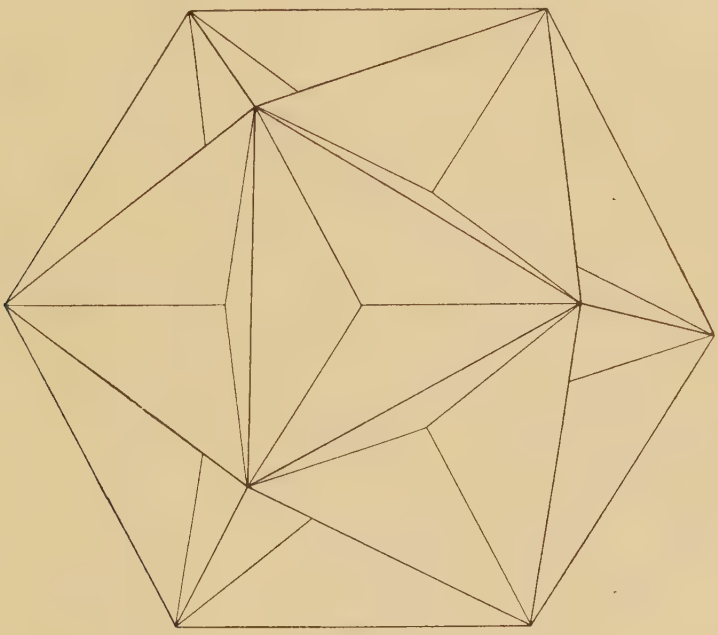

Fig. xiii: The great dodecahedron, $\{5, 5 \mid 3\}$.

The remaining polyhedra that satisfy (5.1) are

$$\{4, 5 \mid 4\}, \quad \{4, 7 \mid 3\}, \quad \{4, 5 \mid 5\},$$

and their reciprocals. The last of these has, of course, the same group as $\{5, 5 \mid 4\}$, which does not satisfy (5.1).

For $\{4, 5 \mid 4\}$ and $\{5, 4 \mid 4\}$, $\mathfrak{G}$ is*

$$S^5 = T^2 = (ST)^4 = (S^2 T)^4 = 1,$$

of order 160. $\{4, 5 \mid 4\}$ can be realized metrically in five dimensions, by taking half the squares of the measure-polytope γ_5. Fig. xiv shows an isometric projection of the vertices and edges of γ_5, in which the 80 squares appear as rhombs of two kinds: 40 of angle $\frac{1}{5}\pi$, and 40 of angle $\frac{2}{5}\pi$. Either

* Coxeter, **5**, 284.

set of 40 are projections of the faces of a $\{4, 5|4\}$. (This partition of the 80 squares is not really unique, but depends on the plane of projection.)

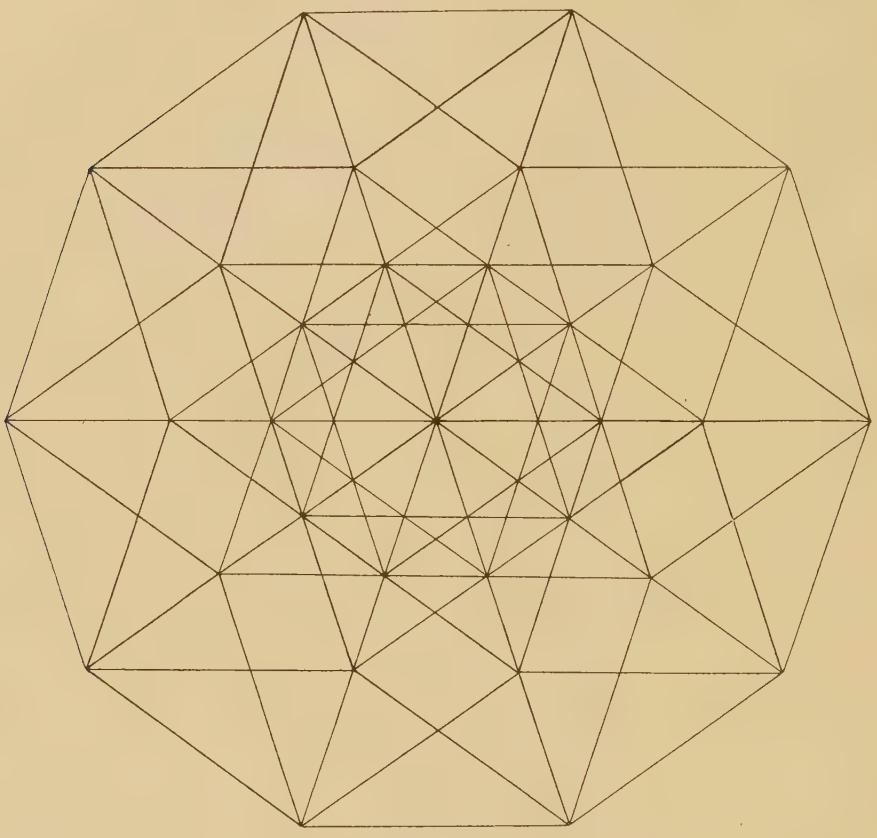

Fig. xiv: Orthogonal projection of $\{4, 5|4\}$.

For $\{4, 7|3\}$ and $\{7, 4|3\}$, $\mathfrak{G}$ is*

$$S^7 = T^2 = (ST)^4 = (S^2T)^3 = 1,$$

the simple group of order 168. Fig. xv shows a conformal representation of $\{4, 7\}$ *qua* partition of the hyperbolic plane, with letters to indicate the identifications that produce $\{4, 7|3\}$. (*Cf.* Fig. viii.)

For $\{4, 5|5\}$, $\{5, 4|5\}$, and $\{5, 5|4\}$, $\mathfrak{G}$ is

$$S^5 = T^2 = (ST)^4 = (S^2T)^5 = 1,$$

* Burnside, 1, 422. $S_7 = S^2$, $S_2 = T$.

or*

$$R^5 = S^5 = (RS)^2 = (RS^{-1})^4 = 1,$$

the simple group of order 360 (*i.e.*, the alternating group of degree six).

The above definition for the 168-group is a special case of

$$S^7 = T^2 = (ST)^l = (S^2 T)^3 = 1.$$

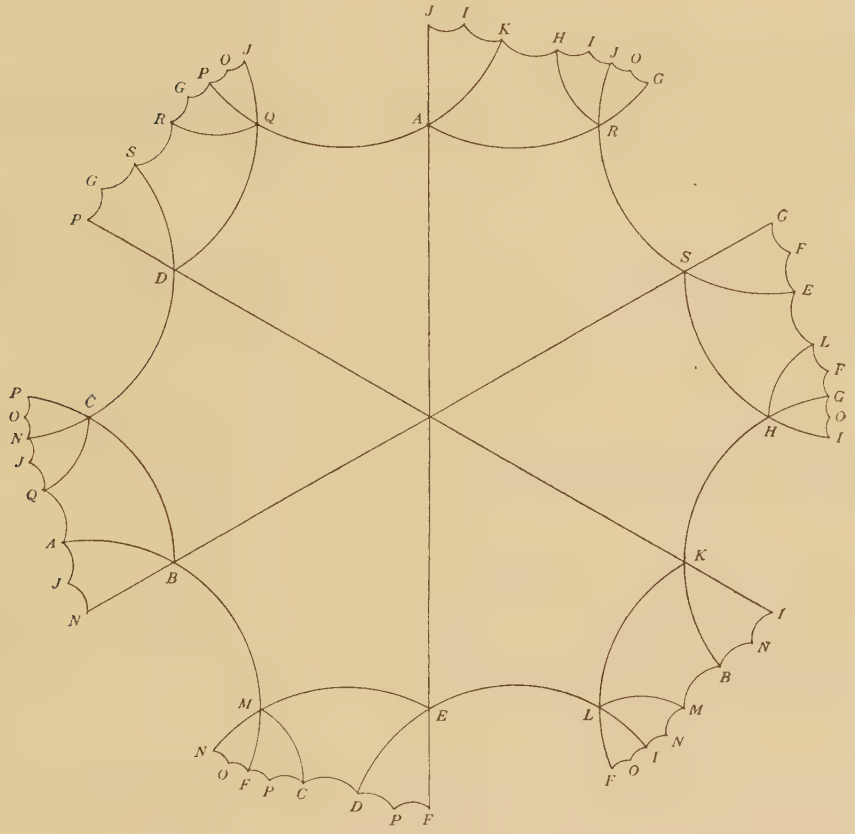

Fig. xv: The topological polyhedron {4, 7|3}.

On putting R^3 for S, this becomes

$$R^7 = T^2 = (RT)^3 = (R^3 T)^l = 1,$$

or, since $R^2 TR = R(TRT)^{-1} = RT^{-1} R^{-1} T$,

$$R^7 = T^2 = (RT)^3 = (RT^{-1} R^{-1} T)^l = 1.$$

* Todd and Coxeter, 1, 31 (5); Brahana, 1, 274.

In the form

$$S^3 = T^2 = (ST)^7 = (S^{-1}T^{-1}ST)^l = 1,$$

this family of groups has been extensively studied by Brahana and Sinkov*. By putting $l = 6$ and then $l = 7$, we see that, for the three polyhedra

$$\{6, 7|3\}, \quad \{7, 6|3\}, \quad \{7, 7|3\},$$

$\mathfrak{G}$ is the simple group of order 1092.

There remain, in Fig. vi, one pair of lattice points that are "only just outside the region", namely the points corresponding to the polyhedra

$$\{4, 9|3\}, \quad \{9, 4|3\}.$$

After enormous labour, I succeeded in enumerating the 272 co-sets of a cyclic subgroup of order 9, thus proving that the group

$$S^9 = T^2 = (ST)^4 = (S^2T)^3 = 1$$

is of order 2448. Then Sinkov completed its identification with the simple group of that order, by citing the linear fractional substitutions

$$S = \left(7\ \frac{x+1}{x+4}\right), \qquad T = (-x) \pmod{17}.$$

With the help of (4.3), we can now write out Table I. In describing the various groups, we use the symbols S_n and A_n to denote the symmetric and alternating groups of degree n.

6. *A further extension: the polyhedron* $\{q_1, m|q_2, q_3, \ldots\}$.

In §5, we derived the topological polyhedron $\{l, m|n\}$ from $\{l, m\}$ by identifying two edges that belong to a certain "chain". We may generalize our results by using a chain of edges which leaves, at each vertex, (say) j on the right (and $m-j$ on the left). When $j = 1$ this is a face; when $j = 2$ it is a "hole". We define a generalized skew polyhedron

(6.1) $$\{q_1, m|q_2, q_3, \ldots\}$$

* Brahana, 2; Sinkov, 1.

by identifying points q_j steps along the j-th chain. In the most general case we would allow j to take all values from 1 to $[\frac{1}{2}m]$. In certain cases, such a skew polyhedron can be realized metrically in Euclidean space of m (or fewer) dimensions. Its vertex figure is then a skew m-gon whose sides, first diagonals, second diagonals, etc., are of lengths

$$2\cos(\pi/q_1), \quad 2\cos(\pi/q_2), \quad 2\cos(\pi/q_3), \quad \ldots.$$

Since these generalized skew polyhedra do not necessarily occur in reciprocal pairs, it is convenient to replace R by TS^{-1}, as in (5.2). A glance at Fig. xii makes it clear that the group takes the form

$$S^m = T^2 = (S^j T)^{q_j} = 1 \quad (j = 1, 2, \ldots),$$

or, in terms of R and S,

$$R^{q_1} = S^m = (RS)^2 = (RS^{-1})^{q_2} = (RS^{-2})^{q_3} = \ldots = 1.$$

When all the q's are even, the only finite polyhedra of this kind are*

$$\{4, m \mid 4^{[\frac{1}{2}m]-1}\}$$

and

$$\{4, m \mid 4^{\frac{1}{2}m-2}, 2p\} \quad (m \text{ even}).$$

These can both be realized metrically in Euclidean m-space, the former by squares from the measure-polytope γ_m, and the latter by squares from the generalized prism (or "rectangular product") of $\frac{1}{2}m$ $2p$-gons:

$$[\{2p\}, \{2p\}, \ldots] \quad \text{or} \quad \{2p\}^{\frac{1}{2}m}.$$

The polyhedron $\{4, m \mid 4^{[\frac{1}{2}m]-1}\}$, with $g = 2^m m$, has $2^{m-2}m$ faces (squares), $2^{m-1}m$ edges, and 2^m vertices (viz., all the edges and vertices of γ_m). Accordingly, its genus is $2^{m-3}(m-4)+1$. (The cases $m = 4$ and $m = 5$ were considered earlier. See Figs. ix, xiv.) The properties of $\{4, m \mid 4^{\frac{1}{2}m-2}, 2p\}$ (m even) can be deduced similarly from the fact that $g = (2p)^{\frac{1}{2}m} m$.

The vertex figure of $\{4, m \mid 4^{[\frac{1}{2}m]-1}\}$ is a skew m-gon whose sides and all kinds of diagonals are of length $\sqrt{2}$. In fact, the vertex figure of γ_m is a regular simplex $\alpha_{m-1}\sqrt{2}$, whose Petrie polygon† is the vertex figure of the

* Coxeter, **5**, 283, (4.2), (4.5). We write 4^n for a row of n 4's.

† Coxeter, **1**, 203.

polyhedron. The vertex figure of $\{4, m \mid 4^{\frac{1}{2}m-2}, 2p\}$ (m even) differs from this in having all pairs of *opposite* vertices distant $2\cos(\pi/2p)$. When we make p infinite, this becomes the Petrie polygon of the cross-polytope $\beta_{\frac{1}{2}m}\sqrt{2}$. Now, this cross-polytope is the vertex figure of $\delta_{\frac{1}{2}m+1}$, the net of measure-polytopes, *i.e.*, the infinite polytope whose vertices are all the lattice points in $\frac{1}{2}m$ dimensions. Hence, writing r for $\frac{1}{2}m$, we have the infinite skew polyhedron

$$\{4,\ 2r \mid 4^{r-2}\},$$

whose faces belong to δ_{r+1}. This can be regarded as a generalization of $\{4, 4\}$ and $\{4, 6 \mid 4\}$.

There is also a generalization of $\{6, 3\}$ and $\{6, 4 \mid 4\}$: the infinite polyhedron

$$\{6, m \mid 4^{[\frac{1}{2}m]-1}\}.$$

Its faces are all the hexagons of the uniform $(m-1)$-dimensional space-filling* whose graph consists of an m-gon with every vertex ringed. The remaining plane faces of this space-filling are squares, which form, when $m = 5$, another infinite polyhedron:

$$\{4, 5 \mid 6\}.$$

When the parity of the q's is unrestricted, very little is known about the possible polyhedra (6.1). One line of investigation is suggested by the fact that

$$\{6, 6 \mid 3, 4\}$$

can be realized metrically by taking all the equatorial hexagons of all the bounding cuboctahedra of the "truncated tessaract" $t_1\gamma_4$. The corresponding group is simply

$$S^6 = T^2 = (S^2T)^3 = (S^3T)^4 = 1,$$

since these relations imply $(ST)^6 = 1$. This is the "hyper-pyritohedral" group† $[(3, 3)', 4]$, of order 192, since it is derivable from $[3, 3, 4]$‡ by putting

$$S = R_1R_2R_4, \quad T = R_1R_3.$$

---

* Coxeter, **4**, 334.

† Coxeter, **5**, 295.

‡ *I.e.*,

$$R_1^2 = R_2^2 = R_3^2 = R_4^2 = (R_1R_2)^3 = (R_2R_3)^3 = (R_3R_4)^4 = (R_1R_3)^2 = (R_1R_4)^2 = (R_2R_4)^2 = 1.$$

The hexagons cut one another diagonally, producing "false vertices". This polyhedron, therefore, is a kind of generalized Kepler-Poinsot polyhedron. Four-dimensional space admits nineteen reciprocal pairs of such skew star-polyhedra; but they lie beyond the scope of the present work.

In conclusion, we mention the case when none of the q's are specified save q_1 and q_3, *i.e.*, the case of the polyhedron*

$$\{l, m|, q\} \quad (m \geqslant 6)$$

with q-gonal "second holes". The group is

$$S^m = T^2 = (ST)^l = (S^3 T)^q = 1, \tag{6.2}$$

or

$$R^l = S^m = (RS)^2 = (RS^{-2})^q = 1,$$

or

$$R^l = T^2 = (RT)^m = (R^2 TRT)^q = 1.$$

When $l = 3$, this last definition becomes†

$$R^3 = T^2 = (RT)^m = (R^{-1} TRT)^q = 1.$$

In particular, one of Brahana's results‡ shows that $\{3, 8|, 6\}$ is infinite.

When $m = 7$, we write S^2 for S in (6.2), obtaining

$$S^7 = T^2 = (ST)^q = (S^2 T)^l = 1.$$

Thus $\{l, 7|, q\}$ *has the same group as* $\{q, 7|l\}$. $\{3, 7|, 6\}$ and $\{3, 7|, 7\}$ provide the extraordinary phenomenon of two polyhedra which have the same number of vertices, edges, and faces (and therefore the same genus), *and the same group*, but which still are distinct.

When m is even, R and S^2 generate a self-conjugate sub-group of index 2:

$$R^l = S'^{\frac{1}{2}m} = (RS')^l = (RS'^{-1})^q = 1,$$

so that, when we put $s_1 = R^{-1}$, $s_2 = RS'$,

$$s_1{}^l = s_2{}^l = (s_1 s_2)^{\frac{1}{2}m} = (s_1{}^2 s_2)^q = 1.$$

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* $m = 3$ implies $q = 2$; $m = 4$ implies $q = l$; and $\{l, 5|, q\}$ is the same as $\{l, 5|q\}$.

† Brahana, **2**.

‡ Brahana, **3**, 901.

It seems likely that the only non-trivial cases when this group is finite are those considered by G. A. Miller*, and one immediate consequence of these, namely:

$$l = 3, \quad \tfrac{1}{2}m = q = 4,$$

$$l = 3, \quad \tfrac{1}{2}m = 4, \quad q = 5,$$

$$l = 3, \quad \tfrac{1}{2}m = 5, \quad q = 4,$$

$$l = 4, \quad \tfrac{1}{2}m = q = 3.$$

In the first case, Miller gives the permutations

$$s_1 = (abc)\,(def), \quad s_2 = (aeh)\,(cdg).$$

Since S transforms s_1 into s_2, and $S^2 = S' = s_1 s_2$, we have

$$R = (acb)\,(dfe), \quad S = (agbcdehf).$$

Hence the group of $\{3, 8\,|\,, 4\}$ is the "extended 168-group"†, *i.e.*, the group of all linear fractional transformations modulo 7. By putting $s_1' = S'$, $s_2' = S'^{-1} R$, we obtain the 168-group itself in the alternative form

$$s_1'^4 = s_2'^4 = (s_1' s_2')^3 = (s_1'^2 s_2')^3 = 1,$$

with $$s_1' = (abdh)\,(cefg), \quad s_2' = (ahfd)\,(bcge).$$

In this case, s_1' is transformed into s_2' by $(afcebd)$; so the group of $\{4, 6\,|\,, 3\}$ is again the extended 168-group.

Du Val has pointed out that the polyhedron $\{3, 7\,|\,, 4\}$ can be metrically realized (albeit singularly) in seven dimensions. Its faces are the fifty-six triangles of the regular simplex α_7; its edges and vertices are those of the simplex, each counted three times. The six triangles that meet at an edge of α_7 have to be paired in a special manner.

* Miller, 2.

† Van der Waerden calls this group PGL (2, 7), PSL (2, 7) being the 168-group itself, which Dickson calls $LF(2, 7)$.

TABLE I. Finite polyhedra $\{l, m|n\}$.

Polyhedron	f	e	v	p	$\mathfrak{G}$	g
$\{3, 3\|3\} = \{3, 3\}$*	4	6	4	0	$A_4 \sim LF(2, 3)$	12
$\{3, 4\|4\} = \{3, 4\}$*	8	12	6	0	S_4	24
$\{4, 3\|4\} = \{4, 3\}$*	6	12	8	0		
$\{3, 5\|5\} = \{3, 5\}$*	20	30	12	0	$A_5 \sim LF(2, 5)$	60
$\{5, 3\|5\} = \{5, 3\}$*	12	30	20	0		
$\{5, 5\|3\} = \{5, \frac{5}{2}\}$*	12	30	12	4		
$\{4, 4\|n\}$†	n^2	$2n^2$	n^2	1		$4n^2$
$\{4, 5\|4\}$‡	40	80	32	5		160
$\{5, 4\|4\}$	32	80	40	5		
$\{4, 6\|3\}$†	30	60	20	6	S_5	120
$\{6, 4\|3\}$†	20	60	30	6		
$\{4, 7\|3\}$	42	84	24	10	$LF(2, 7)$	168
$\{7, 4\|3\}$	24	84	42	10		
$\{4, 5\|5\}$	90	180	72	10	A_6	360
$\{5, 4\|5\}$	72	180	90	10		
$\{5, 5\|4\}$	72	180	72	19		
$\{4, 8\|3\}$†	288	576	144	73		1152
$\{8, 4\|3\}$†	144	576	288	73		
$\{6, 7\|3\}$	182	546	156	105	$LF(2, 13)$	1092
$\{7, 6\|3\}$	156	546	182	105		
$\{7, 7\|3\}$	156	546	156	118		
$\{4, 9\|3\}$	612	1224	272	171	$LF(2, 17)$	2448
$\{9, 4\|3\}$	272	1224	612	171		

TABLE II. Finite polyhedra $\{l, m|, q\}$.

Polyhedron	f	e	v	p	$\mathfrak{G}$	g
$\{3, 6\|, q\}$	$2q^2$	$3q^2$	q^2	1		$6q^2$
$\{3, 2q\|, 3\}$	$2q^2$	$3q^2$	$3q$	$\frac{1}{2}(q-1)(q-2)$		$6q^2$
$\{3, 7\|, 4\}$	56	84	24	3	$LF(2, 7)$	168
$\{3, 8\|, 4\}$	112	168	42	8	$PGL(2, 7)$	336
$\{4, 6\|, 3\}$	84	168	56	15		
$\{3, 7\|, 6\}$	364	546	156	14	$LF(2, 13)$	1092
$\{3, 7\|, 7\}$	364	546	156	14		
$\{3, 8\|, 5\}$	720	1080	270	46		2160
$\{3, 10\|, 4\}$	720	1080	216	73		

* Metrically realizable in three dimensions.

† Metrically realizable in four dimensions.

‡ Metrically realizable in five dimensions.

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SOME THEOREMS ON INTEGRAL FUNCTIONS BOUNDED AT A SEQUENCE OF POINTS*

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1. *Introduction.* Let $f(z)$ be an integral function. Let

$$M(r, f) = \max_{|z| \leqslant r} |f(z)|.$$

The order ρ, the lower type $l(f)$, and the upper type $\kappa(f)$ of $f(z)$ are defined by the relations

$$\rho = \overline{\lim_{r \to \infty}} \frac{\log \log M(r, f)}{\log r}; \quad l(f) = \underline{\lim_{r \to \infty}} \frac{\log M(r, f)}{r^\rho} \leqslant \overline{\lim_{r \to \infty}} \frac{\log M(r, f)}{r^\rho} = \kappa(f).$$

What is here termed the upper type is usually known simply as the type. When $\kappa(f) = 0$, the function is said to be of minimal type.

1.1. The object of this paper is to discuss some properties of integral functions which assume a set of bounded values at a prescribed sequence of points. It is known† that an integral function of order one and minimal type bounded at the points $z = \pm n$, $n = 1, 2, \ldots$, reduces to a constant. Some incomplete generalizations of this last proposition have been proved by the author elsewhere‡. This paper contains some direct generalizations of the result quoted above for functions of finite order.

1.2. Let $z_1, z_2, z_3, \ldots$ denote a sequence of distinct complex numbers whose sole limit point is infinity. We suppose throughout that $\{z_n\}$ are arranged according to non-decreasing moduli, points with the same modulus being arranged according to increasing amplitudes. We denote by $n(r)$ the

* The main results of this paper form part of a thesis submitted for the D.Sc. Degree of the Madras University.

† Pólya suggested the problem and Tschakaloff and others proved it. See *Jahrsbericht der Deutschen Math. Vereinigung*, 40 (1931), *80*, problem 105; *ibid.*, 43 (1933), *10, 11, 67*.

‡ "On integral functions of order one and of finite type", *Jour. Indian Math. Soc.* (2), 2 (1936), 1–12; "On integral functions of finite order bounded at a sequence of points", *ibid.*, 53–66. We shall refer to these in the sequel as I and II respectively.

number of points z_n lying in the circle $|z| \leqslant r$. We shall speak of the exponent of convergence ρ of $\{z_n\}$ as its order, for shortness. In this paper it is supposed that $0 < \rho < \infty$. We denote by $\sigma(z)$ the canonical product with simple zeros at $z = z_n$. It is well known that $\sigma(z)$ is an integral function of order ρ.

1.3. The properties of a function $f(z)$ bounded at the set of points $\{z_n\}$ are, in general, controlled on the one hand by the order and type of $f(z)$ itself and on the other by the properties of the canonical product $\sigma(z)$. The chief properties of $\sigma(z)$ that come into discussion are (i) the nature of the points in the neighbourhood of which $\sigma(z)$ is very near its maximum modulus $M(r, \sigma)$, and (ii) the asymptotic behaviour of $\sigma'(z_n)$ as $n \to \infty$, which measures, roughly, the relative distribution of the points z_n. For $0 < \rho < \frac{1}{2}$, the property (i) does not come into effective play, since there are circles on which the minimum modulus of $\sigma(z)$ is as large as the maximum modulus. But, for $\rho \geqslant \frac{1}{2}$, the nature of distribution of the maximum-modulus-points becomes important, and only the simplest case, where $\{z_n\}$ are distributed along one or more lines, is considered in this paper. As regards property (ii) we require a definite measure of the behaviour of $\sigma'(z_n)$. We denote by μ the lower bound of numbers h such that

$$\Sigma \left| \frac{1}{\sigma'(z_n)\, z_n^{h+1}} \right|$$

converges. Let $\mu^+ = \mu$ if $\mu > 0$, and $\mu^+ = 0$ otherwise. Then we call μ^+ the "index* of distribution" (I.D.) of the sequence $\{z_n\}$. The number μ can vary from $-\infty$ to $+\infty$. When it is not $+\infty$, the I.D. is finite. Its actual value is immaterial; it is sufficient for the purpose of this paper if it is finite.

1.4. We shall first give some examples where the I.D. can be determined. For this purpose we establish

LEMMA 1. *Let $z_n = \lambda_n$ be a real positive sequence of order ρ, $0 < \rho < 1$. Let $n(r)/r^\rho \to D > 0$ as $r \to \infty$. Let*

$$\delta(\rho) = \overline{\lim_{n\to\infty}} \frac{1}{\lambda_n^{\rho}} \log \frac{1}{|\sigma'(\lambda_n)|}$$

and

$$\delta_1(\rho) = \overline{\lim_{n\to\infty}} \frac{1}{\lambda_n^{\rho}} \log \left| \frac{1}{S'(\lambda_n^{\rho})} \right|,$$

* When $z_n = \lambda_n$ is real and positive, there is another number, known as the index of condensation associated with $\{\lambda_n\}$; see V. Bernstein, "Séries de Dirichlet", *Borel Tracts* (1933), 22–27. Compare Lemma 1 of this paper.

where $$S(z)=\prod_{n=1}^{\infty}\left(1-\frac{z^2}{\lambda_n^{2\rho}}\right).$$

*Then** $$\delta_1(\rho)=\delta(\rho)+\pi D\cot\pi\rho.$$

Proof. Let

$$v_n(\rho)=\frac{1}{\lambda_n{}^{\rho}}\log\prod_{m\neq n}\left|\frac{1-(\lambda_n^{2\rho}/\lambda_m^{2\rho})}{1-(\lambda_n/\lambda_m)}\right|. \tag{1}$$

It is sufficient to prove that $\lim_{n\to\infty} v_n(\rho)=-\pi D\cot\pi\rho$. Now the function $(1-x^{2\rho})/(1-x)$ is positive and continuous for $0\leqslant x<\infty$. Hence we can write $v_n(\rho)$ as

$$v_n(\rho)=\frac{1}{\lambda_n{}^{\rho}}\left[\int_0^{\infty}\log\frac{1-(\lambda_n^{2\rho}/t^{2\rho})}{1-(\lambda_n/t)}\,dn(t)-\int_{\lambda_{n-1}}^{\lambda_{n+1}}\log\frac{1-(\lambda_n^{2\rho}/t^{2\rho})}{1-(\lambda_n/t)}\,dn(t)\right].$$

Now, for $\lambda_{n-1}\leqslant t\leqslant\lambda_{n+1}$, the ratio λ_n/t is very near one, since $\lambda_n/\lambda_{n-1}\to 1$ as $n\to\infty$. Using this and the fact that $n(t)/t\to 0$, $n(t)/t^{2\rho}\to 0$ as $n\to\infty$, we get, on integration by parts,

$$\lim_{n\to\infty} v_n(\rho)=-\lim_{n\to\infty}\frac{1}{\lambda_n{}^{\rho}}\int_0^{\infty}\frac{n(t)}{t}\left[\frac{2\rho\lambda_n^{2\rho}}{t^{2\rho}-\lambda_n^{2\rho}}-\frac{\lambda_n}{t-\lambda_n}\right]dt.$$

Putting $n(t)/t^{\rho}=N(t)$, $t=\lambda_n v$, we get

$$\lim_{n\to\infty} v_n(\rho)=-\lim_{n\to\infty}\int_0^{\infty}N(\lambda_n v)\left[\frac{2\rho}{v^{2\rho}-1}-\frac{1}{v-1}\right]v^{\rho-1}\,dv.$$

Since $N(\lambda_n v)\to D$ as $\lambda_n v\to\infty$, it is easy to see, by considering the two intervals $0\leqslant v\leqslant 1/\sqrt{\lambda_n}$ and $1/\sqrt{\lambda_n}\leqslant v<\infty$, that

$$\begin{aligned}\lim_{n\to\infty} v_n(\rho)&=-D\int_0^{\infty}\left[\frac{2\rho}{v^{2\rho}-1}-\frac{1}{v-1}\right]v^{\rho-1}\,dv\\&=-D\left[\int_1^{\infty}+\int_0^{1}\right]\\&=-D\int_1^{\infty}\left[\left(\frac{2\rho}{v^{2\rho}-1}-\frac{1}{v-1}\right)v^{\rho-1}+v^{-\rho-1}\left(\frac{2\rho v^{2\rho}}{1-v^{2\rho}}-\frac{v}{1-v}\right)\right]dv\\&=-D\int_1^{\infty}\frac{v^{-\rho}-v^{\rho-1}}{v-1}\,dv=-D\int_0^{\infty}\frac{v^{\rho-1}\,dv}{1-v}\\&=-D\pi\cot\pi\rho.\end{aligned} \tag{2}$$

This proves the lemma.

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* The number $\delta_1(\rho)$ is the index of condensation of the sequence $\{\lambda_n^{\rho}\}$. See Bernstein, *loc. cit.*, 27. It must be noted that the index of condensation as defined by Bernstein cannot be negative.

1.5. Noting that $\delta_1(\rho)=0$ when* $\varliminf_{n\to\infty}(\lambda_n^{\rho\rho}-\lambda_{n-1}^{\rho\rho})>0$, we deduce, from Lemma 1,

LEMMA 2. *Let $z_n=\lambda_n$ be such that $n(r)/r^\rho\to D$ as $r\to\infty$ while*

$$\varliminf_{n\to\infty}(\lambda_n^{\rho\rho}-\lambda_{n-1}^{\rho\rho})>0 \quad (0<\rho<\tfrac{1}{2}).$$

Then

$$\overline{\lim_{n\to\infty}}\frac{1}{\lambda_n{}^\rho}\log\frac{1}{|\sigma'(\lambda_n)|}=-\pi D\cot\pi\rho<0, \tag{3}$$

so that $\mu=-\infty$ for $\{z_n\}=\{\lambda_n\}$, and therefore the I.D. is finite.

1.6. Remembering that $\delta_1(\rho)\geqslant 0$, we deduce

LEMMA 3. *If $z_n=\lambda_n$ be such that $n(r)/r^\rho\to D$, $\frac{1}{2}<\rho<1$, then*

$$\overline{\lim_{n\to\infty}}\frac{1}{\lambda_n{}^\rho}\log\frac{1}{|\sigma'(\lambda_n)|}\geqslant-\pi D\cot\pi\rho>0,$$

so that the I.D. is infinite†.

1.7. We now establish a general lemma which contains implicitly all the results proved later.

LEMMA 4. *Let $f(z)$ be an integral function which is bounded at the sequence of points $\{z_n\}$ of order ρ and finite I.D. Suppose that, for every integer $p\geqslant p_0>0$, the relation*

$$\frac{[f(z)]^p}{\sigma(z)}=C_\lambda(z,p)+\sum_{n=1}^{\infty}\frac{[f(z_n)]^p}{\sigma'(z_n)}\frac{1}{z-z_n}\left(\frac{z}{z_n}\right)^q \tag{4}$$

holds, where $q\geqslant 0$ is an integer greater than the I.D. and $C_\lambda(z,p)$ is a polynomial whose degree λ does not depend on p although its coefficients may do so. Then $f(z)$ reduces to a constant.

Proof. Let S be any closed bounded point set in the z-plane containing no z_n. Then we get, from (4),

$$\frac{[f(z)]^{p_0}e^{f(z)}}{\sigma(z)}=\lim_{h\to\infty}\left\{\sum_{l=0}^{h}\frac{C_\lambda(z,p_0+l)}{l!}+\sum_{n=1}^{\infty}\frac{1}{\sigma'(z_n)}\frac{1}{z-z_n}\left(\frac{z}{z_n}\right)^q\left[\sum_{l=0}^{h}\frac{[f(z_n)]^{p_0+l}}{l!}\right]\right\} \tag{5}$$

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* Bernstein, *loc. cit.*, 293.

† It is shown in Theorem 6 that the I.D. for $z_n=\lambda_n$ cannot be finite for $\frac{1}{2}<\rho<1$ under the wider assumption $\varliminf_{n\to\infty}\frac{n(r)}{r^\rho}>0$.

uniformly on the set S. Since q is greater than the I.D. and $f(z_n) = O(1)$, we find that the double series

$$\sum_{n,\,l} \left| \frac{[f(z_n)]^{p_0+l}}{l!} \frac{1}{\sigma'(z_n)} \frac{1}{z-z_n} \left(\frac{z}{z_n}\right)^q \right|$$

converges, since, summing with respect to l, we get

$$\sum_{n=1}^{\infty} \frac{|f(z_n)|^{p_0} e^{|f(z_n)|}}{|\sigma'(z_n)(z-z_n)|} \left|\frac{z}{z_n}\right|^q \tag{6}$$

which converges on S on account of the hypothesis on q and $f(z_n)$. Therefore (5) yields

$$\frac{[f(z)]^{p_0} e^{f(z)}}{\sigma(z)} = \lim_{h\to\infty} \sum_{l=0}^{h} \frac{C_\lambda(z,\, p_0+l)}{l!} + \sum_{n=1}^{\infty} \frac{[f(z_n)]^{p_0} e^{f(z_n)}}{\sigma'(z_n)} \frac{1}{z-z_n} \left(\frac{z}{z_n}\right)^q \tag{7}$$

uniformly on S. Let

$$\sum_{l=1}^{h} \frac{C_\lambda(z,\, p_0+l)}{l!} = a_{h,0} z^\lambda + a_{h,1} z^{\lambda-1} + \ldots + a_{h,\lambda}.$$

The limit of this polynomial as $h\to\infty$ exists on S in virtue of (7). Therefore we conclude that $a_{h,l}\to a_l$ as $h\to\infty$ for each $l = 0,\ 1,\ \ldots,\ \lambda$. So (7) gives

$$\frac{[f(z)]^{p_0} e^{f(z)}}{\sigma(z)} = H_\lambda(z) + \sum_{n=1}^{\infty} \frac{[f(z_n)]^{p_0} e^{f(z_n)}}{\sigma'(z_n)} \frac{1}{z-z_n} \left(\frac{z}{z_n}\right)^q. \tag{8}$$

The right-hand side of (8) multiplied by $\sigma(z)$ is an integral function. Using the method of §§4.1–4.3, I, we conclude that this function is of order ρ. Hence $[f(z)]^{p_0} e^{f(z)}$ is of order ρ, which is possible only if $e^{f(z)}$ is of finite order.

But this requires $f(z)$ to be a polynomial, which, being bounded at the points $\{z_n\}$, must be a constant. Therefore the lemma is proved.

2. We first prove a theorem valid for functions of all finite order.

THEOREM 1. *Let $\{\lambda_n\}$ be a real positive sequence such that*

$$\varliminf_{r\to\infty} \frac{n(r)}{r^\rho} > 0,$$

where ρ is the order of the sequence. When $\rho > \frac{1}{2}$, determine the integer q from the inequality $\rho \leqslant q < \rho+1$. Let

$$\left.\begin{aligned} \sigma_\rho(z) &= \prod_{n=1}^{\infty} \left(1-\frac{z}{\lambda_n}\right) \quad (0<\rho\leqslant\tfrac{1}{2});\\ \sigma_\rho(z) &= \prod_{n=1}^{\infty} \left(1-\frac{z^{2q}}{\lambda_n^{2q}}\right) \quad (\rho>\tfrac{1}{2}). \end{aligned}\right\} \tag{9}$$

Let $\{\lambda_n\}$ *be of finite I.D. for* $0<\rho\leqslant\frac{1}{2}$ *and* $\{\lambda_n^{2q}\}$, *which is of order* $\rho/2q\leqslant\frac{1}{2}$, *be of finite I.D. for* $\rho>\frac{1}{2}$. *Let* $f(z)$ *be an integral function of order* ρ *and minimal type bounded at* $\{z_n\}$, *the zeros of* $\sigma_p(z)$ *given by* (9). *Then* $f(z)$ *reduces to a constant.*

Proof. It is shown in Theorem 5 of I and Theorems 4 and 7 of II that, under the hypothesis of the present theorem, there exists a positive constant δ such that, for *every* function $g(z)$ of order ρ satisfying the conditions $g(z_n)=O(1)$ and $\kappa(g)<\delta$, we must have the relation

$$\frac{g(z)}{\sigma(z)}=C_{p-1}(z)+\sum_{n=1}^{\infty}\frac{g(z_n)}{\sigma'(z_n)}\frac{1}{z-z_n}\left(\frac{z}{z_n}\right)^p, \tag{10}$$

where $p\geqslant 0$ is an integer greater than the I.D. of $\{\lambda_n\}$ if $0<\rho\leqslant\frac{1}{2}$ and of $\{\lambda_n^{2q}\}$ if $\rho>\frac{1}{2}$; and $C_{p-1}(z)$ is a polynomial of degree $p-1$ at most. Now, if $f(z_n)=O(1)$, $[f(z_n)]^h=O(1)$ for all $h=0, 1, 2, \ldots$. Also, if $\kappa(f)=0$, then also $\kappa(f^h)=0$ for $h=0, 1, 2, \ldots$. Hence, if we take

$$g=[f(z)]^h \quad (h=0, 1, 2, \ldots)$$

in (10), the theorem follows from Lemma 4.

2.1. Using Lemma 2, from Theorem 1 we deduce

THEOREM 2. *Let* $\{\lambda_n\}$, $0<\lambda_1<\lambda_2\ldots<\lambda_n\to\infty$, *be such that*

$$\text{(i)}\ \lim_{r\to\infty}\frac{n(r)}{r^\rho}=D>0; \quad \textit{and} \quad \text{(ii)}\ \underline{\lim_{n\to\infty}}\,(\lambda_n{}^\rho-\lambda_{n-1}^\rho)>0,$$

$\rho>0$ *being* non-integral. *Let* $\{z_n\}$ *denote the set of points composed of* $\{\lambda_n\}$ *if* $0<\rho<\frac{1}{2}$ *and composed of* $\{\lambda_n\omega^\mu\}$, $\mu=0, 1, \ldots, 2q-1$, *where* ω *is a primitive* $2q$*-th root of unity, if* $\rho\geqslant\frac{1}{2}$, q *being an integer determined by the inequalities* $\rho\leqslant q<\rho+1$. *Let* $f(z)$ *be a function of order* ρ *and minimal type bounded at the set of points* $\{z_n\}$. *Then* $f(z)$ *reduces to a constant.*

3. We now confine ourselves to the special case $0<\rho<\frac{1}{2}$. We prove

THEOREM 3. *Let* $\{z_n\}$ *be a sequence of points of order* ρ, $0<\rho<\frac{1}{2}$, *and finite I.D. Let* $\sigma(z)$ *be such that* $k(\sigma)>0$. *Let* $f(z)$ *be an integral function of order* ρ *and minimal type bounded at the set of points* $\{z_n\}$. *Then* $f(z)$ *must be a constant.*

Proof. Let $p\geqslant 0$ be an integer greater than the I.D. of $\{z_n\}$. Let $g(z)$ be any function of order ρ such that $\kappa(g)<k(\sigma)\cos\pi\rho$. Consider the

integral

$$\frac{1}{2\pi i}\int_{C_R}\frac{g(x)}{\sigma(x)}\frac{1}{x-z}\left(\frac{z}{x}\right)^p dx, \tag{11}$$

where C_R is the circle $|z|=R$, R being such that the circle does not pass through any z_n. The integral (11) is equal to

$$\frac{g(z)}{\sigma(z)}-\left[C_{p-1}(z)+\sum_{z_n\subset C_R}\frac{g(z_n)}{\sigma'(z_n)}\frac{1}{z-z_n}\left(\frac{z}{z_n}\right)^p\right],$$

where $C_{p-1}(z)$ is a polynomial, of degree $p-1$ at most, whose coefficients depend on g only. Since p is greater than the I.D. of $\{z_n\}$, it follows that

$$\frac{g(z)}{\sigma(z)}=C_{p-1}(z)+\sum_{n=1}^{\infty}\frac{g(z_n)}{\sigma'(z_n)}\frac{1}{z-z_n}\left(\frac{z}{z_n}\right)^p, \tag{12}$$

when $g(z_n)=O(1)$, provided that the integral (11) tends to zero as $R\to\infty$ through some sequence of values $\{R_n\}$. To prove that this is so, we make use of a theorem due to Wiman* and Valiron† that, for $0<\rho<1$,

$$\overline{\lim_{r\to\infty}}\frac{\log m(r,\sigma)}{k(\sigma)r^\rho}\geqslant\cos\pi\rho, \tag{13}$$

where $m(r,\sigma)=\min_{|z|=r}|\sigma(z)|$. If $0<\rho<\frac{1}{2}$, $\cos\pi\rho>0$. Let $\epsilon>0$ be given. Then (13) shows that there exists a sequence

$$\{R_n\}\quad(R_1<R_2<R_3\ldots<R_n\to\infty),$$

such that, on $|z|=R_n$,

$$\begin{aligned}|\sigma(z)|&\geqslant m(R_n,\sigma)\\&\geqslant e^{(\cos\pi\rho-\epsilon)k(\sigma)R_n^\rho}.\end{aligned} \tag{14}$$

Using (14) in conjunction with the hypothesis that $\kappa(g)<k(\sigma)\cos\pi\rho$, we easily see that, as $R\to\infty$ through the sequence $\{R_n\}$, the integral (11) tends to zero uniformly in any finite region of the z-plane. Therefore, if $g(z_n)=O(1)$ and $\kappa(g)<k(\sigma)\cos\pi\rho$, the relation (12) holds. As in Theorem 1, if $f(z_n)=O(1)$, $[f(z_n)]^h=O(1)$, $h=0,\ 1,\ 2,\ \ldots$; and if $\kappa(f)=0$, we have $\kappa(f^h)=0<\kappa(\sigma)\cos\pi\rho$ for $h=0,\ 1,\ 2,\ \ldots$. Hence we can apply Lemma 4 and conclude that $f(z)$ must be a constant.

3.1. If a sequence $\{z_n\}$ is of order ρ, $0<\rho<1$, while $|z_p|\neq|z_q|$ if $p\neq q$, it is easy to see that the I.D. of $\{z_n\}$ is finite when the I.D. of the real sequence $\{|z_n|\}$ is finite. Using this and Lemma 2, we get

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* A. Wiman, *Math. Annalen*, 76 (1915), 197–211.

† G. Valiron, *Lectures on the general theory of integral functions* (Toulouse, 1923), 128.

THEOREM 4. *Let* $\{\lambda_n\}$, $0<\lambda_1<\lambda_2\ldots<\lambda_n\to\infty$, *be such that* $n(r)/r^\rho\to D$ *as* $r\to\infty$, $0<\rho<\frac{1}{2}$. *Let* $\varliminf_{n\to\infty}(\lambda_n{}^\rho-\lambda_{n-1}^\rho)>0$. *Let* $f(z)$ *be a function of order* ρ *and minimal type bounded at the sequence of points* $\{z_n\}$, *where* $|z_n|=\lambda_n$. *Then* $f(z)$ *must be a constant. In particular,* $f(z)$ *is a constant if* $f(z_n)=O(1)$, $|z_n|=n^{1/\rho}$.

3.2. When $z_n=\lambda_n$ we can prove a more precise result:

THEOREM 5. *Let* $\{\lambda_n\}$, $0<\lambda_1<\lambda_2\ldots<\lambda_n\to\infty$, *be such that*

$$\lim_{r\to\infty}\frac{n(r)}{r^\rho}=D>0 \quad (0<\rho<\tfrac{1}{2}).$$

Let $\varliminf(\lambda_n{}^\rho-\lambda_{n-1}^\rho)>0$. *Let* $f(z)$ *be an integral function of order* ρ *such that* $f(\lambda_n)=O(1)$. *If* $\kappa(f)<\pi D\cot\pi\rho$, *then* $f(z)$ *must be a constant. In particular, if* $f(n^{1/\rho})=O(1)$ *and* $\kappa(f)<\pi\cot\pi\rho$, *then* $f(z)$ *is a constant.*

Proof. Here

$$\sigma(z)=\prod_{n=1}^{\infty}\left(1-\frac{z}{\lambda_n}\right),$$

and, by Lemma 3,

$$\overline{\lim_{n\to\infty}}\frac{1}{\lambda_n{}^\rho}\log\frac{1}{|\sigma'(\lambda_n)|}=-\pi D\cot\pi\rho<0. \tag{15}$$

It is known (II, Theorem 3) that $l(\sigma)=\kappa(\sigma)=\pi D\operatorname{cosec}\pi\rho$, so that $\kappa(f)<\pi D\cot\pi\rho=k(\sigma)\cos\pi\rho$. Using this and (15), we deduce from the proof of Theorem 3 that we can take $g(z)=f(z)$ in (13) with $p=0$. Therefore

$$\frac{f(z)}{\sigma(z)}=\sum_{n=1}^{\infty}\frac{f(\lambda_n)}{\sigma'(\lambda_n)}\frac{1}{z-\lambda_n}. \tag{16}$$

A similar reasoning shows, in virtue of (15), that

$$\frac{z^m f(z)}{\sigma(z)}=\sum_{n=1}^{\infty}\frac{\lambda_n{}^m f(\lambda_n)}{\sigma'(\lambda_n)}\frac{1}{z-\lambda_n}. \tag{17}$$

Now let $\chi(z)=c_0+c_1z+c_2z^2+\ldots$ be any function of order ρ such that $\kappa(\chi)<\pi D\cot\pi\rho$. Then it is easy to see that

$$\sum_{m=1}^{\infty}c_m|z|^m<e^{\eta|z|^\rho} \quad (\eta<\pi D\cot\pi\rho).$$

Therefore, since $f(\lambda_n) = O(1)$, the double series

$$\sum_{n,\,m} \left| \frac{c_m \lambda_n{}^m f(\lambda_n)}{\sigma'(\lambda_n)} \frac{1}{z-\lambda_n} \right|$$

converges in virtue of (15). Therefore (17) gives

$$\frac{f(z)\chi(z)}{\sigma(z)} = \sum_{n=1}^{\infty} \frac{f(\lambda_n)\chi(\lambda_n)}{\sigma'(\lambda_n)} \frac{1}{z-\lambda_n}. \tag{18}$$

We can take $\chi(z) = z^m f(z)$ in (18), $m = 0, 1, 2, \ldots$, since

$$\kappa(z^m f) = \kappa(f) < \pi D \cot \pi\rho.$$

Therefore we get

$$\frac{z^m [f(z)]^2}{\sigma(z)} = \sum_{n=1}^{\infty} \frac{\lambda_n{}^m [f(\lambda_n)]^2}{\sigma'(\lambda_n)} \frac{1}{z-\lambda_n}. \tag{19}$$

Starting from (19), we can repeat the argument and prove successively that

$$\frac{[f(z)]^p}{\sigma(z)} = \sum_{n=1}^{\infty} \frac{[f(\lambda_n)]^p}{\sigma'(\lambda_n)} \frac{1}{z-\lambda_n}. \tag{20}$$

Using (20), we conclude, in virtue of Lemma 4, that $f(z)$ reduces to a constant.

3.3. In conclusion we prove a property of the I.D. in the range $\frac{1}{2} < \rho < 1$:

THEOREM 6. *Let* $\{\lambda_n\}$, $0 < \lambda_1 < \lambda_2 \ldots < \lambda_n \to \infty$, *be a sequence of order* ρ, $\frac{1}{2} < \rho < 1$. *Let* $\lim_{r\to\infty} \frac{n(r)}{r^\rho} > 0$. *Then the I.D. of the sequence* $\{\lambda_n\}$ *cannot be finite.*

It is to be noticed that, when $\lim_{r\to\infty} \frac{n(r)}{r^\rho} = D > 0$, this result is included in Lemma 3.

Proof. Let $$\sigma(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z}{\lambda_n}\right).$$

Since $\lim_{r\to\infty} \frac{n(r)}{r^\rho} > 0$, it follows that $l(\sigma) > 0$. Suppose the I.D. to be finite for some ρ, $\frac{1}{2} < \rho < 1$. Let $f(z)$ be any function of order ρ and minimal type bounded at the points $z = \lambda_n$. It is shown in Theorem 4 of II that there exists a $\delta > 0$ such that, for *every* function $g(z)$ of order ρ satisfying the

conditions $g(\lambda_n) = O(1)$, $\kappa(g) < \delta$, we have the relation

$$\frac{g(z)}{\sigma(z)} = C_{p-1}(z) + \sum_{n=1}^{\infty} \frac{g(\lambda_n)}{\sigma'(\lambda_n)} \frac{1}{z-\lambda_n} \left(\frac{z}{\lambda_n}\right)^p, \tag{21}$$

where $p \geqslant 0$ is an integer greater than the I.D., which is supposed to be finite, and $C_{p-1}(z)$ is a polynomial of degree $p-1$ at most. Since

$$\kappa(f) = \kappa(f^h) = 0 < \delta \quad (h = 0, 1, 2, \ldots)$$

and

$$[f(\lambda_n)]^h = O(1),$$

we can take $g(z) = [f(z)]^h$, $h = 0, 1, 2, \ldots$, in (21) and conclude from Lemma 4 that $f(z)$ reduces to a constant. Now consider the function

$$f(z) = \frac{\sin \sqrt{z}}{\sqrt{z}}.$$

It is a function of order $\frac{1}{2}$ and so is of *minimal type of any higher order* $\rho > \frac{1}{2}$. It is bounded along the whole of the positive real axis and therefore also at the set $\{\lambda_n\}$ in question. So $\sin \sqrt{z}/\sqrt{z}$ must reduce to a constant, which is obviously not the case. Hence the assumption that the I.D. is finite leads to a contradiction. So the I.D. cannot be finite.

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ON WARING'S PROBLEM : TWO CUBES AND ONE SQUARE

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Introduction.

The object of this paper is to give proofs of the following results:

Theorem 1. *Almost all positive integers are representable as*

$$x^3+y^3+z^2, \tag{1}$$

where x, y, z are positive integers.

Theorem 2. *Almost all positive integers not congruent to* 3 (mod 9) *are representable in the form*

$$x^3+y^3+6z^2, \tag{2}$$

and almost all positive integers not congruent to 6 (mod 9) *are representable in the form*

$$x^3+y^3+12z^2. \tag{3}$$

Consequently almost all positive integers are representable in one of the two forms

$$x^3+y^3+(z+1)^3-(z-1)^3, \quad x^3+y^3+(z+2)^3-(z-2)^3,$$

and so we have

Theorem 3. *Almost all positive integers are representable as the sum of four integer cubes (positive or negative).*

The phrase "almost all positive integers" has its usual meaning in connection with Waring's problem, namely that the number of positive integers not exceeding ξ which do not have the property in question is

$o(\xi)$ as $\xi \to \infty$. As regards the numbers (mod 9) excluded in Theorem 2, it is immediately seen that they are not representable in the relevant form, since a cube is congruent to 0 or 1 or -1 (mod 9) and $6z^2 \equiv 0$ or 6 (mod 9) and $12z^2 \equiv 0$ or 3 (mod 9).

The work in this paper is a fairly straightforward application of the Hardy-Littlewood method, except as regards the singular series, where problems new in this connection arise. The singular series is essentially a partial sum of the Dirichlet series

$$M(s) = \prod_{p \equiv 1 \,(\mathrm{mod}\, 3)} \left(1 + \left(\frac{n}{p}\right)\{2 + \chi(4n) + \bar{\chi}(4n)\}\, p^{-s}\right)$$

at $s = 1$, where χ, $\bar{\chi}$ are the non-principal cubic characters (mod p). The function $M(s)$ can be expressed in terms of the zeta-functions of certain algebraic fields, with a factor which is absolutely convergent at $s = 1$. It appears to be difficult to obtain a positive lower bound for our partial sum by completing it to the infinite series $M(1)$ without making use of some unproved hypothesis concerning the zeros of these zeta-functions†. We avoid this difficulty by proving that, for our purpose, the partial sum can be replaced by the first factors of a "singular product", for which a sufficiently good lower bound can be obtained.

Notation.

P is any sufficiently large positive integer. W is 1 or 6 or 12 [according as we are considering the form (1) or (2) or (3)]. All small Latin letters (whether with or without suffixes) except c, e, and i denote positive integers, and p denotes a prime throughout. $c_1, c_2, \ldots$ denote positive absolute constants. We use the abbreviations

$$e(\alpha) = e^{2\pi i \alpha}, \quad e_q(\alpha) = e^{2\pi i \alpha/q}.$$

Let

$$T(\alpha) = T(\alpha, P) = \sum_{x=1}^{P} e(\alpha x^3),$$

$$U(\alpha) = U(\alpha, P, W) = \sum_{x=1}^{P_2^3} e(\alpha W x^2),$$

$$V(\alpha) = T^2(\alpha)\, U(\alpha) = \sum_n r(n)\, e(\alpha n),$$

† See the Appendix.

so that, for $n \leqslant P^3$, $r(n)$ is the number of representations of n in the form (1) or (2) or (3) according to the value of W. Let

$$S^{(3)}_{a,q} = \sum_{x=1}^{q} e_q(ax^3), \qquad S^{(2)}_{a,q} = \sum_{x=1}^{q} e_q(ax^2),$$

$$I^{(3)}(\beta) = \tfrac{1}{3} \sum_{n=1}^{P^3} n^{-\frac{2}{3}} e(\beta n), \qquad I^{(2)}(\beta) = \tfrac{1}{2} \sum_{n=1}^{P^3} n^{-\frac{1}{2}} e(\beta n),$$

$$T^*(\alpha, a, q) = q^{-1} S^{(3)}_{a,q} I^{(3)}\left(\alpha - \frac{a}{q}\right),$$

$$U^*(\alpha, a, q) = q^{-1} S^{(2)}_{Wa,q} I^{(2)}\left(W\left(\alpha - \frac{a}{q}\right)\right),$$

$$V^*(\alpha, a, q) = T^{*2}(\alpha, a, q)\, U^*(\alpha, a, q),$$

$$A(n, q) = q^{-3} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} (S^{(3)}_{a,q})^2 (S^{(2)}_{Wa,q})\, e_q(-an).$$

Let $\delta = \frac{1}{100}$, $R = [P^{1+\delta}]$, $X = \frac{1}{2} \log R$. Let

$$\mathfrak{S}(R, n) = \sum_{q=1}^{R} A(n, q),$$

$$\mathfrak{S}'(R, n) = \prod_{p \leqslant X} \left(\sum_{\substack{\nu=0 \\ p^\nu \leqslant X}}^{\infty} A(n, p^\nu) \right).$$

We divide the interval $0 \leqslant \alpha \leqslant 1$ into "Farey arcs" surrounding all rational points a/q with $(a, q) = 1$, $a \leqslant q \leqslant P^{2-\delta}$, the arc surrounding a/q extending as far as the mediants $(a+a_1)/(q+q_1)$, $(a+a_2)/(q+q_2)$ of a/q and the neighbouring Farey fractions a_1/q_1, a_2/q_2 of order $P^{2-\delta}$. The interval $(0, 1)$ is covered exactly once by these arcs, when we make the obvious convention about the arc surrounding $1/1$. The points of the Farey arc surrounding a/q are of the form $\alpha = a/q + \beta$ with

$$-\vartheta_1 q^{-1} P^{-2+\delta} \leqslant \beta \leqslant \vartheta_2 q^{-1} P^{-2+\delta},$$

where $\frac{1}{2} \leqslant \vartheta_1 \leqslant 1$, $\frac{1}{2} \leqslant \vartheta_2 \leqslant 1$.

If $q \leqslant P^{1+\delta}$ we call the arc surrounding a/q a major arc and denote it by $\mathfrak{M}_{a,q}$, and if $P^{1+\delta} < q \leqslant P^{2-\delta}$ we call the arc a minor arc and denote it by $\mathfrak{m}_{a,q}$.

We divide each major arc into two parts:

(I) $$|\beta| \leqslant P^{-3(1-\delta)},$$

(II) $$|\beta| \geqslant P^{-3(1-\delta)}.$$

It may happen that the whole of an arc belongs to (I).

Throughout the paper the constants implied by the use of the symbol O depend on ϵ, and on ϵ only.

The approximation of $T(\alpha)$, $U(\alpha)$.

Let

$$S^{(2)}(a, b, q) = \sum_{x=1}^{q} e_q(ax^2+bx), \qquad S^{(3)}(a, b, q) = \sum_{x=1}^{q} e_q(ax^3+bx).$$

LEMMA 1. *If* $(a, q) = 1$,

$$S^{(2)}(aW, b, q) = O(q^{\frac{1}{2}}).$$

Proof. We have

$$|S^{(2)}(Wa, b, q)|^2 = \sum_{x=1}^{q}\sum_{y=1}^{q} e_q\Big(Wa(y^2-x^2)+b(y-x)\Big)$$

$$= \sum_{x=1}^{q}\sum_{z=1}^{q} e_q\Big(Wa(z^2+2zx)+bz\Big),$$

on writing $y \equiv x+z \pmod q$. The sum over x is zero unless $2Wz \equiv 0 \pmod q$. Hence

$$|S^{(2)}(Wa, b, q)|^2 \leqslant q \sum_{\substack{z=1 \\ q \mid 2Wz}}^{q} 1 \leqslant 2Wq.$$

LEMMA 2. *If* $(a, q) = 1$,

$$S^{(3)}(a, b, q) = O\Big(q^{\frac{2}{3}+\epsilon}(q, b)\Big).$$

Proof. If $(q_1, q_2) = 1$, we have

$$S^{(3)}(a, b, q_1q_2) = \sum_{x_1=1}^{q_1}\sum_{x_2=1}^{q_2} e_{q_1q_2}\Big(a(x_1q_2+x_2q_1)^3+b(x_1q_2+x_2q_1)\Big)$$

$$= S^{(3)}(aq_2^2, b, q_1)\, S^{(3)}(aq_1^2, b, q_2).$$

Hence it is sufficient to prove that for any prime $p \nmid a$ and any positive integer ν,

$$S^{(3)}(a, b, p^\nu) = O\Big(p^{\frac{2}{3}\nu}(p^\nu, b)\Big), \tag{4}$$

since then

$$S^{(3)}(a, b, q) = q^{\frac{2}{3}}(q, b) \prod_{p \mid q} O(1) = O\Big(q^{\frac{2}{3}+\epsilon}(q, b)\Big).$$

For $\nu = 1$, (4) is a special case of a result due to Mordell†. We therefore suppose that $\nu \geqslant 2$.

† *Quart. J. of Math.*, 3 (1932), 161–167. Actually the exponent $\frac{2}{3}$ can be replaced by $\frac{5}{8}$; for this see Davenport, *Journal für Math.*, 169 (1933), 158–176, Theorem 1.

Let $\tau = [\frac{1}{2}(\nu+1)]$, so that $2\tau \geqslant \nu$ and $\tau \leqslant \frac{2}{3}\nu$. Then, replacing x by $p^\tau y+z$, we obtain

$$S^{(3)}(a, b, p^\nu) = \sum_{z=1}^{p^\tau} e_{p^\nu}(az^3+bz) \sum_{y=1}^{p^{\nu-\tau}} e_{p^{\nu-\tau}}\Big((3az^2+b)y\Big)$$

$$= p^{\nu-\tau} \sum_{\substack{z=1 \\ 3az^2+b \equiv 0 (p^{\nu-\tau})}}^{p^\tau} e_{p^\nu}(az^3+bz),$$

so that

$$|S^{(3)}(a, b, p^\nu)| \leqslant p^{\nu-\tau} \frac{p^\tau}{p^{\nu-\tau}} N,$$

where N is the number of solutions of the congruence

$$3az^2+b \equiv 0 \pmod{p^{\nu-\tau}}. \tag{5}$$

Suppose first that $p \neq 3$. Let $b = p^\mu b'$, where $p \nmid b'$ and $\mu \geqslant 0$. If $\mu \geqslant \nu-\tau$, we have

$$S^{(3)}(a, b, p^\nu) = O(p^\nu) = O(p^{\frac{2}{3}\nu} p^{\nu-\tau}) = O\Big(p^{\frac{2}{3}\nu}(p^\nu, b)\Big).$$

Hence we may suppose that $\mu < \nu-\tau$. In order that (5) may have any solutions at all it is necessary that $\mu = 2\mu_1$, and then $z = p^{\mu_1} z'$ and

$$3az'^2+b' \equiv 0 \pmod{p^{\nu-\tau-2\mu_1}}, \qquad z' \leqslant p^{\nu-\tau-\mu_1}. \tag{6}$$

The number of values of z' satisfying (6) is $O(p^{\mu_1})$. Hence $N = O(p^{\mu_1})$ and

$$S^{(3)}(a, b, p^\nu) = O(p^{\tau+\mu_1}) = O(p^{\tau+\mu}) = O\Big(p^{\frac{2}{3}\nu}(p^\nu, b)\Big).$$

Suppose now that $p = 3$. Let $b = 3^\mu b'$, where $3 \nmid b'$. If $\mu \geqslant \nu-\tau-1$, we have

$$S^{(3)}(a, b, 3^\nu) = O(3^\nu) = O(3^{\nu-1}) = O(3^{\frac{2}{3}\nu} 3^{\nu-\tau-1}) = O\Big(3^{\frac{2}{3}\nu}(3^\nu, b)\Big).$$

Hence we may suppose that $\mu < \nu-\tau-1$. Then $\mu = 2\mu_1+1$, and $z = 3^{\mu_1} z'$ and

$$az'^2+b' \equiv 0 \pmod{3^{\nu-\tau-\mu}}, \qquad z' \leqslant 3^{\nu-\tau-\mu_1}.$$

The rest of the proof is practically as before.

Lemma 3. *Let $f(x)$ be any polynomial with integral coefficients, and let*

$$F(n) = \sum_{x=1}^{q} e_q\Big(f(x)+nx\Big).$$

Then

$$\sum_{x=1}^{m} e_q\Big(f(x)\Big) = \frac{m}{q} F(0)+O\Big(\sum_{n=1}^{q-1} \frac{1}{n}\Big(|F(n)|+|F(-n)|\Big)\Big)+O(\log q).$$

This is Lemma 4 of our paper "On an exponential sum", *Proc. London Math. Soc.* (2), 41 (1936), 449–453.

LEMMA 4. *If* $(a, q) = 1$,

$$\sum_{x=1}^{m} e_q(ax^3) = \frac{m}{q} S^{(3)}_{a,q} + O(q^{\frac{2}{3}+\epsilon}), \tag{7}$$

$$\sum_{x=1}^{m} e_q(Wax^2) = \frac{m}{q} S^{(2)}_{Wa,q} + O(q^{\frac{1}{2}+\epsilon}). \tag{8}$$

Proof. These results follow from Lemma 3 on taking $f(x)$ to be ax^3 or Wax^2 and using Lemmas 1, 2. In the case of (7), for example, we obtain

$$\begin{aligned}
\sum_{x=1}^{m} e_q(ax^3) &= \frac{m}{q} S^{(3)}_{a,q} + O\Big(\sum_{n=1}^{q-1} \frac{1}{n} q^{\frac{2}{3}+\epsilon}(q, n)\Big) + O(\log q) \\
&= \frac{m}{q} S^{(3)}_{a,q} + O\Big(q^{\frac{2}{3}+\epsilon} \sum_{d \mid q} d \sum_{m \leqslant (q-1)/d} \frac{1}{md}\Big) + O(\log q) \\
&= \frac{m}{q} S^{(3)}_{a,q} + O\Big(q^{\frac{2}{3}+\epsilon} d(q) \log q\Big) + O(\log q) \\
&= \frac{m}{q} S^{(3)}_{a,q} + O(q^{\frac{2}{3}+2\epsilon}).
\end{aligned}$$

LEMMA 5. *If* $(a, q) = 1$, *and* $\alpha = a/q + \beta$, *then*

$$T(\alpha) = T^*(a, q) + O\Big(q^{\frac{2}{3}+\epsilon}(1 + P^3|\beta|)\Big), \tag{9}$$

$$U(\alpha) = U^*(a, q) + O\Big(q^{\frac{1}{2}+\epsilon}(1 + P^3|\beta|)\Big). \tag{10}$$

Proof. The proofs are almost the same in the two cases, so we prove only (9). Let

$$S_0 = 0, \quad S_n = \sum_{x \leqslant n^{\frac{1}{3}}} e_q(ax^3).$$

Then, by partial summation,

$$\begin{aligned}
T(\alpha) &= \sum_{x=1}^{P} e_q(ax^3)\, e(\beta x^3) \\
&= \sum_{n=1}^{P^3} (S_n - S_{n-1})\, e(\beta n) \\
&= \sum_{n=1}^{P^3-1} S_n \Big(e(\beta n) - e\big(\beta(n+1)\big)\Big) + S_{P^3} e(\beta P^3).
\end{aligned}$$

By Lemma 4,

$$S_n = \frac{n^{\frac{1}{3}}}{q} S^{(3)}_{a,q} + O(q^{\frac{2}{3}+\epsilon}).$$

Hence

$$T(\alpha) = q^{-1} S^{(3)}_{a,q} \sum_{n=1}^{P^3-1} n^{\frac{1}{3}} \Big(e(\beta n) - e\big(\beta(n+1)\big)\Big)$$
$$+ q^{-1} S^{(3)}_{a,q} P e(\beta P^3) + O\Big(q^{\frac{2}{3}+\epsilon} \sum_{n=1}^{P^3-1} |1 - e(\beta)|\Big) + O(q^{\frac{2}{3}+\epsilon})$$
$$= q^{-1} S^{(3)}_{a,q} \sum_{n=1}^{P^3} \Big(n^{\frac{1}{3}} - (n-1)^{\frac{1}{3}}\Big) e(\beta n) + O\Big(q^{\frac{2}{3}+\epsilon}(1+P^3|\beta|)\Big).$$

Now

$$n^{\frac{1}{3}} - (n-1)^{\frac{1}{3}} = \tfrac{1}{3} n^{-\frac{2}{3}} + O(n^{-\frac{5}{3}}).$$

Hence

$$T(\alpha) = q^{-1} S^{(3)}_{a,q} I^{(3)}(\beta) + O\Big(q^{\frac{2}{3}+\epsilon}(1+P^3|\beta|)\Big)$$
$$= T^*(a, a, q) + O\Big(q^{\frac{2}{3}+\epsilon}(1+P^3|\beta|)\Big).$$

Lemma 6. *If* $(a, q) = 1$,

$$|S^{(2)}_{Wa\ q}| < c_1 q^{\frac{1}{2}}, \tag{11}$$

$$|S^{(3)}_{a,q}| < c_2 q^{\frac{2}{3}}. \tag{12}$$

Proof. The first result is a special case of Lemma 1. For the second, see Landau, *Vorlesungen über Zahlentheorie*, 1, Satz 315. This book will in future be referred to simply as Landau.

Lemma 7. *If* $|\beta| \leqslant \frac{1}{2}$, *we have*

$$I^{(3)}(\beta) = O(|\beta|^{-\frac{1}{3}}),$$
$$I^{(2)}(W\beta) = O(|\beta|^{-\frac{1}{2}}).$$

Proof. The proofs are similar in the two cases, so we prove the result for $I^{(3)}(\beta)$ only. If $|\beta| \leqslant P^{-3}$ the result is obvious, since $|I^{(3)}(\beta)| \leqslant P$. If $|\beta| \geqslant P^{-3}$, we have

$$I^{(3)}(\beta) = \tfrac{1}{3} \sum_{n \leqslant |\beta|^{-1}} n^{-\frac{2}{3}} e(\beta n) + \tfrac{1}{3} \sum_{|\beta|^{-1} < n \leqslant P^3} n^{-\frac{2}{3}} e(\beta n)$$
$$= O(|\beta|^{-\frac{1}{3}}) + O\Big(|\beta|^{\frac{2}{3}} |1 - e(\beta)|^{-1}\Big),$$

by applying Abel's lemma to the second sum. Hence the result.

Minor arcs.

LEMMA 8. $$\sum_{\mathfrak{m}} \int_{\mathfrak{m}} |V(\alpha)|^2 d\alpha = O(P^{4-\delta+\epsilon}).$$

Proof. By (10), (11), and the fact that $|I^{(2)}(W\beta)| \leqslant P^{\frac{3}{2}}$, we have

$$U(\alpha) = O(q^{-\frac{1}{2}} P^{\frac{3}{2}}) + O\Big(q^{\frac{1}{2}+\epsilon}(1+P^3|\beta|)\Big)$$

$$= O(q^{-\frac{1}{2}} P^{\frac{3}{2}}) + O\Big(q^{\frac{1}{2}+\epsilon}(1+P^3 q^{-1} P^{2+\delta})\Big).$$

For a minor arc, $P^{1+\delta} < q \leqslant P^{2-\delta}$. Hence, on a minor arc,

$$U(\alpha) = O(P^{1-\frac{1}{2}\delta}) + O(P^{1-\frac{1}{2}\delta+2\epsilon}) + O(P^{1+\delta-(\frac{1}{2}-\epsilon)(1+\delta)}) = O(P^{1-\frac{1}{2}\delta+2\epsilon}).$$

Hence it is sufficient to prove that

$$\sum_{\mathfrak{m}} \int_{\mathfrak{m}} |T(\alpha)|^4 d\alpha = O(P^{2+\epsilon}).$$

Now $$\int_0^1 |T(\alpha)|^4 d\alpha = \sum_{n=1}^{2P^3} \Big(r_2(n)\Big)^2,$$

where $r_2(n)$ is the number of representations of n as $x_1^3+x_2^3$ with $x_1 \leqslant P$, $x_2 \leqslant P$. Since $r_2(n) = O(n^\epsilon) = O(P^{3\epsilon})$, we have

$$\int_0^1 |T(\alpha)|^4 d\alpha = O\Big(P^{3\epsilon} \sum_{n=1}^{2P^3} r_2(n)\Big) = O(P^{2+3\epsilon}).$$

Major arcs.

LEMMA 9. *On* $\mathfrak{M}_{a,q}$, *if* $\alpha = a/q+\beta$, $\beta \neq 0$, *we have*

$$T(\alpha) = O(P^{\frac{1}{2}(1+\delta)+\epsilon} q^{-\frac{1}{2}-\epsilon} |\beta|^{-\frac{1}{2}-\epsilon}).$$

Proof. On $\mathfrak{M}_{a,q}$ we have

$$q \leqslant P^{1+\delta}, \quad |\beta| \leqslant q^{-1} P^{-2+\delta}.$$

There exist a_1, q_1, with $(a_1, q_1) = 1$, such that

$$q_1 \leqslant \frac{2}{q|\beta|},$$

$$\left|\alpha - \frac{a_1}{q_1}\right| \leqslant \frac{q|\beta|}{2q_1}. \tag{13}$$

Let $\beta_1 = \alpha - a_1/q_1$. Then

$$|\beta_1| \leqslant \tfrac{1}{2} q_1^{-1} P^{-2+\delta}. \tag{14}$$

It is impossible that $q_1 \leqslant P^{2-\delta}$, for then, by (14), α would be a point of the Farey arc surrounding a_1/q_1, which is only possible if $q_1 = q$, $a_1 = a$, $\beta_1 = \beta$, and this is contrary to (13). Hence

$$q_1 > P^{2-\delta}. \tag{15}$$

Let $$S_0 = 0, \quad S_n = \sum_{x \leqslant n^{\frac{1}{3}}} e_{q_1}(a_1 x^3).$$

As in the proof of Lemma 5,

$$T(\alpha) = \sum_{n=1}^{P^3-1} S_n \left(e(\beta_1 n) - e\left(\beta_1 (n+1)\right)\right) + S_{P^3} e(\beta_1 P^3).$$

By Landau, Satz 267, we have

$$S_n = O\left(P^{\epsilon} q_1^{\epsilon} (P^{\frac{3}{4}} + P q_1^{-\frac{1}{4}} + P^{\frac{1}{4}} q_1^{\frac{1}{4}})\right),$$

for $n \leqslant P^3$. By (15),

$$P^{\frac{3}{4}} \leqslant P^{\frac{1}{4}(1+\delta)} q_1^{\frac{1}{4}}, \quad P q_1^{-\frac{1}{4}} \leqslant P^{\frac{1}{4}(1+\delta)} q_1^{\frac{1}{4}}.$$

Hence $$T(\alpha) = O\left(P^{\epsilon} q_1^{\epsilon} P^{\frac{1}{4}(1+\delta)} q_1^{\frac{1}{4}} (1 + P^3 |\beta_1|)\right).$$

By (14), $$|\beta_1| \leqslant \tfrac{1}{2} q_1^{-1} P^{-2+\delta} < P^{-3}.$$

Hence $$T(\alpha) = O(P^{\frac{1}{4}(1+\delta)+\epsilon} q_1^{\frac{1}{4}+\epsilon})$$

$$= O(P^{\frac{1}{4}(1+\delta)+\epsilon} q^{-\frac{1}{4}-\epsilon} |\beta|^{-\frac{1}{4}-\epsilon}).$$

Lemma 10. *On the part I of a major arc* $\mathfrak{M}_{a,q}$,

$$V(\alpha) - V^*(a, a, q) = O(P^{\frac{5}{2}+7\delta+\epsilon} q^{-\frac{1}{2}}).$$

Proof. By Lemma 5, we have, on I,

$$T - T^* = O(P^{3\delta} q^{\frac{2}{3}+\epsilon}), \tag{16}$$

$$U - U^* = O(P^{3\delta} q^{\frac{1}{2}+\epsilon}). \tag{17}$$

By Lemma 6, noting that $|I^{(3)}(\beta)| \leqslant P$ and $|I^{(2)}(W\beta)| \leqslant P^{\frac{3}{2}}$, we have

$$T^* = O(P q^{-\frac{1}{3}}), \tag{18}$$

$$U^* = O(P^{\frac{3}{2}} q^{-\frac{1}{2}}). \tag{19}$$

By (16), (18),

$$T + T^* = O(P^{1+4\delta} q^{-\frac{1}{3}+\epsilon}). \tag{20}$$

By (17, (19),

$$U = O(P^{\frac{3}{2}} q^{-\frac{1}{2}+\epsilon}). \tag{21}$$

By (16), (17), (18), (20), (21),

$$\begin{aligned} V - V^* &= (T-T^*)(T+T^*)\,U + T^{*2}(U-U^*) \\ &= O(P^{3\delta} q^{\frac{3}{2}+\epsilon} P^{1+4\delta} q^{-\frac{1}{2}+\epsilon} P^{\frac{3}{2}} q^{-\frac{1}{2}+\epsilon}) + O(P^2 q^{-\frac{3}{2}} P^{3\delta} q^{\frac{1}{2}+\epsilon}) \\ &= O(P^{\frac{5}{2}+7\delta} q^{-\frac{1}{2}+3\epsilon}) + O(P^{2+3\delta} q^{-\frac{1}{2}+\epsilon}) \\ &= O(P^{\frac{5}{2}+7\delta} q^{-\frac{1}{2}+3\epsilon}). \end{aligned}$$

LEMMA 11. *On the part II of a major arc* $\mathfrak{M}_{a,q}$,

$$V(\alpha) - V^*(\alpha, a, q) = O(P^{\frac{1}{2}(1+\delta)+\epsilon} q^{-1} |\beta|^{-1}) + O(P^{\frac{7}{2}+\frac{1}{2}\delta+\epsilon} |\beta|^{\frac{1}{2}}).$$

Proof, By Lemma 5,

$$U - U^* = O(P^3 q^{\frac{1}{2}+\epsilon} |\beta|). \tag{22}$$

By Lemma 9,

$$T = O(P^{\frac{1}{4}(1+\delta)+\epsilon} q^{-\frac{1}{4}} |\beta|^{-\frac{1}{4}-\epsilon}). \tag{23}$$

By Lemmas 6, 7,

$$T^* = O(q^{-\frac{1}{3}} |\beta|^{-\frac{1}{3}}), \tag{24}$$

$$U^* = O(q^{-\frac{1}{2}} |\beta|^{-\frac{1}{2}}). \tag{25}$$

Since
$$\begin{aligned} q^{-\frac{1}{3}} |\beta|^{-\frac{1}{3}} &\leqslant q^{-\frac{1}{3}} |\beta|^{-\frac{1}{3}} (P^{3(1-\delta)} |\beta|)^{\frac{1}{12}} \\ &= P^{\frac{1}{4}(1-\delta)} q^{-\frac{1}{3}} |\beta|^{-\frac{1}{4}} \leqslant P^{\frac{1}{4}(1+\delta)+\epsilon} q^{-\frac{1}{4}} |\beta|^{-\frac{1}{4}-\epsilon}, \end{aligned}$$

we have, from (23), (24),

$$|T| + |T^*| = O(P^{\frac{1}{4}(1+\delta)+\epsilon} q^{-\frac{1}{4}} |\beta|^{-\frac{1}{4}-\epsilon}). \tag{26}$$

By (22), (25), (26),

$$\begin{aligned} |V - V^*| &\leqslant (|T| + |T^*|)^2 (|U^*| + |U - U^*|) \\ &= O\Big(P^{\frac{1}{2}(1+\delta)+2\epsilon} q^{-\frac{1}{2}} |\beta|^{-\frac{1}{2}-2\epsilon} (q^{-\frac{1}{2}} |\beta|^{-\frac{1}{2}} + P^3 q^{\frac{1}{2}+\epsilon} |\beta|) \Big) \\ &= O(P^{\frac{1}{2}(1+\delta)+2\epsilon} q^{-1} |\beta|^{-1-2\epsilon}) + O(P^{\frac{7}{2}+\frac{1}{2}\delta+2\epsilon} q^{\epsilon} |\beta|^{\frac{1}{2}-2\epsilon}). \end{aligned}$$

Hence we have the result, since $|\beta|^{-\epsilon} = O(P^{3\epsilon})$ and $q^{\epsilon} = O(P^{2\epsilon})$.

LEMMA 12.

$$\sum_{\mathfrak{M}} \int_{\mathfrak{M}} |V(\alpha) - V^*(\alpha, a, q)|^2 d\alpha = O(P^{4-\delta}).$$

Proof. The contribution of the parts I is, by Lemma 10,

$$O\Big(\sum_{q=1}^{P^{1+\delta}}\sum_{a=1}^{q} P^{5+14\delta+2\epsilon}q^{-\frac{1}{3}}P^{-3(1-\delta)}\Big) = O(P^{2+17\delta+2\epsilon}\sum_{q=1}^{P^{1+\delta}} q^{\frac{2}{3}})$$

$$= O(P^{2+17\delta+2\epsilon+\frac{5}{3}(1+\delta)}) = O(P^{4-\delta}),$$

since $\delta = \frac{1}{100}$ and ϵ is arbitrarily small.

By Lemma 11, the contribution of the part II of a single major arc $\mathfrak{M}_{a,q}$ is

$$O\Big(P^{1+\delta+2\epsilon}q^{-2}\int_{P^{-3,1-\delta)}}^{\infty}\beta^{-2}d\beta\Big)+O\Big(P^{7+\delta+2\epsilon}\int_{0}^{q^{-1}P^{-2+\delta}}\beta\,d\beta\Big)$$

$$= O(P^{1+\delta+2\epsilon+3(1-\delta)}q^{-2})+O(P^{7+\delta+2\epsilon}q^{-2}P^{-4+2\delta})$$

$$= O(P^{4-2\delta+2\epsilon}q^{-2}).$$

Hence the total contribution of the parts II is

$$O\Big(P^{4-2\delta+2\epsilon}\sum_{q=1}^{P^{1+\delta}} q.q^{-2}\Big) = O(P^{4-2\delta+2\epsilon}\log P)$$

$$= O(P^{4-\delta}).$$

Some further lemmas.

Let $||\xi||$ denote the distance of a real number ξ from the nearest integer, so that $0 \leqslant ||\xi|| \leqslant \frac{1}{2}$.

Lemma 13. *If $q \leqslant R$, and Q is a positive integer, and $q+Q$, then*†

$$\sum_{\substack{a=1\\(a,q)=1}}^{q} \min\Big(R, \frac{1}{||Qa/q||-1/q}\Big) = O(R\log R).$$

Proof. Put $d=(Q,q)$, $Q=Q'd$, $q=q'd$, so that $(Q',q')=1$ and $q'>1$. Then the sum in question does not exceed

$$\sum_{\substack{a=1\\q'+a}}^{q} \min\Big(R, \frac{1}{||Q'a/q'||-1/dq'}\Big) = d\sum_{a=1}^{q'-1}\min\Big(R, \frac{1}{||Q'a/q'||-1/dq'}\Big)$$

$$= d\sum_{n=1}^{q'-1}\min\Big(R, \frac{1}{||n/q'||-1/dq'}\Big)$$

$$\leqslant 2d\sum_{n\leqslant\frac{1}{2}q'}\min\Big(R, \frac{q'}{n-1/d}\Big).$$

† $\min(R, 1/0) = R$.

If $d = 1$, this is

$$2R+2\sum_{2\leqslant n\leqslant \frac{1}{2}q'}\frac{q'}{n-1} = O(R)+O(q'\log q') = O(R\log R).$$

If $d > 1$, it does not exceed

$$2d\sum_{n\leqslant \frac{1}{2}q'}\frac{q'}{n-\frac{1}{2}} = O(dq'\log q') = O(R\log R).$$

LEMMA 14.

$$\sum_{q=1}^{R} q^{-1}\sum_{\substack{a=1\\(a,\,q)=1}}^{q}\left(\sum_{Q=1}^{R}\sum_{\substack{A=1\\(A,\,Q)=1\\A,\,Q\neq a,\,q}}^{Q}\min\left(R,\ \frac{1}{|A-Qa/q|-1/q}\right)\right)^2 = O(R^4\log^2 R).$$

Proof. We note that, under the conditions of summation,

$$\left|A-\frac{Qa}{q}\right|\geqslant\frac{1}{q}.$$

For $q = 1$ the inner sum is

$$\sum_{Q=2}^{R}\sum_{\substack{A=1\\(A,\,Q)=1}}^{Q}\min\left(R,\ \frac{1}{|A-Q|-1}\right)\leqslant\sum_{Q=1}^{R}\left(2R+2\sum_{n=1}^{Q}\frac{1}{n}\right)$$
$$= O(R^2).$$

Hence the contribution of $q = 1$ to the sum of the lemma is $O(R^4)$.

For $q\geqslant 2$, $q\,|\,Q$, we have, on writing $Q/q = Q'$,

$$\sum_{\substack{A=1\\(A,\,Q)=1\\A\neq Q'a}}^{Q}\min\left(R,\ \frac{1}{|A-Q'a|-1/q}\right)\leqslant 2\sum_{n=1}^{Q}\frac{1}{n-\frac{1}{2}} = O(\log R).$$

For $q\geqslant 2$, $q\nmid Q$, we have

$$\sum_{\substack{A=1\\(A,\,Q)=1}}^{Q}\min\left(R,\ \frac{1}{|A-Qa/q|-1/q}\right)$$
$$\leqslant 2\min\left(R,\ \frac{1}{\|Qa/q\|-1/q}\right)+2\sum_{n=1}^{Q}\frac{1}{n-1/q}$$
$$\leqslant 2\min\left(R,\ \frac{1}{\|Qa/q\|-1/q}\right)+O(\log R).$$

Hence we have (for $q \geqslant 2$)

$$(27)\quad \sum_{Q=1}^{R} \sum_{\substack{A=1 \\ (A,Q)=1 \\ A,Q \neq a,q}}^{Q} \min\left(R, \frac{1}{|A-Qa/q|-1/q}\right) \leqslant 2 \sum_{\substack{Q=1 \\ q \nmid Q}}^{R} \min\left(R, \frac{1}{\|Qa/q\|-1/q}\right) + O(R \log R).$$

In particular, the sum on the left of (27) is $O(R^2)$.

In the sum of the lemma (but with the lower limit 2 instead of 1 for q) we substitute for one of the inner sums the upper bound $O(R^2)$, and for the other the upper bound given by (27). We obtain

$$O\left(R^2 \sum_{q=2}^{R} q^{-1} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \sum_{\substack{Q=1 \\ q \nmid Q}}^{R} \min\left(R, \frac{1}{\|Qa/q\|-1/q}\right)\right) + O\left(R^2 \sum_{q=2}^{R} q^{-1} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} R \log R\right).$$

The last term is $O(R^4 \log R)$. By Lemma 13, the first sum is

$$O\left(R^2 \sum_{q=2}^{R} q^{-1} \sum_{Q=1}^{R} R \log R\right) = O(R^4 \log^2 R).$$

This establishes Lemma 14.

Lemma 15.

$$\int_0^1 \left| \sum_{Q=1}^{R} \sum_{\substack{A=1 \\ (A,Q)=1}}^{Q}{}' V^*(a, A, Q) \right|^2 da = O(P^{4-\frac{1}{3}}),$$

where the dash on the sign of summation means that if a is on a major arc $\mathfrak{M}_{a,q}$, the term $Q = q$, $A = a$ is to be omitted from the sum.

Proof. We make a Farey dissection of order R, and denote the typical Farey arc of this dissection by $\mathrm{M}_{a,q}$. Each point a/q is also the centre of a major arc $\mathfrak{M}_{a,q}$ of the former Farey dissection, and conversely, and $\mathfrak{M}_{a,q}$ is entirely inside $\mathrm{M}_{a,q}$.

The integral of the lemma is

$$\sum_{q=1}^{R} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \int_{\mathrm{M}_{a,q}-\mathfrak{M}_{a,q}} \left| \sum_{Q=1}^{R} \sum_{\substack{A=1 \\ (A,Q)=1}}^{Q} V^*(a, A, Q) \right|^2 da + \sum_{q=1}^{R} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \int_{\mathfrak{M}_{a,q}} \left| \sum_{\substack{Q=1 \\ A,Q \neq a,q}}^{R} \sum_{\substack{A=1 \\ (A,Q)=1}}^{Q} V^*(a, A, Q) \right|^2 da.$$

The first sum does not exceed

$$2\sum_{q=1}^{R}\sum_{\substack{a=1\\(a,q)=1}}^{q}\int_{\mathrm{M}_{a,q}-\mathfrak{M}_{a,q}}|V^*(\alpha,a,q)|^2\,d\alpha$$

$$+2\sum_{q=1}^{R}\sum_{\substack{a=1\\(a,q)=1}}^{q}\int_{\mathrm{M}_{a,q}-\mathfrak{M}_{a,q}}\left|\sum_{Q=1}^{R}\sum_{\substack{A=1\\(A,Q)=1\\A,Q\neq a,q}}^{Q}V^*(\alpha,A,Q)\right|^2 d\alpha.$$

Hence the integral of the lemma does not exceed

$$2\sum_{q=1}^{R}\sum_{\substack{a=1\\(a,q)=1}}^{q}\int_{\mathrm{M}_{a,q}-\mathfrak{M}_{a,q}}|V^*(\alpha,a,q)|^2\,d\alpha$$

$$+2\sum_{q=1}^{R}\sum_{\substack{a=1\\(a,q)=1}}^{q}\int_{\mathrm{M}_{a,q}}\left|\sum_{Q=1}^{R}\sum_{\substack{A=1\\(A,Q)=1\\A,Q\neq a,q}}^{Q}V^*(\alpha,A,Q)\right|^2 d\alpha$$

$$=2\Sigma_1+2\Sigma_2,$$

say.

By Lemmas 6, 7,

$$V^*(\alpha,a,q)=O(q^{-\frac{7}{8}}|\alpha-a/q|^{-\frac{7}{8}}). \tag{28}$$

Hence

$$\Sigma_1=O\left(\sum_{q=1}^{R}q\int_{\frac{1}{2}q^{-1}P^{-2+\delta}}^{\infty}q^{-\frac{7}{8}}\beta^{-\frac{7}{8}}\,d\beta\right)$$

$$=O\left(\sum_{q=1}^{R}q^{1-\frac{7}{8}}(q^{-1}P^{-2+\delta})^{-\frac{1}{8}}\right)$$

$$=O\left(P^{\frac{8}{3}-\frac{4}{3}\delta}\sum_{q=1}^{R}1\right)=O(P^{4-\frac{1}{3}\delta}).$$

In order to apply (28) in Σ_2, we must first obtain a lower bound for $|\alpha-A/Q|$ when α is a point of the arc $\mathrm{M}_{a,q}$ and $Q, A\neq q, a$. First, we have

$$\left|\alpha-\frac{A}{Q}\right|\geqslant\left|\frac{a}{q}-\frac{A}{Q}\right|-\left|\alpha-\frac{a}{q}\right|$$

$$\geqslant\left|\frac{a}{q}-\frac{A}{Q}\right|-q^{-1}R^{-1}\geqslant\left|\frac{a}{q}-\frac{A}{Q}\right|-q^{-1}Q^{-1},$$

since, on $\mathrm{M}_{a,q}$, $|\alpha-a/q|\leqslant q^{-1}R^{-1}$. Secondly, $\mathrm{M}_{A,Q}$ contains all points of the interval

$$\left(\frac{A}{Q}-\tfrac{1}{2}Q^{-1}R^{-1},\quad\frac{A}{Q}+\tfrac{1}{2}Q^{-1}R^{-1}\right),$$

and a does not lie on $\mathfrak{M}_{A,Q}$. Hence

$$|\alpha - A/Q| \geqslant \tfrac{1}{2} Q^{-1} R^{-1}.$$

Thus we have

$$Q\left|\alpha - \frac{A}{Q}\right| \geqslant \tfrac{1}{2} \max\left(\left|A - \frac{Qa}{q}\right| - \frac{1}{q}, \frac{1}{R}\right). \tag{29}$$

Now, for any ξ, η with $\xi \geqslant 0$, $\eta > 0$, we have

$$\left(\max(\xi, \eta)\right)^{-\frac{7}{8}} \leqslant \eta^{-\frac{1}{8}} \min(\xi^{-1}, \eta^{-1}). \tag{30}$$

By (28), (29), (30),

$$V^*(a, A, Q) = O\left(R^{\frac{1}{8}} \min\left(R, \frac{1}{|A - Qa/q| - 1/q}\right)\right).$$

Substituting this in Σ_2 we have

$$\Sigma_2 = O\left(\sum_{q=1}^{R} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} q^{-1} R^{-1} \left(\sum_{Q=1}^{R} \sum_{\substack{A=1 \\ (A,Q)=1 \\ A,\, Q \neq a,\, q}}^{Q} R^{\frac{1}{8}} \min\left(R, \frac{1}{|A - Qa/q| - 1/q}\right)\right)^2\right)$$

$$= O(R^{-1+\frac{1}{8}} R^4 \log^2 R) = O(P^{4-\frac{1}{8}}),$$

by Lemma 14.

Lemma 16.

$$\int_0^1 \left| V(\alpha) - \sum_{Q=1}^{R} \sum_{\substack{A=1 \\ (A,Q)=1}}^{Q} V^*(a, A, Q) \right|^2 d\alpha = O(P^{4-\delta+\epsilon}).$$

Proof. The integral is

$$\sum_{\mathfrak{m}} \int_{\mathfrak{m}} \left| V(\alpha) - \sum_{Q=1}^{R} \sum_{\substack{A=1 \\ (A,Q)=1}}^{Q} V^*(a, A, Q) \right|^2 d\alpha$$

$$+ \sum_{a,q} \int_{\mathfrak{M}_{a,q}} \left| V(\alpha) - \sum_{Q=1}^{R} \sum_{\substack{A=1 \\ (A,Q)=1}}^{Q} V^*(a, A, Q) \right|^2 d\alpha$$

$$\leqslant 2 \sum_{\mathfrak{m}} \int_{\mathfrak{m}} |V(\alpha)|^2 d\alpha + 2 \sum_{\mathfrak{m}} \int_{\mathfrak{m}} \left| \sum_{Q=1}^{R} \sum_{\substack{A=1 \\ (A,Q)=1}}^{Q} V^*(a, A, Q) \right|^2 d\alpha$$

$$+ 2 \sum_{a,q} \int_{\mathfrak{M}_{a,q}} |V(\alpha) - V^*(a, a, q)|^2 d\alpha + 2 \sum_{a,q} \int_{\mathfrak{M}_{a,q}} \left| \sum_{Q=1}^{R} \sum_{\substack{A=1 \\ (A,Q)=1 \\ A,\, Q \neq a,\, q}}^{Q} V^*(a, A, Q) \right|^2 d\alpha.$$

The first of these sums is $O(P^{4-\delta+\epsilon})$ by Lemma 8. The third sum is $O(P^{4-\delta})$ by Lemma 12. The second and fourth sums together give exactly

$$\int_0^1 \left| \sum_{Q=1}^{R} \sum_{\substack{A=1 \\ (A,Q)=1}}^{Q}{}' V^*(\alpha, A, Q) \right|^2 d\alpha,$$

where the dash has the same meaning as in Lemma 15, and by Lemma 15 this is $O(P^{4-\frac{1}{3}})$. Hence we have the result.

LEMMA 17.

$$\sum_{n=W+2}^{P^3} \Big(r(n)-\psi(n)\mathfrak{S}(R,n)\Big)^2 = O(P^{4-\delta+\epsilon}), \tag{31}$$

where, for $W+2 \leqslant n \leqslant P^3$,

$$c_3 n^{\frac{1}{6}} < \psi(n) < c_4 n^{\frac{1}{6}}. \tag{32}$$

Proof. We have

$$\Big(I^{(3)}(\alpha-A/Q)\Big)^2 I^{(2)}\Big(W(\alpha-A/Q)\Big)$$

$$= \tfrac{1}{18} \sum_{n_1=1}^{P^3} \sum_{n_2=1}^{P^3} \sum_{n_3=1}^{P^3} n_1^{-\frac{2}{3}} n_2^{-\frac{2}{3}} n_3^{-\frac{1}{2}} e\Big((\alpha-A/Q)(n_1+n_2+Wn_3)\Big)$$

$$= \sum_{n=W+2}^{(W+2)P^3} \psi(n)\, e\Big((\alpha-A/Q)n\Big),$$

where

$$\psi(n) = \tfrac{1}{18} \sum_{\substack{n_1=1 \\ n_1+n_2+Wn_3=n}}^{P^3} \sum_{n_2=1}^{P^3} \sum_{n_3=1}^{P^3} n_1^{-\frac{2}{3}} n_2^{-\frac{2}{3}} n_3^{-\frac{1}{2}}. \tag{33}$$

Hence

$$V^*(\alpha, A, Q) = \sum_{n=W+2}^{(W+2)P^3} \psi(n)\, e(n\alpha)\, Q^{-3} (S^{(3)}_{A,Q})^2 (S^{(2)}_{WA,Q})\, e_Q(-An).$$

Summing over A, Q, we have

$$\sum_{Q=1}^{R} \sum_{\substack{A=1 \\ (A,Q)=1}}^{Q} V^*(\alpha, A, Q) = \sum_{n=W+2}^{(W+2)P^3} \psi(n)\, \mathfrak{S}(R,n)\, e(n\alpha).$$

Substituting this in Lemma 16, we obtain (31), on omitting the terms with $n > P^3$.

If $n \leqslant P^3$, the upper limits for n_1, n_2, n_3 in (33) can be omitted. If $n \geqslant W+2$, the number of representations of n as $n_1+n_2+Wn_3$ is greater than $c_5 n^2$, and in each representation

$$n_1^{-\frac{2}{3}} n_2^{-\frac{2}{3}} n_3^{-\frac{1}{2}} \geqslant n^{-\frac{2}{3}-\frac{2}{3}-\frac{1}{2}} = n^{-2+\frac{1}{6}}.$$

This proves one half of (32). For the other half we observe that

$$\sum_{n_1+n_2=m} n_1^{-\frac{2}{3}}\, n_2^{-\frac{2}{3}} \leqslant 2 \sum_{n_1 \leqslant \frac{1}{2}m} n_1^{-\frac{2}{3}} (\tfrac{1}{2}m)^{-\frac{2}{3}} < c_6 m^{-\frac{1}{3}}$$

and

$$\sum_{m+Wn_3=n} m^{-\frac{1}{3}} n_3^{-\frac{1}{2}} \leqslant \sum_{m \leqslant \frac{1}{2}n} m^{-\frac{1}{3}} (n/2W)^{-\frac{1}{2}} + \sum_{n_3 \leqslant n/2W} (\tfrac{1}{2}n)^{-\frac{1}{3}} n_3^{-\frac{1}{2}} < c_7 n^{\frac{1}{6}}.$$

Lemmas for the singular series.

Lemma 18. $$|A(n, q)| < c_8 q^{-\frac{1}{3}}.$$

Proof. By Lemma 6 and the definition of $A(n, q)$, we have

$$|A(n, q)| \leqslant q^{-3} \sum_{a=1}^{q} c_1 q^{\frac{1}{2}} (c_2 q^{\frac{2}{3}})^2 = c_8 q^{-\frac{1}{3}}.$$

Lemma 19. *If* $(q_1, q_2) = 1$,

$$A(n, q_1 q_2) = A(n, q_1) A(n, q_2).$$

Proof. We have

$$A(n, q_1 q_2) = (q_1 q_2)^{-3} \sum_{\substack{a_1=1 \\ (a_1, q_1)=1}}^{q_1} \sum_{\substack{a_2=1 \\ (a_2, q_2)=1}}^{q_2} e_{q_1 q_2}\big(-n(a_1 q_2 + a_2 q_1)\big) \times (S^{(3)}_{a_1 q_2 + a_2 q_1,\, q_1 q_2})^2 S^{(2)}_{W(a_1 q_2 + a_2 q_1),\, q_1 q_2}. \tag{34}$$

Now

$$\begin{aligned} S^{(3)}_{a_1 q_2 + a_2 q_1,\, q_1 q_2} &= \sum_{x=1}^{q_1 q_2} e_{q_1 q_2}\big((a_1 q_2 + a_2 q_1) x^3\big) \\ &= \sum_{x=1}^{q_1 q_2} e_{q_1}(a_1 x^3)\, e_{q_2}(a_2 x^3) \\ &= \sum_{x_1=1}^{q_1} \sum_{x_2=1}^{q_2} e_{q_1}(a_1 x_1^3)\, e_{q_2}(a_2 x_2^3) \\ &= S^{(3)}_{a_1,\, q_1} S^{(3)}_{a_2,\, q_2}. \end{aligned}$$

Similarly $$S^{(2)}_{W(a_1q_2+a_2q_1),\,q_1q_2} = S^{(2)}_{Wa_1,\,q_1} S^{(2)}_{Wa_2,\,q_2}.$$

Substituting in (34), we obtain the result.

LEMMA 20. *Let p be a prime $\neq 2$ and $p \nmid a$. Then*

$$S^{(2)}_{a,\,p^\nu} = \begin{cases} p^{\frac{1}{2}\nu} & \text{if } \nu \text{ is even;} \\ (a/p)\, S^{(2)}_{1,\,p}\, p^{\frac{1}{2}(\nu-1)} & \text{if } \nu \text{ is odd;} \end{cases}$$

and

$$(S^{(2)}_{1,\,p})^2 = \left(\frac{-1}{p}\right) p.$$

Proof. By Landau, Satz 312, if $\nu > 2$,

$$S^{(2)}_{a,\,p^\nu} = p\, S^{(2)}_{a,\,p^{\nu-2}}.$$

Also, by Landau, Satz 307,

$$S^{(2)}_{a,\,p^2} = p.$$

By Landau, Satz 310,

$$S^{(2)}_{a,\,p} = \left(\frac{a}{p}\right)\tau,$$

where τ is independent of a, so that $\tau = S^{(2)}_{1,\,p}$. The last of the results enunciated follows from Landau, Satz 211.

LEMMA 21. *Let p be a prime $\neq 3$ and $p \nmid a$. Then*

$$S^{(3)}_{a,\,p^\nu} = \begin{cases} p^{\frac{2}{3}\nu} & \text{if } \nu \equiv 0 \pmod 3; \\ 0 & \text{if } \nu \equiv 1 \pmod 3,\ p \equiv 2 \pmod 3; \\ S^{(3)}_{a,\,p}\, p^{\frac{2}{3}(\nu-1)} & \text{if } \nu \equiv 1 \pmod 3,\ p \equiv 1 \pmod 3; \\ p^{\frac{1}{3}(2\nu-1)} & \text{if } \nu \equiv 2 \pmod 3. \end{cases}$$

Also, if $p \equiv 1 \pmod 3$,

$$S^{(3)}_{a,\,p} = \chi(a)\,\tau(\bar\chi) + \bar\chi(a)\,\tau(\chi),$$

where χ, $\bar\chi$ are the non-principal cubic characters (mod p),

$$\tau(\chi) = \sum_{x=1}^{p} \chi(x)\, e_p(x),$$

and

$$|\tau(\chi)| = |\tau(\bar\chi)| = p^{\frac{1}{2}}.$$

Proof. By Landau, Satz 312, if $\nu > 3$,

$$S^{(3)}_{a,\,p^\nu} = p^2\, S^{(3)}_{a,\,p^{\nu-3}}.$$

Also, by Landau, Satz 307,

$$S^{(3)}_{a,\,p^2} = p, \quad S^{(3)}_{a,\,p^3} = p^2.$$

If $p \equiv 2 \pmod 3$, the congruence $x^3 \equiv y \pmod p$ has exactly one solution for given y, hence

$$S^{(3)}_{a,\,p} = \sum_{x=1}^{p} e_p(ax^3) = \sum_{y=1}^{p} e_p(ay) = 0.$$

If $p \equiv 1 \pmod 3$, then, by Landau, Satz 310,

$$S^{(3)}_{a,\,p} = \chi(a)\tau(\bar{\chi}) + \bar{\chi}(a)\tau(\chi),$$

where χ, $\bar{\chi}$ are the non-principal cubic characters (mod p).

By Landau, Satz 308,

$$|\tau(\chi)| = |\tau(\bar{\chi})| = p^{\frac{1}{2}}.$$

This establishes all the assertions of Lemma 21.

Lemma 22. *If* $p \mid n$, $p \neq 2, 3$, *and* $\nu \leqslant 5$, *then*

$$|A(n, p^\nu)| \leqslant 4p^{-1}.$$

Proof. We consider each value of ν separately, and apply Lemmas 20, 21.

(1) If $p \equiv 2 \pmod 3$, then $A(n, p) = 0$. If $p \equiv 1 \pmod 3$, then

$$A(n, p) = p^{-3} \sum_{a=1}^{p-1} \left(\frac{Wa}{p}\right) S^{(2)}_{1,\,p} \left(\chi(a)\tau(\bar{\chi}) + \bar{\chi}(a)\tau(\chi)\right)^2$$

$$= 0,$$

since none of the characters

$$\left(\frac{a}{p}\right)\chi^2(a), \quad \left(\frac{a}{p}\right)\chi\bar{\chi}(a) = \left(\frac{a}{p}\right), \quad \left(\frac{a}{p}\right)\bar{\chi}^2(a)$$

is a principal character.

(2) We have

$$|A(n, p^2)| = \left| p^{-6} \sum_{\substack{a=1 \\ p \nmid a}}^{p^2} e_{p^2}(-na)\, p^3 \right| \leqslant p^{-1}.$$

(3) We have

$$A(n, p^3) = p^{-9} \sum_{\substack{a=1 \\ p+a}}^{p^3} e_{p^3}(-na)\, p^4 \left(\frac{Wa}{p}\right) p\, S^{(2)}_{1,p}.$$

This is zero if $p^3 | n$. It is also zero if $n = pn'$, where $p+n'$, for

$$\sum_{\substack{a=1 \\ p+a}}^{p^3} e_{p^2}(-n'a)\left(\frac{a}{p}\right) = p \sum_{a_1=1}^{p-1} \sum_{a_2=1}^{p} \left(\frac{a_1}{p}\right) e_{p^2}\big(-n'(a_1+pa_2)\big) = 0.$$

Hence we are left with the case $n = p^2 n''$, where $p+n''$. In this case

$$|A(n, p^3)| = \left| p^{-4} S^{(2)}_{1,p}\, p^2 \sum_{a=1}^{p-1} \left(\frac{Wa}{p}\right) e_p(-n''a) \right|$$

$$= \left| p^{-2} S^{(2)}_{1,p} \left(\frac{-Wn''}{p}\right) S^{(2)}_{1,p} \right| = p^{-1}.$$

(4) If $p \equiv 2 \pmod 3$, $A(n, p^4) = 0$. If $p \equiv 1 \pmod 3$,

$$|A(n, p^4)| = \left| p^{-12} \sum_{\substack{a=1 \\ p+a}}^{p^4} e_{p^4}(-na)\, p^4 p^2 (S^{(3)}_{a,p})^2 \right| \leqslant 4p^{-1}.$$

(5) We have

$$|A(n, p^5)| = \left| p^{-15} \sum_{\substack{a=1 \\ p+a}}^{p^5} e_{p^5}(-na)\, p^6 p^2\, S^{(2)}_{1,p} \left(\frac{Wa}{p}\right) \right| < p^{-1}.$$

LEMMA 23. *If $p \neq 2$, 3, and $p+n$, then, for $\nu > 1$,*

$$A(n, p^\nu) = 0.$$

Proof. By Lemmas 20, 21, if $a \equiv a' \pmod p$, $p+a$,

$$S^{(3)}_{a,p^\nu} = S^{(3)}_{a',p^\nu}, \quad S^{(2)}_{Wa,p^\nu} = S^{(2)}_{Wa',p^\nu}.$$

Hence

$$A(n, p^\nu) = p^{-3\nu} \sum_{a_1=1}^{p-1} \sum_{a_2=1}^{p^{\nu-1}} e_{p^\nu}\big(-n(a_1+pa_2)\big)\, (S^{(3)}_{a_1,p^\nu})^2\, S^{(2)}_{Wa_1,p^\nu} = 0.$$

LEMMA 24. *If $p \neq 2$, 3, and $p+n$, then*

$$A(n, p) = \begin{cases} 0 & \text{if } p \equiv 2 \pmod 3; \\ \left(\dfrac{Wn}{p}\right)\{2 + \chi(4n) + \bar{\chi}(4n)\} p^{-1} & \text{if } p \equiv 1 \pmod 3; \end{cases}$$

where χ, $\bar{\chi}$ are the non-principal cubic characters (mod p).

Proof. The first result follows at once from Lemma 21. For the second part we have†, if $p+a$,

$$\begin{aligned}(S^{(3)}_{a,p})^2 &= \sum_{x=1}^{p}\sum_{y=1}^{p} e_p\left(a(x^3+y^3)\right)\\ &= \sum_{x=1}^{p}\sum_{y=1}^{p} e_p\left(a\left((x+y)^3+(x-y)^3\right)\right)\\ &= \sum_{x=1}^{p}\sum_{y=1}^{p} e_p(2ax^3+6axy^2)\\ &= p+S^{(2)}_{1,p}\sum_{x=1}^{p-1}\left(\frac{6ax}{p}\right)e_p(2ax^3).\end{aligned}$$

Using this and Lemma 20, we have

$$\begin{aligned}A(n,p) &= p^{-3}\left(\frac{W}{p}\right)S^{(2)}_{1,p}\sum_{a=1}^{p-1}\left(\frac{a}{p}\right)p\,e_p(-na)\\ &\quad+p^{-3}\left(\frac{W}{p}\right)(S^{(2)}_{1,p})^2\sum_{a=1}^{p-1}\left(\frac{a}{p}\right)e_p(-na)\sum_{x=1}^{p-1}\left(\frac{6ax}{p}\right)e_p(2ax^3)\\ &= p^{-2}(S^{(2)}_{1,p})^2\left(\frac{-nW}{p}\right)+p^{-3}(S^{(2)}_{1,p})^2\left(\frac{6W}{p}\right)\sum_{a=1}^{p}\sum_{x=1}^{p-1}\left(\frac{x}{p}\right)e_p\left(a(2x^3-n)\right)\\ &\quad-p^{-3}(S^{(2)}_{1,p})^2\left(\frac{6W}{p}\right)\sum_{x=1}^{p-1}\left(\frac{x}{p}\right)\\ &= p^{-2}(S^{(2)}_{1,p})^2\left(\frac{-nW}{p}\right)+p^{-2}(S^{(2)}_{1,p})^2\left(\frac{3W}{p}\right)\sum_{2x^3\equiv n\,(\mathrm{mod}\,p)}\left(\frac{2x}{p}\right).\end{aligned}$$

By Lemma 20, $$(S^{(2)}_{1,p})^2 = \left(\frac{-1}{p}\right)p.$$

Also, if $2x^3\equiv n \pmod p$, then $(2x/p)=(n/p)$. Hence

$$A(n,p)=p^{-1}\left(\frac{nW}{p}\right)+p^{-1}\left(\frac{-3W}{p}\right)\left(\frac{n}{p}\right)\sum_{2x^3\equiv n\,(\mathrm{mod}\,p)}1.$$

Since $(-3/p)=1$ for $p\equiv 1 \pmod 3$, and since the number of solutions of $2x^3\equiv n \pmod p$ is $1+\chi(4n)+\bar{\chi}(4n)$, we have the result stated.

The transition to the singular product.

For any positive integer q let $q^*=\prod_{p\mid q}p$, and let $\nu(q)$ denote the number of different prime factors of q.

† A proof can, of course, be obtained by using the last part of Lemma 21, but then we require the relations between the Gaussian sums for the characters of order 6.

LEMMA 25. *For any quadratfrei m, and any n,*

$$\sum_{\substack{q \geqslant m \\ q^* = m}} |A(n, q)| \leqslant c_9^{\nu(m)} m^{-1}.$$

In particular, for any q and n,

$$|A(n, q)| \leqslant \frac{c_9^{\nu(q)}}{q^*}.$$

Proof. If $m = 1$, the only value for q is $q = 1$, and the result is obviously true. Now suppose that $m = p_1 \dots p_s$, where $p_1, \dots, p_s$ are different primes. The possible values for q are $q = p_1^{l_1} \dots p_s^{l_s}$. By Lemma 19,

$$\sum_{\substack{q \geqslant m \\ q^* = m}} |A(n, q)| = \prod_{j=1}^{s} \left(\sum_{l=1}^{\infty} |A(n, p_j^l)| \right).$$

If $p = 2$ or 3, then, by Lemma 18,

$$\sum_{l=1}^{\infty} |A(n, p^l)| < \sum_{l=1}^{\infty} c_8 p^{-\frac{1}{8}l} < c_{10} p^{-1}.$$

If $p > 3$, $p \mid n$, then, by Lemmas 18, 22,

$$\sum_{l=1}^{\infty} |A(n, p^l)| < \sum_{l=1}^{5} 4p^{-1} + \sum_{l=6}^{\infty} c_8 p^{-\frac{1}{8}l} < c_{11} p^{-1}.$$

If $p > 3$, $p \nmid n$, then, by Lemmas 23, 24,

$$\sum_{l=1}^{\infty} |A(n, p^l)| < c_{12} p^{-1}.$$

This establishes the result.

LEMMA 26. *If $q_1 \neq q_2$, then, for all n_1, n_2,*

$$\left| \sum_{n=n_1}^{n_2} A(n, q_1) A(n, q_2) \right| \leqslant \frac{q_1 q_2}{q_1^* q_2^*} c_9^{\nu(q_1)} c_9^{\nu(q_2)}.$$

Proof. We have

$$A(n, q_1) A(n, q_2)$$

$$= (q_1 q_2)^{-3} \sum_{\substack{a_1 = 1 \\ (a_1, q_1) = 1}}^{q_1} \sum_{\substack{a_2 = 1 \\ (a_2, q_2) = 1}}^{q_2} (S^{(3)}_{a_1, q_1})^2 S^{(2)}_{W a_1, q_1} (S^{(3)}_{a_2, q_2})^2 S^{(2)}_{W a_2, q_2}$$

$$\times e_{q_1 q_2}\left(-n(a_1 q_2 + a_2 q_1)\right).$$

Now $a_1q_2+a_2q_1 \not\equiv 0 \pmod{q_1q_2}$, for this would imply $q_1 | q_2$ and $q_2 | q_1$, contrary to hypothesis. Hence

$$\sum_{n=n_1+1}^{n_1+q_1q_2} A(n, q_1q_2) = 0$$

for all n_1. Therefore

$$\left| \sum_{n=n_1}^{n_2} A(n, q_1) A(n, q_2) \right| \leqslant \sum_{n=1}^{q_1q_2} |A(n, q_1) A(n, q_2)|$$

$$\leqslant \frac{q_1q_2}{q_1^* q_2^*} c_9^{\nu(q_1)} c_9^{\nu(q_2)},$$

by Lemma 25.

Lemma 27. $\sum_{q=1}^{R} \frac{q}{q^*} c_9^{\nu(q)} = O(R^{1+\epsilon})$ *for any* $\epsilon > 0$.

Proof. For any $\sigma > 0$ we have

$$\sum_{q=1}^{\infty} \frac{1}{q^*} c_9^{\nu(q)} q^{-\sigma} = \prod_p \left\{ 1 + \frac{c_9}{p} \sum_{l=1}^{\infty} p^{-l\sigma} \right\}$$

$$= \prod_p \left\{ 1 + \frac{c_9}{p(p^\sigma - 1)} \right\},$$

the product (and therefore the series) being convergent. Hence, for any $\epsilon > 0$,

$$\sum_{q=1}^{R} \frac{1}{q^*} c_9^{\nu(q)} q^{-\epsilon} = O(1),$$

and so

$$\sum_{q=1}^{R} \frac{q}{q^*} c_9^{\nu(q)} = O(R^{1+\epsilon}).$$

Lemma 28. *For any* n,

$$\sum_{q>X} \Big(A(n, q) \Big)^2 < c_{13} X^{-\frac{1}{3}}.$$

Proof. We have

$$\sum_{q>X} \Big(A(n, q) \Big)^2 = \sum_{\substack{q>X \\ q^*<X}} \Big(A(n, q) \Big)^2 + \sum_{\substack{q>X \\ q^* \geqslant X}} \Big(A(n, q) \Big)^2.$$

By Lemmas 18, 25,

$$\sum_{\substack{q>X \\ q^*<X}} \Big(A(n, q) \Big)^2 < c_8 X^{-\frac{1}{6}} \sum_{\substack{q>X \\ q^*<X}} |A(n, q)| \leqslant c_8 X^{-\frac{1}{6}} \sum_{m<X} c_9^{\nu(m)} m^{-1}.$$

Now
$$\sum_{m<X} c_9^{\nu(m)} m^{-1} = O(X^{\epsilon})$$

for any $\epsilon > 0$. Hence

$$\sum_{\substack{q>X \\ q^*<X}} \Big(A(n, q)\Big)^2 < c_{14} X^{-\frac{1}{7}}.$$

By Lemma 25,

$$\begin{aligned}\sum_{\substack{q>X \\ q^*\geqslant X}} \Big(A(n, q)\Big)^2 &\leqslant \sum_{m\geqslant X} \Big(\sum_{\substack{q\geqslant m \\ q^*=m}} |A(n, q)| \Big)^2 \\ &\leqslant \sum_{m\geqslant X} (c_9^{\nu(m)} m^{-1})^2 \\ &= O(X^{-1+\epsilon})\end{aligned}$$

for any $\epsilon > 0$. This proves Lemma 28.

LEMMA 29.

$$\sum_{n=1}^{P^3} \Big(\mathfrak{S}(R, n) - \mathfrak{S}'(R, n)\Big)^2 = O\Big(P^3 (\log P)^{-\frac{1}{7}}\Big).$$

Proof. For sufficiently large P we have [denoting by $\pi(X)$ the number of primes not exceeding X]

$$X^{\pi(X)} < X^{2X/\log X} = e^{2X} = R.$$

Hence we can write

$$\mathfrak{S}(R, n) - \mathfrak{S}'(R, n) = \sum_{q=1}^{R} \theta(q) A(n, q),$$

where $\theta(q)$ is 0 or 1, and $\theta(q) = 0$ for $q \leqslant X$. We have, therefore,

$$\begin{aligned}&\sum_{n=1}^{P^3} \Big(\sum_{q=1}^{R} \theta(q) A(n, q) \Big)^2 \\ &\quad = \sum_{n=1}^{P^3} \sum_{q=1}^{R} \theta(q) \Big(A(n, q)\Big)^2 + \sum_{n=1}^{P^3} \sum_{\substack{q_1=1 \\ q_1\neq q_2}}^{R} \sum_{q_2=1}^{R} \theta(q_1)\theta(q_2) A(n, q_1) A(n, q_2) \\ &\quad = \Sigma_1 + \Sigma_2,\end{aligned}$$

say. By Lemma 28,

$$\Sigma_1 \leqslant \sum_{n=1}^{P^3} \sum_{q>X} \Big(A(n, q)\Big)^2 < c_{13} P^3 X^{-\frac{1}{7}} = O\Big(P^3 (\log P)^{-\frac{1}{7}}\Big).$$

By Lemmas 26, 27,

$$\Sigma_2 = O\Big(\sum_{q_1=1}^{R}\sum_{q_2=1}^{R}\frac{q_1 q_2}{q_1^* q_2^*}\,c_9^{\nu(q_1)}c_9^{\nu(q_2)}\Big)$$

$$= O\Big(\sum_{q=1}^{R}\frac{q}{q^*}\,c_9^{\nu(q)}\Big)^2$$

$$= O(R^{2(1+\epsilon)}) = O\Big(P^3(\log P)^{-\frac{1}{2}}\Big).$$

The singular product.

Lemma 30. *There exist* c_{15}, c_{16}, c_{17} *such that*

$$\prod_{c_{15}\leqslant p\leqslant X}\Big(\sum_{\substack{\nu=0\\ p^\nu\leqslant X}}^{\infty}A(n,p^\nu)\Big) > c_{16}(\log\log P)^{-c_{17}}.$$

Proof. By Lemmas 18, 22, if $p>3$ and $p\,|\,n$,

$$\sum_{\substack{\nu=0\\ p^\nu\leqslant X}}^{\infty}A(n,p^\nu) \geqslant 1-\frac{20}{p}-\sum_{\nu=6}^{\infty}c_8 p^{-\frac{1}{6}\nu} > 1-\frac{c_{18}}{p}.$$

By Lemmas 23, 24, if $p>3$ and $p\nmid n$,

$$\sum_{\substack{\nu=0\\ p^\nu\leqslant X}}^{\infty}A(n,p^\nu) \geqslant 1-\frac{4}{p}.$$

Hence, if $c_{15} > 2c_{18}$,

$$\prod_{c_{15}\leqslant p\leqslant X}\Big(\sum_{\substack{\nu=0\\ p^\nu\leqslant X}}^{\infty}A(n,p^\nu)\Big) > \prod_{2c_{18}\leqslant p\leqslant X}\Big(1-\frac{c_{18}}{p}\Big)$$

$$> c_{16}(\log X)^{-c_{17}}$$

$$> c_{16}(\log\log P)^{-c_{17}}.$$

This establishes Lemma 30.

We introduce the (temporary) notation

$$S'^{(3)}_{a,\,p^\nu} = \sum_{\substack{x=1\\ p\nmid x}}^{p^\nu} e_{p^\nu}(ax^3),$$

$$B(n,p^\nu) = p^{-3\nu}\sum_{\substack{a=1\\ p\nmid a}}^{p^\nu} S'^{(3)}_{a,\,p^\nu}S^{(3)}_{a,\,p^\nu}S^{(2)}_{Wa,\,p^\nu}e_{p^\nu}(-na).$$

LEMMA 31.

$$\sum_{\nu=1}^{l} A(n, p^\nu) \geqslant -\frac{1}{p} + \sum_{\nu=1}^{l} B(n, p^\nu).$$

Proof. We have

$$\begin{aligned} 1 + \sum_{\nu=1}^{l} A(n, p^\nu) &= \sum_{\nu=0}^{l} p^{-3\nu} \sum_{\substack{a=1 \\ p \nmid a}}^{p^\nu} (S^{(3)}_{a, p^\nu})^2 S^{(2)}_{Wa, p^\nu} e_{p^\nu}(-na) \\ &= p^{-3l} \sum_{\nu=0}^{l} \sum_{\substack{a=1 \\ p \nmid a}}^{p^\nu} (S^{(3)}_{ap^{l-\nu}, p^l})^2 S^{(2)}_{Wap^{l-\nu}, p^l} e_{p^l}(-nap^{l-\nu}) \\ &= p^{-3l} \sum_{a=1}^{p^l} (S^{(3)}_{a, p^l})^2 S^{(2)}_{Wa, p^l} e_{p^l}(-na) \\ &= p^{-2l} N(n, p^l), \end{aligned}$$

where $N(n, p^l)$ is the number of solutions of the congruence

$$x^3 + y^3 + Wz^2 \equiv n \pmod{p^l}.$$

By exactly the same argument we find that

$$1 - \frac{1}{p} + \sum_{\nu=1}^{l} B(n, p^\nu) = p^{-2l} N'(n, p^l),$$

where $N'(n, p^l)$ is the number of solutions of

$$x^3 + y^3 + Wz^2 \equiv n \pmod{p^l}, \quad p \nmid x.$$

The truth of the result enunciated is now obvious.

LEMMA 32. *For* $p \neq 3$,

$$\sum_{\nu=0}^{l} A(n, p^\nu) > c_{19}.$$

Proof. If $p \neq 3$, we have, for $\nu > 1$, $p \nmid a$,

$$\begin{aligned} S'^{(3)}_{a, p^\nu} = \sum_{\substack{x=1 \\ p \nmid x}}^{p^\nu} e_{p^\nu}(ax^3) &= \sum_{y=1}^{p} \sum_{\substack{z=1 \\ p \nmid z}}^{p^{\nu-1}} e_{p^\nu}\left(a(p^{\nu-1}y + z)^3\right) \\ &= \sum_{y=1}^{p} \sum_{\substack{z=1 \\ p \nmid z}}^{p^{\nu-1}} e_{p^\nu}(az^3)\, e_p(3ayz^2) \\ &= 0. \end{aligned}$$

Hence, by Lemma 31,

$$\sum_{\nu=0} A(n, p^\nu) \geqslant 1 - p^{-1} + B(n, p).$$

For $p \equiv 2 \pmod 3$, $S^{(3)}_{a,p} = 0$ by Lemma 21, hence, in this case,

$$B(n, p) = 0,$$

and the result is established.

For $p \equiv 1 \pmod 3$, we observe that

$$S'^{(3)}_{a,p} = S^{(3)}_{a,p} - 1.$$

Hence

$$|B(n,p)| = \left| A(n,p) - p^{-3} \sum_{a=1}^{p-1} S^{(3)}_{a,p} S^{(2)}_{Wa,p} e_p(-na) \right|$$

$$\leqslant |A(n,p)| + 2p^{-2}(p-1).$$

By part (1) of the proof of Lemma 22, $A(n, p) = 0$ if $p \mid n$. By Lemma 24, $|A(n, p)| \leqslant 4p^{-1}$ if $p \nmid n$. Hence

$$|B(n,p)| \leqslant 4p^{-1} + 2p^{-1} - 2p^{-2},$$

and

$$\sum_{\nu=0}^{l} A(n, p^\nu) \geqslant 1 - p^{-1} - |B(n,p)|$$

$$\geqslant 1 - 7p^{-1} + 2p^{-2}$$

$$\geqslant 1 - \tfrac{7}{7} + \tfrac{2}{49}$$

$$> c_{19}.$$

Lemma 33. *For $l \geqslant 2$,*

$$\sum_{\nu=0}^{l} A(n, 3^\nu) \geqslant \tfrac{2}{3} + B(n, 9).$$

Proof. The argument at the beginning of the proof of the previous Lemma shows that if $3 \nmid a$, $S'^{(3)}_{a,3^\nu} = 0$ for $\nu > 2$. Also, if $3 \nmid a$,

$$S^{(3)}_{a,3} = 1 + e_3(a) + e_3(8a) = 0.$$

Hence $B(n, 3) = 0$. The result now follows from Lemma 31.

Lemma 34. *If $W = 1$, then, for all n,*

$$\tfrac{2}{3} + B(n, 9) \geqslant \tfrac{4}{9}.$$

Proof. Let $\zeta = e_9(1)$. Then if $3 \nmid a$,

$$S'^{(3)}_{a,9} = 3(\zeta^a + \zeta^{-a}),$$

$$S^{(3)}_{a,9} = 3(1 + \zeta^a + \zeta^{-a}),$$

$$S^{(2)}_{a,9} = 3 + 2(\zeta^a + \zeta^{4a} + \zeta^{7a}) = 3. \tag{35}$$

Hence $$B(n, 9) = \tfrac{1}{27} \sum_{\substack{a=1 \\ 3+a}}^{9} \zeta^{-na}(2+\zeta^a+\zeta^{-a}+\zeta^{2a}+\zeta^{-2a}).$$

It is easily seen that

$$\sum_{\substack{a=1 \\ 3+a}}^{9} \zeta^{-ma} = \begin{cases} 0 & \text{if } 3+m, \\ -3 & \text{if } 3|m, \quad 9+m, \\ 6 & \text{if } 9|m. \end{cases} \tag{36}$$

Now of the numbers n, n, $n-1$, $n+1$, $n-2$, $n+2$, exactly two are divisible by 3. Hence

$$B(n, 9) \geqslant \frac{-6}{27} = -\tfrac{2}{3}+\tfrac{4}{9}.$$

LEMMA 35. *If* $W = 6$ *or* 12, *and* $n \not\equiv -W \pmod 9$, *then*

$$\tfrac{2}{3}+B(n, 9) \geqslant \tfrac{1}{9}.$$

Proof. Suppose that $W = 6$. Then instead of (35) we have

$$S^{(2)}_{Wa,\,9} = 3+6\zeta^{6a}.$$

Hence

$$B(n, 9) = \tfrac{1}{27} \sum_{\substack{a=1 \\ 3+a}}^{9} \zeta^{-na}(2+\zeta^a+\zeta^{-a}+\zeta^{2a}+\zeta^{-2a})(1+2\zeta^{6a}).$$

Suppose first that $n \not\equiv 0 \pmod 3$. By the first part of (36),

$$B(n, 9) = \tfrac{1}{27} \sum_{\substack{a=1 \\ 3+a}}^{9} \zeta^{-na}(\zeta^a+3\zeta^{-a}+\zeta^{2a}+3\zeta^{-2a}+2\zeta^{4a}+2\zeta^{-4a}).$$

Since $$\zeta^a+\zeta^{-2a}+\zeta^{4a} = \zeta^{-a}+\zeta^{2a}+\zeta^{-4a} = 0$$

for $3+a$, we have

$$B(n, 9) = \tfrac{1}{27} \sum_{\substack{a=1 \\ 3+a}}^{9} \zeta^{-na}(2\zeta^{-a}+2\zeta^{-2a}+\zeta^{4a}+\zeta^{-4a}).$$

Of the numbers $n+1$, $n+1$, $n+2$, $n+2$, $n-4$, $n+4$, not all can be divisible by 3. Hence

$$B(n, 9) \geqslant -\tfrac{15}{27} = -\tfrac{2}{3}+\tfrac{1}{9}.$$

Suppose next that $n \equiv 0 \pmod 3$. By the first part of (36),

$$B(n, 9) = \tfrac{2}{27} \sum_{\substack{a=1 \\ 3+a}}^{9} \zeta^{-na}(1+2\zeta^{6a}).$$

Hence $$B(6, 9) = \tfrac{2}{27}(-3+12) > 0,$$

and $$B(9, 9) = \tfrac{2}{27}(6-6) = 0.$$

Thus the lemma is established in the case $W = 6$. For the case $W = 12$, we observe that, if we write $B_W(n, 9)$ to indicate the dependence of $B(n, 9)$ on W, then

$$B_{12}(n, 9) = B_6(n', 9)$$

if $n \equiv -n'$ (mod 9).

Lemma 36. *If $l \geqslant 2$, the inequality*

$$\sum_{\nu=0}^{l} A(n, 3^\nu) > c_{20}$$

holds (a) for all n if $W = 1$, (b) for all n not congruent to $-W$ (mod 9) if $W = 6$ or 12.

Proof. Lemmas 33, 34, 35.

Lemma 37. *The inequality*

$$\mathfrak{S}'(R, n) > c_{21}(\log\log P)^{-c_{17}}$$

holds (a) for all n if $W = 1$, (b) for all n not congruent to $-W$ (mod 9) if $W = 6$ or 12.

Proof. By Lemmas 30, 32, 36,

$$\mathfrak{S}'(R, n) > c_{20}\Big(\prod_{p<c_{15}} c_{19}\Big)\, c_{16}(\log\log P)^{-c_{17}}$$

$$= c_{21}(\log\log P)^{-c_{17}}.$$

Proof of Theorems 1, 2.

By Lemma 17,

$$\sum_{n=W+2}^{P^3} \Big(r(n) - \psi(n)\mathfrak{S}(R, n)\Big)^2 = O(P^{4-\delta+\epsilon}).$$

By Lemma 29 and the inequality (32),

$$\sum_{n=W+2}^{P^3} \Big(\psi(n)\mathfrak{S}(R, n) - \psi(n)\mathfrak{S}'(R, n)\Big)^2 = O\Big(P^4(\log P)^{-\frac{1}{2}}\Big).$$

Hence

$$\sum_{n=W+2}^{P^3} \Big(r(n) - \psi(n)\mathfrak{S}'(R, n)\Big)^2 = O\Big(P^4(\log P)^{-\frac{1}{2}}\Big).$$

Let a dash attached to the sign of summation denote (*a*) if $W = 1$, no restriction on n, (*b*) if $W = 6$ or 12 the restriction $n \not\equiv -W \pmod 9$. Then

$$\sum_{\substack{n=W+2 \\ r(n)=0}}^{P^3}{}' \Big(\psi(n)\,\mathfrak{S}'(R,n)\Big)^2 = O\Big(P^4(\log P)^{-\frac{1}{7}}\Big).$$

By Lemma 37 and the inequality (32) this implies

$$\sum_{\substack{n=W+2 \\ r(n)=0}}^{P^3}{}' n^{\frac{1}{3}} = O\Big(P^4(\log P)^{-\frac{1}{7}}(\log\log P)^{2c_{17}}\Big)$$

$$= O\Big(P^4(\log P)^{-\frac{1}{8}}\Big).$$

Let $Z = [P^3(\log P)^{-\frac{1}{11}}]$. Then

$$\sum_{\substack{n=1 \\ r(n)=0}}^{P^3}{}' 1 \leqslant Z + \sum_{\substack{n=Z \\ r(n)=0}}^{P^3}{}' 1$$

$$\leqslant Z + Z^{-\frac{1}{3}} \sum_{\substack{n=Z \\ r(n)=0}}^{P^3}{}' n^{\frac{1}{3}}$$

$$= O(Z) + O\Big(Z^{-\frac{1}{3}} P^4(\log P)^{-\frac{1}{8}}\Big)$$

$$= O\Big(P^3(\log P)^{-\frac{1}{11}}\Big).$$

This proves Theorems 1 and 2.

Appendix. The infinite singular series.

We give here the connection (referred to in the Introduction) between the infinite singular series and certain L-series in algebraic fields. For simplicity we suppose that $W = 1$ and that $4n$ is neither a square nor a cube.

By Lemmas 19, 23, 24 we have

$$\sum_{q=1}^{\infty} A(n,q)\,q^{1-s} = \phi(s)\,M(s), \tag{37}$$

where $M(s)$ is as defined in the Introduction and

$$\phi(s) = \prod_{p \mid 6n} \Big\{ \sum_{\nu=0}^{\infty} A(n, p^\nu) \Big\}.$$

We denote by n_3 the least positive integer congruent to 0, 1, or -1 (mod 9) for which $27.4n/n_3$ is a cube, and by n_2 the least positive integer congruent to 0 or 1 (mod 4) for which $4n/n_2$ is a square.

We introduce the four algebraic fields

$$K_2 = \mathbf{P}(\sqrt{-3}), \quad K_4 = K_2(\sqrt{n_2}), \quad K_6 = K_2(\sqrt[3]{n_3}), \quad K_{12} = K_2(\sqrt{n_2}, \sqrt[3]{n_3}),$$

and we denote by $\zeta_r(s)$ the Dedekind ζ-function of K_r. Then it follows from the general theory of relatively Abelian fields that

$$\zeta_4(s) = \zeta_2(s)\, L_A(s), \tag{38}$$

$$\zeta_6(s) = \zeta_2(s)\, L_B{}^2(s), \tag{39}$$

$$\zeta_{12}(s) = \zeta_2(s)\, L_A(s)\, L_B{}^2(s)\, L_C{}^2(s), \tag{40}$$

where $L_A(s)$, $L_B(s)$, $L_C(s)$ are Abelian L-series in K_2. As such they are integral functions of s, and do not vanish at $s = 1$.

By (38), (39), (40),

$$\left(\frac{\zeta_{12}(s)\,\zeta_4(s)}{\zeta_6(s)\,\zeta_2(s)}\right)^{\frac{1}{2}} = L_A(s)\, L_C(s). \tag{41}$$

If we express the functions $\zeta_r(s)$ in rational terms by using the laws of decomposition of primes into prime ideals in metabelian fields, we find that

$$L_A(s)\, L_C(s) = (1-\gamma_2 3^{-s})^{-1}\,(1-\gamma_2\gamma_3 3^{-s})^{-1} \tag{42}$$

$$\times \prod_{p\equiv 1 \pmod 3} \left(1-\left(\frac{n_2}{p}\right)p^{-s}\right)^{-2} \left(1-\left(\frac{n_2}{p}\right)\chi_p(n_3)\,p^{-s}\right)^{-1} \left(1-\left(\frac{n_2}{p}\right)\bar{\chi}_p(n_3)\,p^{-s}\right)^{-1}$$

$$\times \prod_{\substack{p\equiv 2 \pmod 3 \\ p+n_2}} (1-p^{-2s})^{-2} \prod_{\substack{p\equiv 2 \pmod 3 \\ p+n_2 \\ p\mid n_3}} (1-p^{-2s}),$$

where

$$\gamma_2 = \begin{cases} \left(\dfrac{n_2}{3}\right) & \text{if } 3+n_2, \\ \left(\dfrac{-\frac{1}{3}n_2}{3}\right) & \text{if } 3\mid n_2; \end{cases}$$

$$\gamma_3 = \begin{cases} 1 & \text{if } n_3 \equiv \pm 1 \pmod 9, \\ 0 & \text{if } n_3 \equiv 0 \pmod 9. \end{cases}$$

Hence

$$M(s) = g_1(s)\, g_2(s)\, L_A(s)\, L_C(s), \tag{43}$$

where we collect together in $g_1(s)$ the product over the prime divisors of $3n$. Then $g_2(s)$ is a product which is absolutely convergent for $\sigma \geqslant \frac{1}{2}+\epsilon$ for any $\epsilon > 0$, uniformly in n.

Hence

$$\sum_{q=1}^{\infty} A(n, q) = \phi(1) g_1(1) g_2(1) L_A(1) L_C(1).$$

By using (41) it is possible to express $L_A(1)\, L_C(1)$ in terms of the class-numbers and regulators of the fields K_2, K_4, K_6, K_{12}.

Using the recent work of Siegel† to obtain a lower bound for $L_A(1)$, it is possible to prove that

$$\sum_{q=1}^{\infty} A(n, q) > c(\epsilon)\, n^{-\epsilon}.$$

But only if we assume some hypothesis of the nature of the Riemann hypothesis about the zeros of $L_A(s)$ and $L_C(s)$ does it seem possible to prove the same about $\sum_{q=1}^{R} A(n, q)$.

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THEOREMS ON FOURIER SERIES AND POWER SERIES (III).

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Introduction.

1. Suppose that $f(\theta) \epsilon L$, that is to say that f is integrable in the sense of Lebesgue in $(-\pi, \pi)$; and that

$$\tfrac{1}{2}a_0+\sum_1^\infty(a_n \cos n\theta+b_n \sin n\theta), \quad \sum_{-\infty}^{\infty} c_n e^{n\theta i}$$

are the ordinary and the "complex" Fourier series of f. We suppose, to avoid trivial complications in stating our results, that $c_0 = a_0 = 0$. Let

$$u(\rho, \theta) = \sum_1^\infty(a_n \cos n\theta+b_n \sin n\theta)\,\rho^n, \quad v(\rho, \theta) = \sum_1^\infty(-b_n \cos n\theta+a_n \sin n\theta)\,\rho^n;$$

$$\phi(z) = \phi(z, f) = u+iv = 2\sum_1^\infty c_n z^n \quad (z = \rho e^{i\theta});$$

$$g(\theta) = g(\theta, f) = \Big(\int_0^1 (1-\rho)\,|\phi'(\rho e^{i\theta})|^2\,d\rho\Big)^{\frac{1}{2}}.$$

Let α and $\beta \geqslant \alpha$ be positive constants, and let $\nu_1, \nu_2, \ldots$ be any sequence of positive integers satisfying

$$(1+\alpha)\,\nu_n \leqslant \nu_{n+1} \leqslant (1+\beta)\,\nu_n,$$

and $\nu_1', \nu_2', \ldots$ any other such sequence (with the same α, β). Let

$$\Delta_0 = \sum_1^{\nu_1} c_m e^{m\theta i}, \quad \Delta_n = \sum_{\nu_n < m \leqslant \nu_{n+1}} c_m e^{m\theta i} \quad (n > 0).$$

These Δ_n we may call "long pieces" of the positive part of the complex Fourier series of f (each is of a length comparable with the whole of the

† We are indebted to Dr. A. C. Offord, and also to a Referee, for valuable criticism.

earlier part of the series: the typical case is $\nu_n = 2^n$). The negative part $\sum_{-\infty}^{-1}$ of the Fourier series is similarly divided into "long pieces" $\Delta_{-1}, \ldots, \Delta_{-n}, \ldots$ derived from the sequence $(\nu_1', \nu_2', \ldots)$.

Let p, q, r denote indices satisfying respectively

$$r > 1, \quad 1 < p \leqslant 2 \leqslant q.$$

We use the notation r, r' for a pair of "conjugate" indices which satisfy

$$\frac{1}{r} + \frac{1}{r'} = 1, \qquad r' = \frac{r}{r-1}.$$

Note that p' is a q, q' a p.

Let $A(x, y, \ldots)$, or $A_{x,y,\ldots}$, denote a positive number, not always the same from one occurrence to another, depending only on the parameters $x, y, \ldots$ shown explicitly. Then we have the following theorems, in which the range of integration of θ is always $(-\pi, \pi)$.

THEOREM 1.

$$\sum_{-\infty}^{\infty} \int |\Delta_n|^q \, d\theta \leqslant A(q, \alpha, \beta) \int |f|^q \, d\theta.$$

THEOREM 2.

$$\int |f|^p \, d\theta \leqslant A(p, \alpha, \beta) \sum_{-\infty}^{\infty} \int |\Delta_n|^p \, d\theta.$$

THEOREM 3. *The ratio*

$$\int \Big(\sum_{-\infty}^{\infty} |\Delta_n|^2 \Big)^{\frac{1}{2}r} d\theta \Big/ \int |f|^r \, d\theta$$

lies between two numbers $A(r, \alpha, \beta)$†.

THEOREM 4. *Suppose that* $(\eta_n)(-\infty < n < +\infty)$ *is any set of numbers each of which has one of the two values* $+1$ *and* -1. *Suppose that* $f \in L^r$. *Then*

$$\sum_{-\infty}^{\infty} \eta_m \Delta_m$$

is (when written at length as a "complex" trigonometrical series) the Fourier series of a function F *of* L^r, *and*

$$\int |F|^r \, d\theta \leqslant A(r, \alpha, \beta) \int |f|^r \, d\theta.$$

† More strictly: if either the numerator or the denominator is finite, then so is the other, and the ratio lies between two $A(r, \alpha, \beta)$'s. A similar gloss is needed in Theorem 7.

THEOREM 5.

$$\int_0^1 d\rho \int_{-\pi}^{\pi} (1-\rho)^{q-1} |\phi'(z)|^q d\theta \leqslant A(q) \int_{-\pi}^{\pi} |f|^q d\theta.$$

THEOREM 6.

$$\int_{-\pi}^{\pi} |f|^p d\theta \leqslant A(p) \int_0^1 d\rho \int_{-\pi}^{\pi} (1-\rho)^{p-1} |\phi'(z)|^p d\theta.$$

THEOREM 7. *The ratio*

$$\int_{-\pi}^{\pi} |g|^r d\theta \Big/ \int_{-\pi}^{\pi} |f|^r d\theta$$

lies between two numbers $A(r)$.

THEOREM 7*. *Let*

$$g^*(\theta) = \left(\int_0^1 (1-\rho) \chi^2(\rho, \theta) d\rho \right)^{\frac{1}{2}},$$

where† $$\chi^2(\rho, \theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} |\phi'(\rho e^{i(\theta+t)})|^2 P(\rho, t) dt.$$

Then, for any positive even integer $2k$,

$$\int g^{*2k} d\theta \leqslant A(k) \int |f|^{2k} d\theta.$$

THEOREM 8. *Let* $f \in L^r$, *and let*

$$S(\theta) = \max_{(n)} \left| \sum_{m=0}^{n+1} c_m e^{m\theta i} \right| = \max \left| \sum_0^n \Delta_m \right|,$$

the upper bounds being for fixed θ *and varying* n *in* $0 \leqslant n < \infty$. *Then* (i)

$$\int S^r(\theta) d\theta \leqslant A(r) \int |f|^r d\theta;$$

(ii) *the series* $\Sigma \Delta_m$ *converges p.p. to the sum* $f(\theta)$.

THEOREM 9. *Let* $f \in L^p$, *and let*‡

$$S_n(\theta) = \sum_0^n c_m e^{m\theta i}, \quad S_n^*(\theta) = \sum_0^n \lambda_m c_m e^{m\theta i},$$

† $P(\rho, \theta) = (1-\rho^2)/(1-2\rho\cos\theta+\rho^2)$ is the "Poisson kernel".

‡ $S_n(\theta)$ has no connection with the $S(\theta)$ of Theorem 8. s_n denotes, as usual,

$$\sum_{-n}^{n} c_m e^{m\theta i},$$

which is S_n+S_{-n} (since $c_0 = 0$).

where $$\lambda_m = \lambda_m(p) = \left(\log(|m|+2)\right)^{-1/p}.$$

Let $$T(\theta) = \max_{(n)} |\lambda_n S_n(\theta)|, \quad T^*(\theta) = \max_{(n)} |S_n^*(\theta)|,$$

the upper bounds being for fixed θ and varying n in $-\infty < n < \infty$. Then (i)

$$\int T^p(\theta)\, d\theta \leqslant A(p) \int |f|^p\, d\theta, \quad \int T^{*p}(\theta)\, d\theta \leqslant A(p) \int |f|^p\, d\theta;$$

(ii) *the series $\Sigma \lambda_m c_m e^{m\theta i}$ converges p.p.*

THEOREM 10. *Let $f \in L^q$, and let*

$$Q(\theta) = \max_{(n)} |l_n S_n(\theta)|, \quad Q^*(\theta) = \max_{(n)} \left| \sum_0^n l_m c_m e^{m\theta i} \right|,$$

where $$l_m = \left(\log(|m|+2)\right)^{-1/q'},$$

and the upper bounds are taken for fixed θ and varying n. Then

$$\int Q^q(\theta)\, d\theta \leqslant A(q) \int |f|^q\, d\theta, \quad \int Q^{*q}(\theta)\, d\theta \leqslant A(q) \int |f|^q\, d\theta.$$

2. These results, with 7* omitted, and in slightly modified form, were stated in our paper **5**, and Theorems 1 to 7* were proved in our paper **6**†. Our present task is the proof of the remaining Theorems 8, 9, and 10; but the entire group is connected, and we have restated it in full for convenience of reference, even though we do not have to refer explicitly to all the results proved in (II).

3. We summarize for purposes of reference a number of known results. Some references will be found in (II), and full ones and proofs in Zygmund's *Trigonometrical series*. (We do not here suppose that $c_0 = 0$.)

(A) Hölder's inequality.

(1) $$\left| \frac{1}{2\pi} \int_{-\pi}^{\pi} a(\theta)\, b(\theta)\, d\theta \right| \leqslant M_r(a)\, M_{r'}(b),$$

where $M_r(a)$ means, as usual, $\left(\frac{1}{2\pi} \int |a|^r\, d\theta\right)^{1/r}$.

† See the list of papers at the end. Papers **5** and **6** bore the same general title: they will be referred to as (I) and (II).

(B) The converse of Hölder's inequality.

If $$\left|\frac{1}{2\pi}\int_{-\pi}^{\pi} abd\theta\right| \leqslant UV$$

for every b for which $M_{r'}(b) \leqslant V$, then $M_r(a) \leqslant U$.

(C) Minkowski's inequality.

$$M_r(a+b+\ldots) \leqslant M_r(a)+M_r(b)+\ldots.$$

(D) Suppose that $f \epsilon L^r$, and that $s_n(f)$ denotes the sum to $n+1$ terms of the Fourier series of f,

$$s_n(f) = \tfrac{1}{2}a_0+\sum_1^n(a_m\cos m\theta+b_m\sin m\theta) = \sum_{-n}^{n} c_m e^{m\theta i}.$$

Then (i) if $S = \sum_{\mu}^{\nu} c_m e^{m\theta i}$ is any connected piece (finite or not) of the Fourier series of f, $M_r(S) \leqslant A_r M_r(f)$. In particular, this is true for $S = \sum_0^{\infty}$, $\left(\text{or } \sum_0^{-\infty}\right)$, for $S = \sum_0^m$, and for $S = s_n = \sum_{-n}^{n}$.

(ii) $M_r(S-f) \to 0$ as $\mu \to -\infty$ and $\nu \to \infty$ together; S converges strongly to f with index r.

Conversely, (iii) if $s_n = \sum_{-n}^{n} c_m e^{m\theta i}$ and $M_r(s_n) \leqslant K$ for all n, then there exists an $f \sim \sum_{-\infty}^{\infty} c_m e^{m\theta i}$ with $M_r(f) < A_r K$.

(E) If $$\sigma_n(\theta) = c_0+\sum_{m=0}^{n}\left(1-\frac{m}{n}\right)(c_m e^{m\theta i}+c_{-m}e^{-m\theta i})$$

is the n-th Cesàro mean of the Fourier series of a function $f \epsilon L^r$, then $\max_{(n)}|\sigma_n(\theta)| \epsilon L^r$, and

$$\int_{-\pi}^{\pi} \max|\sigma_n(\theta)|^r d\theta \leqslant A(r)\int_{-\pi}^{\pi}|f|^r d\theta.$$

We have several times to appeal to M. Riesz's theorem† on the convexity of bilinear forms and linear functionals. As we said in (II), the theorem cannot be stated shortly, and we refer the reader to Zygmund's book, where there is a full account of it.

As a matter of general notation we use B's for constants of the type $A(r)$, $A(r, \alpha, \beta)$ occurring in the statement of the theorem under consideration.

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† M. Riesz, 9.

We need the following lemma, which enables us to reduce a proof of "convergence p.p." to one of the much more manageable "boundedness p.p.".

LEMMA 1. *Suppose that*

$$f \sim \Sigma c_m e^{m\theta i} \in L^r.$$

Then there exists a set of numbers μ_m $(m = \ldots -1, 0, 1, 2, \ldots)$, *increasing to* ∞ *as* $|m|$ *increases from* 0 *to* $+\infty$, *and such that*

$$\sum_{-\infty}^{\infty} \mu_m c_m e^{m\theta i} \sim f^* \in L^r.$$

(*The sequence depends on the function* f.)

We can arrange also that μ_m *is constant in each* Δ_r, *so that, e.g. for* $r > 0$,

$$\mu_m = \mu_{\nu_{r+1}} (\nu_r < m \leqslant \nu_{r+1}).$$

This is new (so far as we know).

By (D) (ii) we have

$$(1) \qquad M_r\left(f(\theta) - \sum_{-\lambda_k'}^{\lambda_k} c_m e^{m\theta i}\right) = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \left| f(\theta) - \sum_{-\lambda_k'}^{\lambda_k} \right|^r d\theta\right)^{1/r} < \frac{1}{k^2},$$

provided that the sequences λ_k, λ_k' increase fast enough. Let

$$f_N^* = \sum_{k=1}^{N} \left(f(\theta) - \sum_{-\lambda_k'}^{\lambda_k} c_m e^{m\theta i}\right).$$

Since, by (C) and (1),

$$M_r\left(\sum_{N+1}^{\infty} \left(f - \sum_{-\lambda_k'}^{\lambda_k}\right)\right) \leqslant \sum_{N+1}^{\infty} M_r\left(f - \sum_{-\lambda_k'}^{\lambda_k}\right) < \sum_{N+1}^{\infty} \frac{1}{k^2} \to 0$$

as $N \to \infty$, f_N^* converges strongly with index r to a limit function f^* of L^r. For the m-th Fourier coefficient of f^* we have, therefore,

$$c_m(f^*) = \lim_{N\to\infty} c_m(f_N^*) = \mu_m c_m(f),$$

where μ_m is the number of values of k for which m lies outside the range $(-\lambda_k', \lambda_k)$. μ_m will further be constant in the Δ_r if $-\lambda_k'$ is a subsequence of $-\nu_r'$ and λ_k one of ν_r. The numbers μ_m have the desired properties.

4. The theorems (8, 9, and 10) which we are now to consider have ideas in common which do not occur in (II). We have, however, to appeal to

results from (II). In particular we need a variant of Theorem 3 (or part of it), and in order not to distract our main argument we shall end the present Introduction with a proof† of

THEOREM 3(a). *Let*‡

$$\Delta_n' = \sum_{\nu_n < m \leqslant \nu_{n+1}} \frac{m}{\nu_n} c_m e^{m\theta i} \quad (n>0), \qquad \Delta'_{-n} = \sum_{\nu_n' < m \leqslant \nu'_{n+1}} \frac{m}{\nu_n'} c_{-m} e^{-m\theta i} \quad (n>0),$$

$$\Delta_0' = \sum_{0 < m \leqslant \nu_1} m c_m e^{m\theta i},$$

and let

$$G'^2 = \sum_{-\infty}^{\infty} |\Delta_n'|^2.$$

Then

$$\int G'^r d\theta \leqslant B \int |f|^r d\theta,$$

where B is a constant of type $A(r, \alpha, \beta)$.

5. We prove Theorem 3(a) first for the case where r is an even integer, $r = 2k$. As in the corresponding place in Theorem 3 [(II), § 14], we may suppose f to be a Fourier power series, and all the terms in Δ_0 to be zero;

$$f \sim \sum_{m=\nu_1+1}^{\infty} c_m e^{m\theta i}.$$

For, in the general case, the two parts $\sum_{\nu_1+1}^{\infty}$, $\sum_{-\nu_1'-1}^{-\infty}$ of the series are, by (D), Fourier power series of f_1, f_2 with

$$\int |f_i|^r d\theta \leqslant A(r) \int |f|^r d\theta,$$

so are Δ_1, Δ_{-1}, and the result of Theorem 3(a) is true for the complete f by combination if it is true for the four parts separately. Next [as in (II), § 14] we write (for $n>0$)

$$N = \nu_n; \quad N' = \nu_{n+1};$$

$$\rho_n = 1 - 1/\nu_n = 1 - 1/N, \quad \delta = \delta_n = 1 - \rho_n = 1/\nu_n = 1/N.$$

In (II) we had

$$\Delta_n = \int_{-\pi}^{\pi} \phi'(\rho_n, \theta+t)\chi(t)\,dt, \tag{1}$$

† This would, but for an oversight, have been given at the same time as that of Theorem 3: the two proofs, as the reader will easily believe, run in parallel.

‡ Dashes do not mean differentiations.

where†

$$\phi'(\rho_n,\ \theta+t) = \phi'(\rho_n e^{i(\theta+t)}) = 2\sum_1^\infty mc_m(\rho_n e^{i(\theta+t)})^{m-1},$$

and

$$\chi(t) = \frac{1}{2\pi}\sum_{N+1}^{N'}\frac{\cos mt}{m}\rho_n^{-m}.$$

Here we have

$$\Delta_n' = \int_{-\pi}^{\pi}\phi'(\rho_n,\ \theta+t)\chi_1(t)\,dt, \tag{2}$$

where

$$\chi_1(t) = \frac{1}{2\pi}\sum_{N+1}^{N'}\frac{\cos mt}{N}\rho_n^{-m}.$$

Following the proof in (II), we multiply the expression for χ_1 by $\sin\frac{1}{2}t$, replace $2\cos mt\sin\frac{1}{2}t$ by $\sin(m+\frac{1}{2})t - \sin(m-\frac{1}{2})t$, and sum twice by parts. Since

$$|\Sigma\sin(m+\tfrac{1}{2})t| \leqslant A\,|\operatorname{cosec}\tfrac{1}{2}t|$$

whatever the limits of summation, we find that

$$D_1 = \chi_1\sin\tfrac{1}{2}t - \left[\frac{1}{4\pi}\frac{\rho_n^{-m}}{N}\sin(m-\tfrac{1}{2})t\right]_{m=N+1}^{N'+1}$$

is the difference of two expressions of the form

$$A\left(\sum_\lambda^\mu \sin(m+\tfrac{1}{2})t\right)\Delta\left(\frac{\rho_n^{-\nu}}{N}\right),$$

where λ, μ, ν lie between $N-1$ and $N'+2$, together with a sum majorized by

$$A\,|\operatorname{cosec}\tfrac{1}{2}t|\sum_{N+1}^{N'+1}\left|\Delta^2\left(\frac{\rho_n^{-m}}{N}\right)\right|.$$

It follows that $|D_1| < B\delta^2/|t|$, the same inequality as was proved for the corresponding expression D in (II). Now D_1 differs from D in that χ_1 replaces χ and $1/N$ replaces $1/(N'+1)$, $1/(N+1)$ respectively at the upper and lower values in the differenced square bracket. The latter change is completely insignificant, and, since $|\chi_1| < B$, we can conclude that Δ_n' satisfies the same inequality (6) of (II), § 14, that Δ_n does; viz.

$$|\Delta_n'|^2 \leqslant BM_{2k}^2(f) + BT_n^2 + B\delta_n^2\sum_1^4|V_{ni}|^2 = W_n^2, \tag{3}$$

† $\phi(z)$ is the ϕ of § 1, and ϕ' is its derivative.

say. The precise meaning of the symbols need not concern us further, since it is shown in (II)† that

$$\int (\Sigma W_n^2)^k d\theta \leqslant B \int |f|^{2k} d\theta,$$

and we can therefore conclude that

$$\int G'^{2k} d\theta \leqslant B \int |f|^{2k} d\theta,$$

the desired result for the case $r = 2k$.

6. Next, if (now for general f‡)

$$F' = \sum_{-\infty}^{\infty} \eta_m \Delta_m',$$

it follows, exactly as in (II), § 16, first that

$$\int |F'|^r d\theta \leqslant B \int |f|^r d\theta \tag{1}$$

in the even integer case $r = 2k$. Then (as in § 17), by M. Riesz's "interpolation" theorem on the convexity of linear functionals§, that (1) is true for each range $2k \leqslant r \leqslant 2k+2$, and so for $r \geqslant 2$. Next, by the "conjugacy" argument (of § 17), that it is true also for $r = p$||.

Finally, as in (II), § 18, we "average" the inequality

$$\int \left| \sum_{-n}^{n} \eta_m \Delta_m' \right|^r d\theta \leqslant B \int \left| \sum_{-n}^{n} \Delta_m \right|^r d\theta \leqslant B \int |f|^r d\theta$$

over the 2^{2n+1} distinct sets of (η_m), obtaining

$$\int \left(\sum_{-n}^{n} |\Delta'_m|^2 \right)^{\frac{1}{2}r} d\theta \leqslant B \int |f|^r d\theta,$$

† The proof uses the deep Theorem 7*.

‡ With c_0 not necessarily zero.

§ F' is obtained from f by a "linear" operation, the sets of ν's and η's being fixed.

|| Let $h(-\theta)$ be an arbitrary function of $L^{p'}$, and $H'(-\theta)$ the "F'" of $h(-\theta)$ (with the same set of η's as F'). Then, as in (II),

$$\frac{1}{2\pi} \int s_n(F') h(-\theta)\, d\theta = \frac{1}{2\pi} \int s_n(f) H'(-\theta)\, d\theta,$$

and the differences in the rest of the argument are trivial.

and so, making $n \to \infty$,

$$\int G'^r \, d\theta \leqslant B \int |f|^r \, d\theta.$$

Theorems 8, 9, *and* 10.

7. We shall now no longer suppose that $c_0 = 0$†.

We begin with Theorem 8‡. We may suppose that f is a Fourier power series,

$$f \sim \sum_0^\infty c_m e^{m\theta i};$$

for in the general case $\sum_0^\infty$ is, by (D), the Fourier series of a function f_1 for which

$$\int |f_1|^r \, d\theta \leqslant A(r) \int |f|^r \, d\theta,$$

the same thing is true for $\sum_{-1}^{-\infty}$, and the result of Theorem 8 is true for the sum of two f's if it is true for the f's separately. Then (with the convention $\nu_0 = 1$) we have

$$\sum_1^{\nu_n} c_m e^{m\theta i} = \sigma_{\nu_n}(\theta) + \frac{1}{\nu_n} \sum_{m=0}^{n-1} \nu_m \Delta_m'(\theta),$$

whence, taking moduli and upper bounds for fixed θ and varying n,

$$S(\theta) = \max_{(n)} \left| \sum_1^{\nu_n} c_m e^{m\theta i} \right| \leqslant \max_{(n)} |\sigma_{\nu_n}(\theta)| + \left(\frac{1}{\nu_n} \sum_0^{n-1} \nu_m \right) \max_{(m)} |\Delta'_m|$$

$$\leqslant \max_{(m)} |\sigma_m(\theta)| + BG'(\theta).$$

Part (i) of the theorem follows at once from (E) and Theorem 3(a).

This proof is a short and easy deduction from other results. But one of these is the deep Theorem 3(a), variant of Theorem 3, which needs a long

† There is no point in the restriction in Theorems 8, 9, and 10; and "interpolation" requires f to be unrestricted.

‡ The result of Theorem 8 was known already in the case $r = 2$. Kolmogoroff, **2**. See also Marcinkiewicz, **7**, and Zygmund, *Trigonometrical series*, 251. In (I) we mistakenly attributed it to Hardy and Littlewood, as being a corollary of their result (E). Kolmogoroff's work is earlier.

and difficult proof; and it is worth mentioning that it is possible to prove Theorem 8 independently. The proof is rather like that of Theorem 9 (proceeding, when $r = p$, via the "conjugate" theorem for $r = 2k$) and is of the same order of difficulty. In this independent proof there is a great difference between indices p and q, the latter case being an easy deduction from Theorem 1 (less deep than Theorem 3). When $r = q$ we have

$$\max_{(m)} |\Delta_m'|^q \leqslant \Sigma |\Delta_m'|^q,$$

and, since

$$|\Delta_m'| = \left| \frac{\nu_{m+1}}{\nu_m} \left(\Delta_m(\theta) - \sigma_{\nu_{m+1}}(\theta, \Delta_m) \right) \right| \leqslant B\left(|\Delta_m| + |\sigma_{\nu_{m+1}}(\theta, \Delta_m)| \right),$$

$$\int |\Delta_m'|^q d\theta \leqslant B \int |\Delta_m|^q d\theta + B \int |\sigma_{\nu_{m+1}}(\theta, \Delta_m)|^q d\theta,$$

$$\leqslant B \int |\Delta_m|^q d\theta + B \int |\Delta_m|^q d\theta$$

by a known result†. Hence

$$\int \max_{(m)} |\Delta_m'|^q d\theta \leqslant B \Sigma \int |\Delta_m|^q d\theta,$$

and the result follows from Theorem 1 (less deep than Theorem 3).

Part (ii) follows easily from part (i) and Lemma 1. Still supposing f to be a Fourier power series, we apply part (i) to the function f^* of Lemma 1. Then

$$(1) \qquad \sum_{m=0}^{\nu_{n+1}} \mu_m c_m e^{m\theta i}$$

is bounded (for varying n) p.p. in θ. But, when the μ_m are made constant in the Δ_r, (1) is the same as

$$\sum_{0}^{n} \mu_{\nu_r} \Delta_r.$$

For θ for which this is bounded $\sum_{0}^{n} \Delta_m$ is convergent by Dirichlet's convergence test. This gives the desired result, since the sum of the series must coincide p.p. with the strong limit, which is f.

8. We proceed to Theorem 9. When $p = 2$, part (ii) (asserting the convergence p.p. of $\Sigma \lambda_m c_m e^{m\theta i}$) becomes a well-known theorem of

† Or, of course, by (E).

Kolmogoroff and Seliverstoff†. Part (i) for $p = 2$ was proved by Hardy and Littlewood, but their proof has not been published.

We prove first a lemma needed for both Theorem 9 and Theorem 10.

LEMMA 2. *Suppose that* Λ_m *is either* λ_m *or* l_m *(of Theorems* 9 *and* 10*), and that*

$$W(\theta) = \max_{(n)} |\Lambda_n S_n(\theta)|, \quad W^*(\theta) = \max_{(n)} \left| \sum_0^n \Lambda_m c_m e^{m\theta i} \right|.$$

Then

$$\int W^{*r}(\theta)\, d\theta \leqslant B \int W^r(\theta)\, d\theta + B \int |f|^r\, d\theta,$$

where B *depends only on* r *and the parameter* (p *or* q) *occurring in* Λ_m.

We may suppose that f and W belong to L^r, or there is nothing to prove. Let $n \geqslant 0$. We have, by one partial summation,

$$\sum_0^n \Lambda_m c_m e^{m\theta i} = \Lambda_n S_n(\theta) + \sum_0^{n-1} S_m(\theta)(\Lambda_m - \Lambda_{m+1}),$$

and by another, dropping the arguments θ,

$$\sum_0^n \Lambda_m c_m e^{m\theta i} - \Lambda_n S_n = \Big(\sum_0^{n-1} S_m\Big)(\Lambda_{n-1} - \Lambda_n) + \sum_{m=0}^{n-2} \Big(\sum_0^m S_k\Big)(\Lambda_m - 2\Lambda_{m+1} + \Lambda_{m+2}).$$

Applying (E) to the function $f_1 \sim \sum_0^\infty c_m e^{m\theta i}$, for which $M_r(f_1) \leqslant B M_r(f)$, we have

$$\left| \sum_0^m S_k \right| = |(m+1)\sigma_m(\theta, f_1)| \leqslant (m+1) \max_{(m)} |\sigma_m| \leqslant B(m+1) W_1(\theta),$$

where $M_r(W_1) \leqslant B M_r(f)$; and so, since $\Lambda_m - 2\Lambda_{m+1} + \Lambda_{m+2} \geqslant 0$ and

$$n(\Lambda_{n-1} - \Lambda_n) + \sum_0^{n-2} (m+1)(\Lambda_m - 2\Lambda_{m+1} + \Lambda_{m+2}),$$

$$= \sum_0^{n-1} (\Lambda_m - \Lambda_{m+1}) < B,$$

$$\left| \sum_0^n \Lambda_m c_m e^{m\theta i} \right| \leqslant \max_{(n)} |\Lambda_n S_n| + B W_1 \leqslant W + B W_1.$$

† Plessner, **8**; Kolmogoroff and Seliverstoff, **4**. A less precise result was proved earlier by the latter authors, **3**.

A similar inequality is true for $n \leqslant 0$, and, taking the upper bound, we have

$$W^*(\theta) \leqslant W(\theta) + BW_1(\theta),$$

$$\int W^{*r}(\theta) \leqslant B \int W^r(\theta)\,d\theta + B \int W_1{}^r(\theta)\,d\theta \leqslant B \int W^r(\theta)\,d\theta + B \int |f|^r\,d\theta,$$

the desired result.

9. We prove next a lemma which is crucial for Theorem 9, and likely to be useful in similar investigations.

LEMMA 3. *Let $q \geqslant 2$ be an integer, $N_1, N_2, \ldots, N_q$ positive or negative integers (not zero), $\theta_1, \theta_2, \ldots, \theta_q$ real numbers. Then, if $N_{rs} = \min(|N_r|, |N_s|)$, and N is an arbitrary integer,*

$$(1) \quad \left| \int_{-\pi}^{\pi} e^{iNt} \prod_{r=1}^{q} \left(\frac{1-e^{iN_r(\theta_r - t)}}{1-e^{i(\theta_r - t)}} \right) dt \right| \leqslant A(q) \sum_{r=1}^{q} \prod_{s \neq r} \left(\frac{1}{|\sin \tfrac{1}{2}(\theta_s - \theta_r)| + N_{rs}^{-2}} \right) \dagger.$$

There are difficulties when some of the θ's are nearly equal; we meet them by showing that we can change every θ_r into a θ_r' differing from θ_r by $O(N_r^{-2})$, in such a way that the θ''s are not too nearly equal. Such a change from θ's to θ''s leaves either side of (1) practically unaffected.

We may suppose, without loss of generality, that $|N_1|$ is the greatest (or one of the greatest) $|N_r|$. Let $b = 3q(q-1)$. We shall show that, for a fixed set of θ's, there exists a set of integers $a_1, a_2, \ldots, a_q$ satisfying

$$(2) \qquad a_1 = 0, \quad 0 \leqslant a_r \leqslant b \quad (1 < r \leqslant q),$$

such that, if

$$\theta_r' = \theta_r + a_r N_r^{-2} \quad (1 \leqslant r \leqslant q),$$

then

$$(3) \qquad |\sin \tfrac{1}{2}(\theta_s' - \theta_r')| \geqslant \sin(\tfrac{1}{2} N_{rs}^{-2})$$

for every pair (r, s) with $r \neq s$.

There are $(b+1)^{q-1}$ sets of a's satisfying (2). We inquire for how many of these it can happen that, for a particular pair of different suffixes r, s,

$$(4) \qquad |\sin \tfrac{1}{2}(\theta_s' - \theta_r')| \leqslant \sin(\tfrac{1}{2} N_{rs}^{-2}).$$

Of r and s, select the one which has the smaller $|N|$ in case of inequality (of $|N_r|$ and $|N_s|$), and one which is not 1 in the case of equality. If r is so

† Note that the right-hand side is independent of N, and that it depends on the θ's only through their differences. The result is probably true if N_{rs}^{-2} is replaced by N_{rs}^{-1}, but this is curiously awkward to prove, and is no more effective in applications.

selected, then both r is not 1 and $N_{rs} \geqslant |N_r|$; and a little consideration shows that then, *for given* a_s, (4) *can happen for at most* 3 *out of the* b *possible values of* a_r. *Similarly, if* s *is selected, then, for a given* a_r, (4) *can happen for at most* 3 *of the* b *possible values of* a_s. In any case, at most $3(b+1)^{q-2}$ sets of a's satisfying (2) can have a particular pair of suffixes (r, s) giving (4). Hence at most $\frac{1}{2}q(q-1).3(b+1)^{q-2}$ sets can have *some* such pair (r, s). Since this number is less than the total number b^{q-1} of sets, sets exist which have no such pair, and for which accordingly (3) is true for every pair, the desired result.

We now proceed by induction. The result (of Lemma 3) is easily verified for $q = 2$; we suppose it true for 2, 3, ..., $q-1$, and consider it for q. We call the right-hand side of (1) $K(\theta_1, \theta_2, \ldots, \theta_q;\ N_1, N_2, \ldots, N_q)$, or $K(\theta_1, \theta_2, \ldots, \theta_q)$, or $K_q(\theta)$. We suppose, as we may, that $|N_1|$ is the greatest (or one of the greatest) $|N_r|$, and we write

$$P_r(\theta_r - t) = \frac{1-e^{iN_r(\theta_r-t)}}{1-e^{i(\theta_r-t)}}.$$

Let (θ_r') be the set constructed above, and let

$$D_r(t) = P_r(\theta_r - t) - P_r(\theta_r' - t).$$

Then

$$\prod_{r=1}^{q} P_r(\theta_r - t) - \prod_{r=1}^{q} P_r(\theta_r' - t) = \Pi\left(P_r(\theta_r' - t) + D_r(t)\right) - \Pi P_r(\theta_r' - t)$$

$$= \sum_r D_r(t) \prod_{m \neq r} P_m(\theta_m' - t) + \sum_{r \neq s} D_r(t) D_s(t) \prod_{m \neq r,\, s} P_m(\theta_m' - t) + \ldots.$$

Now $D_r(t)$ is the sum of at most $|N_r|$ terms of type

$$e^{iMt}(e^{-iM\theta_r} - e^{-iM\theta_r'})$$

with $|M| \leqslant |N_r|$, and

$$\int e^{iNt} D_r(t) \prod_{m \neq r} P_m(\theta_m' - t)\, dt$$

is the sum of at most $|N_r|$ terms of type

$$T_{M,r} = (e^{-iM\theta_r} - e^{-iM\theta_r'}) \int e^{i(N+M)t} \prod_{m \neq r} P_m(\theta_m' - t)\, dt.$$

By the hypothesis of our induction, the integral is bounded by

$$K(\theta_1', \ldots, \theta_{r-1}', \theta_{r+1}', \ldots, \theta_q'),$$

and *a fortiori* by $2K_q(\theta')$. Since

$$|e^{-iM\theta_r}-e^{-iM\theta_r'}| \leqslant A|M(\theta_r-\theta_r')| \leqslant BN_r^{-1},$$

we have

$$\sum_M |T_{M,r}| \leqslant BN_rN_r^{-1}K_q(\theta') = BK_q(\theta'),$$

$$\sum_r \sum_M |T_{M,r}| \leqslant BK_q(\theta'),$$

$$\sum_r \left| \int e^{iNt} D_r(t) \prod_{m\neq r} P_m(\theta_m'-t)\,dt \right| \leqslant BK_q(\theta').$$

Next, $D_r(t)D_s(t)$ is of type

$$\Sigma C_M e^{iMt} \Sigma C_{M'} e^{iM't}$$

with at most $|N_r|$ terms in the first factor and $|N_s|$ in the second, and with

$$|C_M| \leqslant B|N_r|^{-1}, \quad |C_{M'}| \leqslant B|N_s|^{-1}.$$

Hence, arguing as above,

$$\left| \int e^{iNt} D_r(t) D_s(t) \prod_{m\neq r,\,s} P_m(\theta_m'-t)\,dt \right| \leqslant \left| \sum_M \sum_{M'} C_M C_{M'} \int e^{i(N+M+M')t} \prod_{m\neq r,\,s} dt \right|$$

$$\leqslant B|N_r||N_s||N_r|^{-1}|N_s|^{-1}K_{q-2}(\theta')$$

$$\leqslant BK_q(\theta'),$$

(where the K_{q-2} has θ_r' and θ_s' missing).

The sum with respect to r, s satisfies a similar inequality, and similarly for the products of D's 3, 4, ... at a time. We have, finally,

$$\left| \int e^{iNt} \prod P_r(\theta_r-t)\,dt - \int e^{iNt} \prod P_r(\theta_r'-t)\,dt \right| \leqslant BK_q(\theta'). \tag{5}$$

Now

$$\prod_{r=1}^{q} \left(\frac{1}{1-e^{i(\theta_r'-t)}} \right) = \sum_{r=1}^{q} C_r \frac{1}{1-e^{i(\theta_r'-t)}},$$

where

$$C_r = \prod_{s\neq r} \left(\frac{1}{1-e^{i(\theta_s'-\theta_r')}} \right),$$

and $e^{iNt}\prod P_r(\theta_r'-t)$ is of the form

$$\sum_r C_r \frac{1-e^{iN_r(\theta_r'-t)}}{1-e^{i(\theta_r'-t)}} \sum_M \pm e^{i\alpha_M - iMt},$$

where the inner sum represents $e^{iNt}\prod_{s\neq r}(1-e^{iN_s(\theta_s'-t)})$ and contains at most B terms. Now

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1-e^{iN_r x}}{1-e^{ix}}\,e^{-iMx}\,dx$$

is either 0 or 1. Hence

$$(6)\qquad \left|\int e^{iNt}\prod P_r(\theta_r'-t)\,dt\right| \leqslant B\sum_{r=1}^{q}|C_r| \leqslant B\sum_r\prod_{s\neq r}\left(\frac{1}{|\sin\tfrac{1}{2}(\theta_s'-\theta_r')|}\right).$$

Now

$$|\sin\tfrac{1}{2}(\theta_s-\theta_r)|+N_{rs}^{-2} \leqslant |\sin\tfrac{1}{2}(\theta_s'-\theta_r')|+A\,|N_r|^{-2}+A\,|N_s|^{-2}+N_{rs}^{-2}$$
$$\leqslant |\sin\tfrac{1}{2}(\theta_s'-\theta_r')|+AN_{rs}^{-2} \leqslant B\,|\sin\tfrac{1}{2}(\theta_s'-\theta_r')|,$$

by (3). Consequently the right-hand side of (6), and *a fortiori* $K_q(\theta')$, are less than $BK_q(\theta)$. From (5) and (6) we have

$$\left|\int e^{iNt}\prod P_r(\theta_r-t)\,dt\right| \leqslant BK_q(\theta),$$

and this establishes the truth of the lemma.

10. Consider now part (i) of Theorem 9. By Lemma 2, it is enough to prove the first inequality, and to this end it is sufficient to prove that the inequality

$$(1)\qquad \int|S_{n_\theta}(\theta)|^p\big(\log(|n_\theta|+2)\big)^{-1}d\theta \leqslant B\int|f|^p\,d\theta$$

holds for every $N\geqslant 2$ and every measurable function n_θ with integral values satisfying $2\leqslant n_\theta\leqslant N$. [For, since $|S_0(\theta)|$ and $|S_1(\theta)|$ do not exceed $2M_1(f)\leqslant 2M_p(f)$, (1) will then be true (with perhaps a new B) when n_θ is restricted by $0\leqslant n_\theta\leqslant N$; by applying now this result to $f(-\theta)$ in place of $f(\theta)$, it will be true under the restriction $-N\leqslant n_\theta\leqslant 0$, and so, by combination, true, with changed B, under the restriction $-N\leqslant n_\theta\leqslant N$; and if finally $T_N{}^p(\theta)$ is the upper bound of the integrand on the left-hand side of (1) for functions n_θ thus restricted, $T_N(\theta)$ increases with N to the limit function $T(\theta)$.]

Let ν_θ be the integer such that

$$(2)\qquad 2^{2^{\nu_\theta}}\leqslant n_\theta < 2^{2^{\nu_\theta+1}},$$

and let us write, for brevity, μ_θ for 2^{ν_θ}. By Theorem 8†, S_{μ_n} is majorized by a function $S(\theta)$ with

$$\int |S|^p \, d\theta \leqslant B \int |f|^p \, d\theta.$$

It is therefore enough to prove (1) with $S_{n_\theta} - S_{\mu_\theta}$ in place of S_{n_θ}. Hence, by (B), it is enough to prove that

$$\int (S_{n_\theta} - S_{\mu_\theta}) \Big(\log (|n_\theta|+2) \Big)^{-1/p} g(\theta) \, d\theta \leqslant B \, M_p(f) \, M_{p'}(g) \tag{3}$$

for every $N \geqslant 2$, every function n_θ restricted by $2 \leqslant n_\theta \leqslant N$, and every g of $L^{p'}$. Finally it is enough to prove this for the special set of p's for which p' is an even integer $q = 2k$. For then (1) holds for this set of p's. But, for a fixed function n_θ, $S_{n_\theta}(\theta)$ is obtained from $f(\theta)$ by a linear operation; it is permissible to apply M. Riesz's convexity (or "interpolation") theorem, the weighted integral operation in the transformed domain being with respect to $\Big(\log (|n_\theta|+2) \Big)^{-1} d\theta$; and the inequality (1) (with a suitable value of B) is true for $p_1 \leqslant p \leqslant p_2$ if it is true for $p = p_1$ and $p = p_2$.

Since $\log (|n_\theta|+2) \geqslant 2^{\nu_\theta}$, it is enough, then, to prove that for $p' = q = 2k$ we have, for all $N \geqslant 2$, all functions n_θ satisfying $2 \leqslant n_\theta \leqslant N$, all f of L^p, and all g of L^q, the inequality

$$\left| \int (S_{n_\theta} - S_{\mu_\theta}) \, 2^{-\nu_\theta/p} \, g(\theta) \, d\theta \right| \leqslant B M_p(f) \, M_q(g).$$

Now the left-hand side is

$$\left| \int_{-\pi}^{\pi} 2^{-\nu_\theta/p} \, g(\theta) \, d\theta \int_{-\pi}^{\pi} f(t) \sum_{m=\mu_\theta+1}^{n_\theta} e^{im(\theta-t)} \, dt \right| = \left| \int f(t) \, G(t) \, dt \right| \leqslant M_p(f) \, M_q(G),$$

where

$$G(t) = \int 2^{-\nu_\theta/p} \, g(\theta) \sum_{\mu_\theta+1}^{n_\theta} e^{im(\theta-t)} \, d\theta \tag{4}$$

$$= \int 2^{-\nu_\theta/p} \, g(\theta) \, \frac{1 - e^{i(n_\theta - \mu_\theta)(\theta-t)}}{1 - e^{i(\theta-t)}} \, e^{i(\mu_\theta+1)(\theta-t)} \, d\theta,$$

and it is enough to prove that

$$\int |G(t)|^q \, dt \leqslant B \int |g(t)|^q \, dt. \tag{5}$$

† μ_θ being a power of 2.

Let E_ν be the set of θ for which $\nu_\theta = \nu$. Then†

$$G(t) = \sum_\nu 2^{-\nu/p} G_\nu(t),$$

where
$$G_\nu(t) = \int_{E_\nu} g(\theta) \frac{1-e^{i(n_\theta-\mu)(\theta-t)}}{1-e^{i(\theta-t)}} e^{i(\mu+1)(\theta-t)} d\theta$$

and $\mu = 2^{2^\nu}$. Now

$$|G(t)|^q = \Big(G(t)\Big)^k \Big(\bar{G}(t)\Big)^k = \Big(\sum 2^{-\nu/p} G_\nu(t)\Big)^k \Big(\sum 2^{-\nu/p} \bar{G}_\nu(t)\Big)^k$$

$$= \sum_{\nu_1, \ldots, \nu_{2k}} 2^{-(\nu_1+\ldots+\nu_{2k})/p} \prod_{r=1}^{k} \int_{E_{\nu_r}} g(\theta_r) \frac{1-e^{i(n_{\theta_r}-\mu_r)(\theta_r-t)}}{1-e^{i(\theta_r-t)}} e^{i(\mu_r+1)(\theta_r-t)} d\theta_r$$

$$\times \prod_{r=1}^{k} \int_{E_{\nu_{k+r}}} \bar{g}(\theta_{k+r}) \frac{1-e^{-i(n_{\theta_{k+r}}-\mu_{k+r})(\theta_{k+r}-t)}}{1-e^{-i(\theta_{k+r}-t)}} e^{-i(\mu_{k+r}+1)(\theta_{k+r}-t)} d\theta_{k+r},$$

the sum being taken over 0 to ∞ for each ν. Hence

$$(6) \quad \int |G(t)|^q dt = \sum_{(\nu)} 2^{-\Sigma\nu/p} \int dt \int_{E_{\nu_1}} d\theta_1 \int_{E_{\nu_2}} d\theta_2 \ldots \int_{E_{2k}} d\theta_{2k}$$

$$\times g(\theta_1) \ldots g(\theta_k) \bar{g}(\theta_{k+1}) \ldots \bar{g}(\theta_{2k}) X,$$

where
$$X = \left(\prod_{r=1}^{2k} \frac{1-e^{iN_r(\theta_r-t)}}{1-e^{i(\theta_r-t)}}\right) e^{iNt+i\Theta},$$

$$N_r = n_{\theta_r} - \mu_r \ (1 \leqslant r \leqslant k), \quad N_r = -(n_{\theta_r} - \mu_r) \ (k+1 \leqslant r \leqslant 2k),$$

and the precise values of N and Θ need not concern us.

We now employ Lemma 3 to give an upper bound for $\int X\,dt$. Since $|N_r| \leqslant n_{\theta_r} < \mu_r{}^2$‡, we have $N_{rs} \leqslant \mu_{rs}^2$, where $\mu_{rs} = \min(\mu_r, \mu_s)$. Hence

$$(7) \qquad \left|\int X\,dt\right| \leqslant B \sum_{r=1}^{2k} U_r,$$

where
$$U_r = U_r(\theta_1, \theta_2, \ldots, \theta_{2k}) = \prod_{s \neq r} \frac{1}{|\sin \frac{1}{2}(\theta_s - \theta_r)| + \mu_{rs}^{-4}}.$$

In the inequality (6) we substitute from (7) and from

$$|g(\theta_1) \ldots g(\theta_k) \bar{g}(\theta_{k+1}) \ldots \bar{g}(\theta_{2k})| \leqslant \sum_{m=1}^{2k} |g(\theta_m)|^{2k},$$

† G_ν is identically zero, of course, beyond some value $A \log\log(N+1)$ of ν.

‡ On account of (2).

and we obtain

$$\int |G(t)|^q dt \leqslant B \sum_{\nu_1, \ldots, \nu_{2k}} 2^{-\Sigma\nu/p} \sum_{m=1}^{2k} \sum_{r=1}^{2k} J_{mr},$$

where

$$J_{mr} = \int_{E_{\nu_1}} \ldots \int_{E_{\nu_{2k}}} |g(\theta_m)|^{2k} U_r d\theta_1 \ldots d\theta_{2k}. \tag{8}$$

We must separate the cases $r \neq m$, $r = m$. When $r \neq m$ we have

$$U_r = \frac{1}{|\sin \tfrac{1}{2}(\theta_m - \theta_r)| + \mu_{rm}^{-4}} \prod_{s \neq r, m} \frac{1}{|\sin \tfrac{1}{2}(\theta_s - \theta_r)| + \mu_{rs}^{-4}}. \tag{9}$$

In the multiple integration in (8) we perform first the integrations with respect to the θ_s with $s \neq r, m$, then that with respect to θ_r, and finally that with respect to θ_m. The first (multiple) integration bears only on the second factor in (9), and turns that into something not greater than

$$\prod_{s \neq r, m} I_{sr} = \prod_{s \neq r, m} \int_{-\pi}^{\pi} \frac{d\theta_s}{|\sin \tfrac{1}{2}(\theta_s - \theta_r)| + \mu_{rs}^{-4}} \leqslant B \prod_{s \neq r, m} \log(\mu_{rs} + 1) \leqslant B \prod_{s \neq r, m} 2^{\nu_{rs}}.$$

Hence

$$J_{mr} \leqslant \int_{E_{\nu_m}} d\theta_m |g(\theta_m)|^{2k} B \prod_{s \neq r, m} 2^{\nu_{rs}} \int_{-\pi}^{\pi} \frac{d\theta_r}{|\sin \tfrac{1}{2}(\theta_m - \theta_r)| + \mu_{rm}^{-4}};$$

and so, the inner integral being $I_{rm} \leqslant B \log(\mu_{rm} + 1) \leqslant B 2^{\nu_{rm}}$,

$$J_{mr} \leqslant B \Big(\prod_{s \neq r} 2^{\nu_{rs}} \Big) \int_{E_{\nu_m}} |g(\theta)|^{2k} d\theta. \tag{10}$$

When $r = m$,

$$J_{mm} \leqslant \int_{E_{\nu_m}} d\theta_m |g(\theta_m)|^{2k} \Big(\prod_{s \neq m} \int_{-\pi}^{\pi} d\theta_s \Big) \prod_{s \neq m} \frac{1}{|\sin \tfrac{1}{2}(\theta_s - \theta_m)| + \mu_{ms}^{-4}}.$$

The inner multiple integral is bounded by

$$\prod_{s \neq m} I_{sm} \leqslant B \prod_{s \neq m} 2^{\nu_{ms}},$$

and we have

$$J_{mm} \leqslant B \Big(\prod_{s \neq m} 2^{\nu_{ms}} \Big) \int_{E_{\nu_m}} |g(\theta)|^{2k} d\theta;$$

that is to say, (10) is true for $r = m$ as well as for $r \neq m$.

Summing up, we have

$$(11) \qquad \int |G(t)|^q\,dt \leqslant B \sum_{\nu_1,\,\ldots,\,\nu_{2k}} \sum_{r=1}^{2k} \sum_{m=1}^{2k} 2^{-\Sigma\nu/p+\sum_{s\neq r}\nu_{rs}} \int_{E_{\nu_{rm}}} |g(\theta)|^{2k}\,d\theta;$$

and in this we may suppose, with a change of B, that the variables ν of the summations are restricted by

$$\nu_1 \geqslant \nu_2 \geqslant \ldots \geqslant \nu_{2k}.$$

Now the exponent of the power of 2 in (11) is

$$-(\nu_1+\nu_2+\ldots+\nu_{2k})(2k-1)/2k+(r-1)\,\nu_r+\nu_{r+1}+\ldots+\nu_{2k}$$

$$\leqslant -\big((\nu_1-\nu_2)+(\nu_1-\nu_3)+\ldots+(\nu_1-\nu_{2k})\big)/2k.$$

This upper bound is the same for every pair r, m. Hence the term

$$\int_{E_\nu} |g(\theta)|^{2k}\,d\theta$$

occurs, for a fixed ν, on the right-hand side of (11) with a total coefficient C_ν not exceeding

$$B \sum_{m=1}^{2k} \Sigma_m{}' 2^{-((\nu_1-\nu_2)+(\nu_1-\nu_3)+\ldots+(\nu_1-\nu_{2k}))/2k},$$

where $\Sigma_m{}'$ denotes a sum over all $\nu_1, \nu_2, \ldots, \nu_{2k}$ for which $\nu_m = \nu$ and

$$\nu_1 \geqslant \nu_2 \geqslant \ldots \geqslant \nu_{2k}.$$

Now

$$\Sigma_1{}' \leqslant \sum_{\nu_2\leqslant\nu} 2^{-(\nu-\nu_2)/2k} \sum_{\nu_3\leqslant\nu} 2^{-(\nu-\nu_3)/2k} \ldots \leqslant B,$$

and, for $m>1$,

$$\Sigma_m{}' \leqslant \sum_{\nu_1\geqslant\nu} 2^{-(\nu_1-\nu)/2k} \sum_{\nu_2\leqslant\nu_1} 2^{-(\nu_1-\nu_2)/2k} \sum_{\nu_3\leqslant\nu_1} 2^{-(\nu_1-\nu_3)/2k} \ldots \leqslant B.$$

Hence $C_\nu \leqslant B$ for all ν; and we have, finally,

$$\int |G(t)|^q\,dt \leqslant B \sum_\nu \int_{E_\nu} |g(\theta)|^{2k}\,d\theta = B \int_{-\pi}^{\pi} |g(\theta)|^{2k}\,d\theta.$$

Part (i) of Theorem 9 is now established. In particular, $\dot{T}^*(\theta)$ is finite p.p., and the Fourier series is bounded p.p. We now apply Lemma 1: the Fourier series of its function f^* being bounded p.p., that of f is convergent p.p.; this is the result of part (ii).

11. We have, finally, to consider Theorem 10. This is of rather minor importance, but should be included for completeness. The special case $q = 2$ is identical with the special case $p = 2$ of Theorem 9, part (i). It is now true but trivial that $\Sigma l_m c_m e^{m\theta i}$ converges p.p., for this asserts the less the larger q; Theorem 10 asserts in compensation, however, that the function majorising the partial sums belongs to a higher L-class.

By Lemma 2, it is enough to prove the result for Q. That result is equivalent to the truth of the inequality

$$(1) \qquad \left(\int \left| \frac{S_{n_\theta}(\theta)}{\log(|n_\theta|+2)} \right|^q \log(|n_\theta|+2)\, d\theta\right)^{1/q} \leqslant A(q)\left(\int |f|^q\, d\theta\right)^{1/q},$$

for some $A(q)$ and every measurable function n_θ whose values are (positive, negative, or zero) integers. Now for a fixed function n_θ, $S_{n_\theta}(\theta)/\log(|n_\theta|+2)$ is obtained from $f(\theta)$ by a linear operation, M. Riesz's convexity theorem is available for "interpolation" [the weighted integral operation in the transformed field being with respect to $(\log|n_\theta|+2)\,d\theta$], and (for the fixed n_θ) the inequality is true for $2 \leqslant q \leqslant \infty$ if it is true when $q = 2$ and when $q = \infty$. In the latter case the inequality is to be interpreted as

$$(2) \qquad \max_{(\theta)} \left| \frac{S_{n_\theta}(\theta)}{\log(|n_\theta|+2)} \right| \leqslant B \max_{(\theta)} |f|$$

(the upper bound on the right-hand side being an "effective" one). But†

$$|S_n(\theta)| \leqslant B \max|f| \log(|n|+2)$$

for all θ and n, and (2) is accordingly true.

The result for $q = 2$ is, as has been explained above, identical with the corresponding result in Theorem 9 for $p = 2$‡.

We conclude by two slight corrections of arguments in (II).

(i) § 10 (p. 73). By a momentary oversight we asserted that the theorem was false for $q = \infty$. Actually it is true, the interpretation being

$$\text{"}(1-\rho)|\phi'| < A \text{ if } |f| \leqslant 1\text{"}.$$

It is therefore possible to interpolate between $q = 2$ and $q = \infty$ without

† With s_n in place of S_n this is well known. For S_n we have

$$|S_n(\theta)| = \left| \frac{1}{2\pi} \int f(\theta+t) \sum_0^n e^{-mti}\, dt \right| \leqslant A \max|f| \int \left| \sum_0^n e^{-mti} \right| dt \leqslant A \max|f| \log(|n|+2).$$

‡ The proof is naturally very much easier in this special case.

having to modify the form of the theorem. (The case $q = 2$ is duly covered by the argument of the text.)

(ii) p. 63. In line 8 substitute $\Phi^2|\phi_1'\phi_\nu'|$ for $|\phi_1\phi_\nu'|$, and in line 12 substitute $g^{2k-4}\Phi^2|\phi_1'\phi_\nu'|$ for $|\phi_1\phi_\nu'|g^{2k-2}$.

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THE GEOMETRICAL INVARIANTS OF ALGEBRAIC LOCI

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Until quite recently the theory of the invariants of an algebraic variety V_d of d dimensions has centred almost entirely round the canonical system of V_{d-1}'s lying on it. The definition of this system follows closely that of the canonical system of curves on a surface, and dates back to Noether†. If $|C|$ is a linear system of ∞^d V_{d-1}'s on V_d, and if C_J is its Jacobian V_{d-1}, then the system $|C_J-(d+1)C|$ is independent of $|C|$ and is the canonical system in question. An alternative expression for the system is $|C'-C|$, where $|C'|$ is the linear system adjoint to $|C|$, cutting each C in a V_{d-2} belonging to the canonical system on C.

It is only within the last few years that any attention has been drawn to other geometrical invariants of algebraic varieties. In 1932 Severi‡ introduced an invariant series of sets of points on an algebraic surface, and still more recently Segre§ has considered a similar series on a V_3, and an invariant system of curves on V_3.

We show here how to define on a given V_d an invariant system of varieties of each dimension from 0 up to $d-1$; the system of dimension $d-1$ being the canonical system already mentioned, and the systems of sets of points and curves being (for $d=2$, 3) the invariant systems of Severi and Segre. We shall refer to these invariant systems as the *canonical systems* of appropriate dimension, and shall denote the canonical system of dimension k by the symbol $\{X_k(V_d)\}$.

† For references see the article in *Encyk. Mat. Wiss.*, III C 6b, § 47, or the later reports by Lefschetz, *Mémorial des Sciences Mathématiques*, 40 (1929), and Rosenblatt, *Atti del Congresso Internationale dei Matematici* (Bologna, 1928), 4, 93.

‡ Severi, *Commentarii Matematici Helvetici*, 4 (1932), 268.

§ B. Segre, *Mem. R. Accademia d'Italia*, 5 (1934), 479.

In order to simplify the formal work somewhat, we shall attach the following conventional meaning to the symbol $X_k(V_d)$ when $k \geqslant d$. The symbol $X_d(V_d)$ shall mean V_d itself; the symbol $X_k(V_d)$, for $k > d$, shall represent zero.

In the case of the canonical system $\{X_{d-1}(V_d)\}$ it is well known that the arithmetical invariants of a general member of the system give invariants of V_d. For the canonical systems of lower dimensions, such invariants do not necessarily arise. For example, on a V_3 the virtual genus of a curve is not necessarily constant as the curve moves in a system of equivalence, as is easily seen by taking V_3 to be a linear space, in which the aggregate of curves of given order forms a single connected system of equivalence†. We do, however, obtain invariants by considering the virtual number of intersections of a set of canonical varieties of such number and dimension that their virtual intersection consists of a finite set of points. In this way the results obtained here lead to significant results in the theory of the *arithmetical* invariants of V_d, which it is hoped to consider in a subsequent paper.

The canonical system of dimension zero on V_d is easily defined in terms of a pencil $|C|$ of V_{d-1}'s with an irreducible base V_{d-2}, (C^2). In fact, if δ denotes the set of double points of the pencil, and $X_0(C)$, $X_0(C^2)$ denote canonical sets on C and (C^2) respectively, then we show in § 2 below that the series of equivalence (possibly virtual) defined by the set

$$\delta - 2X_0(C) - X_0(C^2)$$

is independent of $|C|$; this is the system $\{X_0(V_d)\}$.

To define the canonical varieties of higher dimension we introduce the idea of the *adjoints of various dimensions* of a linear system $|C|$ of V_{d-1}'s on V_d, supposing (in the first place) that $|C|$ is a sufficiently ample linear system and is free from base-points. The definition of these systems is similar to that of the adjoint system of V_{d-1}'s; the adjoint system of dimension k being the set of V_k's on V_d which cut out on a general C a V_{k-1} belonging to $\{X_{k-1}(C)\}$. There is no difficulty in defining (possibly virtual) varieties with this property, and we assume that the aggregate of adjoint V_k's to $|C|$ forms on V_d a system of equivalence $\{A_k(C)\}$. It then appears that the (possibly virtual) system of equivalence $\{A_k(C) - X_k(C)\}$ is independent of $|C|$; this is the canonical system $\{X_k(V_d)\}$ of dimension k on V_d, and may be used to define the system adjoint to V_{d-1}'s on V_d which are not members of sufficiently general linear systems for the preceding definition to apply.

† See, *e.g.*, Severi, *Mem. R. Accademia d'Italia*, 5 (1934), 242.

It may be remarked that, as defined, these systems are *a priori* virtual, but there certainly exist V_d's on which they may be realized effectively. Further, their invariance under birational transformation of the V_d is relative, in general, since the introduction of fundamental varieties of various dimensions in the transformation will in general affect the canonical systems.

1. *Equivalence of V_k's on a V_d.*

We recall the notion of *equivalence* between V_k's on V_d, a concept introduced by Severi†. The ideas underlying the present formulation are due to the author‡. We consider on V_d $d-k$ linear systems $|C_1|, |C_2|, \ldots, |C_{d-k}|$ of V_{d-1}'s and consider the aggregate of V_k's obtained as the complete intersection of $d-k$ V_{d-1}'s, chosen one from each of the $d-k$ systems. This aggregate of V_k's we call a *system of intersection*. Consider now the aggregate of cycles of $2k$ dimensions on the Riemannian R_{2d} of V_d which represent virtual algebraic V_k's on V_d, *i.e.* which are homologous on R_{2d} to the difference of two cycles representing effective V_k's. We call these the *algebraic* $2k$-cycles of R_{2d}. They form an abelian group with addition as the law of composition; let this group be denoted by $\mathfrak{G}$. We define a subgroup $\mathfrak{H}$ of $\mathfrak{G}$ as follows: $\mathfrak{H}$ is generated by all the cycles of the form $\Gamma'-\Gamma''$, where Γ', Γ'' are cycles representing two effective V_k's, A', A'', with the property that for some effective V_k, A say, $A+A'$ and $A+A''$ belong to the same system of intersection. Two V_k's A, B will be called *equivalent* if $A-B$ is represented by a cycle of $\mathfrak{H}$, *i.e.* if the cycles representing A and B lie in the same coset of $\mathfrak{G}$ with respect to $\mathfrak{H}$. When this happens we write $A \equiv B$. The relation of equivalence is symmetrical, reflexive, and transitive, and it is invariant under the operation of group-addition, *i.e.* from $A \equiv B$ and $C \equiv D$ follows $A+C \equiv B+D$. Further, if two V_k's A, B are equivalent on V_d, and a V_{d-l} meets them in V_{k-l}'s A_1, B_1, then $A_1 \equiv B_1$ on V_{d-l}.

A *system of equivalence* is essentially a connected algebraic system of mutually equivalent effective V_k. In the present work we stress the notion of equivalence rather than that of systems of equivalence, since the former is better adapted to deal with varieties which may in general be virtual.

It is of importance to have a criterion for deciding whether two varieties A and B on a V_d are equivalent or not. If A and B are of dimension $d-1$, then it is known that a sufficient condition for the equivalence of A and B

† Severi, *Mem. R. Accademia d'Italia*, 4 (1933), 71.

‡ Todd, *Annals of Mathematics* (2), 35 (1934), 702.

is that they should cut equivalent V_{d-2}'s on the V_{d-1}'s of a sufficiently general linear pencil. It is natural to seek an extension of this result to the case in which A and B are of dimension smaller than $d-1$. The result that suggests itself is that, if A and B are of k dimensions, they are equivalent if they meet the V_{d-1}'s of a sufficiently general linear system $|S|$, free from fundamental varieties, in equivalent V_{k-1}'s. In this paper we shall assume that this is so. That the hypothesis made is probable is suggested from the considerations which follow, though these scarcely amount to a proof. The justification for making the assumption lies in the consequences that follow from it. Algebraic geometry, especially when concerned with loci of three or more dimensions, is still an experimental science, and the results obtained here, by using this hypothesis, seem to be very suggestive.

In order to examine more clearly the nature of the assumption made, we consider in a little more detail the case in which A and B are curves on a V_3; the general case presents no serious new complication. Let us suppose, in the first instance, that A and B cut out on a general surface S_0 of $|S|$ sets of points A_1 and B_1 which belong to a series of intersection. This means that we can find on S_0 two pencils of curves $|f|$, $|g|$, with the property that the complete intersection of one pair of curves f_1, g_1 of the pencils is the set A_1, while the complete intersection of another pair f_2, g_2 is the set B_1. It may be possible to select such a pair of pencils $|f|$ and $|g|$ in more than one way. Suppose that $|f'|$, $|g'|$ are another pair of pencils with the same property. Then, if the curves of $|f'|$ are members of the same complete continuous system as the curves of $|f|$, they must belong to the same linear system. For f_1 and f_1', being algebraically equivalent, cut any g in the same number of points, hence the set A_1 is the complete intersection of f_1' and g_1, and it then follows from the criterion of linear equivalence for curves on a surface that f_1 and f_1' are linearly equivalent. It may happen, however, that $|f'|$ and $|g'|$ consist of curves respectively of the same orders as, but not equivalent to, the curves $|f|$ and $|g|$, *i.e.* that the surface possesses several descriptively similar linear systems which are non-equivalent (*e.g.* the cubic surface, with twenty-seven pencils of conics). It seems likely, however, that this can happen only for a limited type of surface on V_3, and that, if $|S|$ is sufficiently general, this possibility cannot arise.

This being the case, we can now isolate the complete linear systems on S_0 to which $|f|$ and $|g|$ belong and determine *uniquely* on the other surfaces of the pencil the corresponding linear systems. Then, by assigning fixed base points to these systems, lying on the base curve of the pencil, we can reduce the freedom of the systems passing through the sets A_1 and B_1 to zero. If now S_0 is allowed to vary in the pencil, the curves f_1 and g_1 describe

surfaces F_1, G_1 which meet in the curve A, and f_2, g_2 describe surfaces F_2, G_2 meeting in the curve B, these being the complete intersections of the respective pairs of surfaces. Since the surfaces F_1, F_2 are clearly equivalent on V_3, as are G_1, G_2, it follows that the curves A and B belong to a system of intersection.

The extension to the case when the sets A_1 and B_1 do not belong to a series of intersection is simple, since in the most general case in which A_1 and B_1 are equivalent there exists a set D_1 such that A_1+D_1, B_1+D_1 belong to the sum or difference of a finite number of series of intersection. These being, as we have seen, determinable rationally on each S if $|S|$ is sufficiently general, it follows by an obvious process that A and B are such that there exists a curve D such that $A+D$ and $B+D$ belong to the sum or difference of a finite number of systems of intersection on V_3, and are hence equivalent.

This outline may serve to explain how the hypothesis mentioned above is suggested, and may form the lines on which a complete proof can be based.

2. *The canonical series $\{X_0(V_d)\}$ of sets of points on V_d.*

We shall now prove, by induction on d, that if δ denotes the set of double points of the members of a pencil $|C|$ of V_{d-1}'s, with irreducible base (C^2), on V_d (assumed to be free from singularities), and if $X_0(C)$, $X_0(C^2)$ are canonical sets of C and (C^2), then the series of equivalence defined by

$$\delta - 2X_0(C) - X_0(C^2) \qquad (2.1)$$

is independent of $|C|$. The proof is very similar to that of Segre in the case $d = 3$, to which we have already referred, and is essentially the same as the argument establishing the existence of the Zeuthen-Segre invariant I_d of V_d. In fact, the number of points in the set $X_0(V_d)$ is $I_d+(-1)^d \cdot 2d$.

We consider two pencils, $|C|$, $|D|$, of V_{d-1}'s with irreducible base-loci (C^2), (D^2). Let ϕ_1 be the curve of contact of a C and a D, and τ the group of points of stationary contact (lying on ϕ_1). Let δ, δ' be the sets of double points of members of the pencils $|C|$, $|D|$, respectively, and let ρ, ρ' be the sets of double points of the pencils $|C^*|$, $|D^*|$, cut out respectively by $|D|$ on (C^2) and by $|C|$ on (D^2).

Consider a fixed V_{d-1}, C_0, of the pencil $|C|$, and the pencil of V_{d-2}'s cut out on C_0 by $|D|$. Assuming (2.1) to define the canonical series on varieties of dimension less than d, we have

$$X_0(C) \equiv \delta_1 - 2X_0(CD) - X_0(CD^2), \qquad (2.2)$$

where δ_1 is the set of double points of the pencil $|(C_0 D)|$. Again, considering the pencil $|C^*|$ on (C^2), we have

$$X_0(C^2) \equiv \rho - 2X_0(C^2 D) - X_0(C^2 D^2). \tag{2.3}$$

The points δ_1 and ρ make up the intersection of C_0 and ϕ_1; hence

$$(\phi_1 C) \equiv \delta_1 + \rho. \tag{2.4}$$

The set of double points of the linear series cut on ϕ_1 by $|C|$ consists of the set δ of double points of $|C|$, the set ρ of double points of $|(C^2 D)|$, each counted twice since they are base-points of the series in question, the set ρ' of double points of $|(CD^2)|$, and the set τ of stationary points. Hence

$$\delta + 2\rho + \rho' + \tau \equiv 2(\phi_1 C) + X_0(\phi_1). \tag{2.5}$$

From (2.2), (2.3), (2.4), (2.5) we deduce that

$$\delta - 2X_0(C) - X_0(C^2) \equiv X_0(\phi_1) - \tau - \rho - \rho' + 4X_0(CD) + 2X_0(C^2 D) + 2X_0(CD^2) + X_0(C^2 D^2). \tag{2.6}$$

Since the right-hand member of (2.6) is symmetrical in C and D, it follows that

$$\delta - 2X_0(C) - X_0(C^2) \equiv \delta' - 2X_0(D) - X_0(D^2),$$

which establishes the existence of the canonical series $\{X_0(V_d)\}$ on V_d, since (2.1) reduces, when $d = 1$, to the familiar expression for the canonical series on a curve†.

3. *The canonical system $\{X_1(V_3)\}$ of curves on a V_3.*

Having thus established the existence of the system $\{X_0(V_d)\}$ for all values of d, we proceed to consider the canonical curves. The existence of these on a surface is known from quite elementary considerations. We shall establish the existence of a canonical system of curves, first on a V_3, and then on a V_d of any dimension. The proof given here for the V_3 is different from, and somewhat simpler than, that of Segre.

We consider two pencils $|C|$, $|D|$ of surfaces on V_3, and retain the notation of the last section. The equation (2.2) becomes

$$X_0(C) \equiv \delta_1 - 2X_0(CD) - (CD^2), \tag{3.1}$$

† When $d = 2$ the proof requires verbal modification, since the base (C^2) consists of a set of points and is thus "reducible."

since (CD^2) is a set of points, and so $X_0(CD^2) \equiv (CD^2)$. Now $X_0(CD)$ is the canonical series on the curve (CD), and is therefore cut out on (CD) by that curve on the surface D which is adjoint to (CD), namely $(CD)+X_1(D)$, where $X_1(D)$ is a canonical curve of D. Hence

$$X_0(CD) \equiv \Big((CD) . (CD) + X_1(D) \Big)_D \equiv (C^2 D) + \Big(C . X_1(D) \Big). \quad (3.2)$$

In the same way we find that

$$X_0(C^2) \equiv \Big((C^2) . (C^2) + X_1(C) \Big)_C \equiv (C^3) + \Big(C . X_1(C) \Big). \quad (3.3)$$

The equation (2.3) reduces, since $(C^2 D^2) = 0$, to

$$X_0(C^2) \equiv \rho - 2(C^2 D),$$

whence, by (3.3),

$$\rho \equiv \Big(C . (C^2) + X_1(C) + 2(CD) \Big).$$

Hence, from (3.1), (3.2), (2.4),

$$X_0(C) \equiv \Big(C . \phi_1 - (C^2) - 4(CD) - (D^2) - X_1(C) - 2X_1(D) \Big). \quad (3.4)$$

Thus the curves

$$A_1(C) \equiv \phi_1 - (C^2) - 4(CD) - (D^2) - X_1(C) - 2X_1(D) \quad (3.5)$$

cut out on C sets of the canonical series $\{X_0(C)\}$. The system $\{A_1(C)\}$ is called the *adjoint system of curves*† of $|C|$. We see from (3.5) that $A_1(C) - X_1(C)$ is symmetrical in C and D. Hence the system

$$\{X_1(V_3)\} \equiv \{A_1(C) - X_1(C)\}$$

is independent of C. This is the *canonical system of curves* on V_3.

Now let $|S|$ be a net of surfaces on V_3, and let J_1 be its Jacobian curve. On a fixed surface S_0 of $|S|$ the other surfaces cut out a pencil of curves, and the double points of these curves lie at the intersections of S_0 and J_1. Hence

$$\begin{aligned} X_0(S) &\equiv (S . J_1) - 2X_0(S^2) - (S^3) \\ &\equiv \Big(S . J_1 - 2X_1(S) - 3(S^2) \Big) \end{aligned}$$

by virtue of (3.3). Thus

$$J_1 - 2X_1(S) - 3(S^2) \equiv A_1(S) \equiv X_1(S) + X_1(V_3),$$

† The assumption made in §1 is, of course, used here (and in similar places throughout the paper) to deduce the uniqueness of the system $\{A_1(C)\}$.

that is to say

$$X_1(V_3) \equiv J_1 - 3X_1(S) - 3(S^2). \tag{3.6}$$

Since $X_1(S) \equiv (S^2) + (SX)$, where X is a canonical surface of V_3, we deduce from (3.6) the relation

$$X_1(V_3) \equiv J_1 - 3(SX) - 6(S^2)$$

which is used by Segre to define the system $\{X_1(V_3)\}$.

4. *Canonical curves on V_d.*

The extension of the considerations of the preceding section to varieties of any dimension is now quite simple. We assume the existence of the canonical system $\{X_1(V_l)\}$ for $l < d$, and of the adjoint system $\{A_1(V_{l-1})\}$ for a linear system of V_{l-1}'s lying on V_l, and deduce the existence of the system $\{X_1(V_d)\}$.

Considering once again the two pencils $|C|$ and $|D|$ of V_{d-1}'s, we have the relations (2.2), (2.3), (2.4). By the hypothesis of the induction, if α and β are not both zero, we have

$$X_0(C^{1+\alpha} D^\beta) \equiv \left(C^{1+\alpha} D^\beta \,.\, A_1(C^{1+\alpha} D^\beta) \right)_{(C^\alpha D^\beta)},$$

the notation implying that the adjoint curves lie on $(C^\alpha D^\beta)$. Hence

$$X_0(C^{1+\alpha} D^\beta) \equiv \left(C^{1+\alpha} D^\beta \,.\, X_1(C^{1+\alpha} D^\beta) + X_1(C^\alpha D^\beta) \right)_{(C^\alpha D^\beta)}$$

$$\equiv \left(C \,.\, X_1(C^{1+\alpha} D^\beta) + X_1(C^\alpha D^\beta) \right).$$

The equations (2.2), (2.3), (2.4) now give

$$X_0(C) \equiv \left(C \,.\, \phi_1 - X_1(C) - 2X_1(D) - X_1(C^2) - 4X_1(CD) - X_1(D^2) \right.$$
$$\left. - 2X_1(C^2 D) - 2X_1(CD^2) - X_1(C^2 D^2) \right),$$

showing that the system of curves

$$\{A_1(C)\} \equiv \{\phi_1 - X_1(C) - 2X_1(D) - X_1(C^2) - 4X_1(CD) - X_1(D^2)$$
$$- 2X_1(C^2 D) - 2X_1(CD^2) - X_1(C^2 D^2)\}$$

is adjoint to C. Further, the system

$$X_1(V_d) \equiv \{A_1(C) - X_1(C)\},$$

being symmetrical in C and D, is independent of C and defines the canonical

system of curves on V_d. It also follows that

$$\phi_1 \equiv \big(X_1(V_d)+2X_1(C)+X_1(C^2)\big)+2\big(X_1(D)+2X_1(DC)+X_1(DC^2)\big) + \big(X_1(D^2)+2X_1(D^2C)+X_1(D^2C^2)\big). \quad (4.1)$$

It is a simple matter to deduce from this an expression for the Jacobian curve of a net $|S|$ of V_{d-1}'s on V_d. If J_1 is the Jacobian curve in question, a repetition of the argument of the last section shows that

$$X_0(S) \equiv (S.J_1)-2X_0(S^2)-X_0(S^3)$$
$$\equiv \big(S.J_1-2X_1(S)-3X_1(S^2)-X_1(S^3)\big),$$

so that $$A_1(S) \equiv J_1-2X_1(S)-3X_1(S^2)-X_1(S^3).$$

But $A_1(S) \equiv X_1(S)+X_1(V_d)$, and so

$$J_1 \equiv X_1(V_d)+3X_1(S)+3X_1(S^2)+X_1(S^3). \quad (4.2)$$

5. *Canonical V_k's on a V_d.*

The extension of the considerations of the last two sections to prove the existence of an invariant system of V_k's on a V_d for any value of k (less than d) presents no essentially new difficulty. Let us suppose that the canonical system $\{X_{k-1}(V_d)\}$ is defined for all d. On a V_{k+1} the existence of the canonical V_k follows by elementary methods. We assume that the system $\{X_k(V_l)\}$ is defined on varieties of dimension l less than d, and proceed to establish their existence on V_d. As part of the inductive hypothesis we assume the following generalizations of (4.1) and (4.2): (i) that on a V_l (of any dimension l) the Jacobian J_{k-1} of a linear ∞^k system of V_{l-1}'s, $|S|$, is given by

$$J_{k-1} \equiv \sum_{r=0}^{k+1} \binom{k+1}{r} X_{k-1}(S^r), \quad (5.1)\dagger$$

and (ii) that the V_{k-1} of contact of a pencil $|T|$ of V_{l-1}'s and a linear ∞^{k-1} system $|U|$ of V_{l-1}'s is given by

$$\phi_{k-1} \equiv \sum_{r=0}^{k} \binom{k}{r} \big(X_{k-1}(U^r)+2X_{k-1}(U^rT)+X_{k-1}(U^rT^2)\big). \quad (5.2)$$

These reduce respectively to (4.2) and (4.1) when $k=2$.

---

† With the convention that $S^0 = V_l$.

We take on V_d a pencil $|C|$ of V_{d-1}'s and a linear system $|D|$ of freedom k. Let ϕ_k be the locus of contacts of a C and a D. The system $|D|$ cuts out on a fixed C_0 of $|C|$ a linear ∞^k system of V_{d-2}'s whose Jacobian J_{k-1} is, by (5.1), given by

$$J_{k-1} \equiv \sum_{r=0}^{k+1} \binom{k+1}{r} X_{k-1}(D^r C)$$

$$\equiv X_{k-1}(C) + \sum_{r=1}^{k+1} \binom{k+1}{r} X_{k-1}(D^r C). \qquad (5.3)$$

The system $|D|$ cuts on the base (C^2) of $|C|$ a linear ∞^k system of V_{d-3}'s whose Jacobian J'_{k-1} is given by

$$J'_{k-1} \equiv \sum_{r=0}^{k+1} \binom{k+1}{r} X_{k-1}(D^r C^2). \qquad (5.4)$$

It is easily seen that

$$(\phi_k \,.\, C) \equiv J_{k-1} + J'_{k-1}. \qquad (5.5)$$

Now, by the definition of $X_k(V_l)$ for $l < d$,

$$X_{k-1}(CD^r) \equiv \Big((CD^r) \,.\, A_k(CD^r) \Big)_{(D^r)} \equiv \Big((CD^r) \,.\, X_k(CD^r) + X_k(D^r) \Big)_{(D^r)}$$

$$\equiv \Big(C \,.\, X_k(CD^r) + X_k(D^r) \Big),$$

and, in the same way,

$$X_{k-1}(C^2 D^r) \equiv \Big(C \,.\, X_k(C^2 D^r) + X_k(CD^r) \Big).$$

Hence, from (5.3), (5.4), and (5.5),

$$X_{k-1}(C) \equiv \Bigg(C \,.\, \phi_k - \sum_{r=1}^{k+1} \binom{k+1}{r} \Big(X_k(CD^r) + X_k(D^r) \Big)$$

$$- \sum_{r=0}^{k+1} \binom{k+1}{r} \Big(X_k(C^2 D^r) + X_k(CD^r) \Big) \Bigg),$$

and so the system

$$\{A_k(C)\} \equiv \Big\{ \phi_k - X_k(C) - X_k(C^2)$$

$$- \sum_{r=1}^{k+1} \binom{k+1}{r} \Big(X_k(D^r) + 2X_k(D^r C) + X_k(D^r C^2) \Big) \Big\} \qquad (5.6)$$

is the adjoint system of $|C|$ of dimension k.

Consider now the characteristic system cut by $|D|$ on a particular V_{d-1}, D_0, of $|D|$, and the pencil cut by $|C|$ on D_0. The locus of contacts of

these systems is $(\phi_k . D_0)$, so that, from (5.2),

$$(\phi_k . D) \equiv \sum_{r=0}^{k} \binom{k}{r} \left(X_{k-1}(D^{r+1}) + 2X_{k-1}(D^{r+1} C) + X_{k-1}(D^{r+1} C^2) \right)$$

$$\equiv X_{k-1}(D) + 2X_{k-1}(DC) + X_{k-1}(DC^2)$$

$$+ \sum_{r=1}^{k} \binom{k}{r} \left(X_{k-1}(D^{r+1}) + 2X_{k-1}(D^{r+1} C) + X_{k-1}(D^{r+1} C^2) \right).$$

Hence, since

$$X_{k-1}(D^{\alpha+1} C^{\beta}) \equiv \left(D . X_k(D^{\alpha} C^{\beta}) + X_k(D^{\alpha+1} C^{\beta}) \right)$$

if α, β are not both zero, we find that

$$X_{k-1}(D) \equiv \Big(D . \phi_k - 2X_k(C) - X_k(C^2) + X_k(D)$$

$$- \sum_{r=1}^{k+1} \binom{k+1}{r} \left(X_k(D^r) + 2X_k(D^r C) + X_k(D^r C^2) \right) \Big).$$

Hence the system

$$\{A_k(D)\} \equiv \Big\{ \phi_k - 2X_k(C) - X_k(C^2) + X_k(D)$$

$$- \sum_{r=1}^{k+1} \binom{k+1}{r} \left(X_k(D^r) + 2X_k(D^r C) + X_k(D^r C^2) \right) \Big\} \quad (5.7)$$

is adjoint to $|D|$. From (5.6) and (5.7) it follows that

$$A_k(C) - X_k(C) \equiv A_k(D) - X_k(D),$$

which establishes the existence of the invariant system $X_k(V_d)$. Also, from (5.6) or (5.7) we find

$$\phi_k \equiv \sum_{r=0}^{k+1} \binom{k+1}{r} \left(X_k(D^r) + 2X_k(D^r C) + X_k(D^r C^2) \right),$$

which is (5.2) with $k-1$ replaced by k.

To complete the induction it is only necessary to extend (5.1). For this we consider a linear ∞^{k+1} system $|S|$ and its Jacobian J_k. The Jacobian of the characteristic system cut by $|S|$ on a fixed S_0 is $(S_0 J_k)$, so that, by (5.1),

$$(J_k . S) \equiv \sum_{r=0}^{k+1} \binom{k+1}{r} X_{k-1}(S^{r+1}).$$

Now $$X_{k-1}(S^{r+1}) \equiv \left(S . X_k(S^{r+1}) + X_k(S^r) \right) \quad (r > 0),$$

so that

$$(J_k . S) \equiv X_{k-1}(S) + \left(S . \sum_{r=1}^{k+1} \binom{k+1}{r} \left(X_k(S^{r+1}) + X_k(S^r) \right) \right)$$

$$\equiv X_{k-1}(S) + \left(S . \sum_{r=1}^{k+2} \binom{k+2}{r} X_k(S^r) - X_k(S) \right).$$

Hence
$$A_k(S) \equiv J_k + X_k(S) - \sum_{r=1}^{k+2} \binom{k+2}{r} X_k(S^r)$$

and
$$J_k \equiv X_k(V_d) + \sum_{r=1}^{k+2} \binom{k+2}{r} X_k(S^r)$$

$$\equiv \sum_{r=0}^{k+2} \binom{k+2}{r} X_k(S^r).$$

This is (5 . 1) with k instead of $k-1$. The induction is thus complete, and the canonical system $\{X_k(V_d)\}$ is defined for all values of k and d.

[*Note*—Added 25 February, 1937.] Since the above paper was written two notes on the same subject have appeared by M. Eger [*Comptes rendus*, 204 (1937), 92, 217]. In the second of these papers the canonical system of dimension k on a V_d is defined in terms of the Jacobian of $k+1$ pencils of V_{d-1}'s, and a formula equivalent to our (5 . 1) is obtained (equation γ of Eger's note).

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NOTE ON THE CANONICAL SERIES OF A V_d

By J. A. TODD.

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In the course of a recent paper† I have shown how to define an invariant series of equivalence $\{X_0(V_d)\}$ of sets of points on an algebraic V_d, for which I have proposed the name *canonical series*. In this note I propose to extend to the general V_d a property of this series which was established by Severi‡ for $d = 2$.

Consider a V_d which possesses at least one everywhere finite simple integral u. We are interested in the set of stationary points or *Jacobian set* of the integral u. It is assumed that the V_d is so general that this set consists of a finite number of points, thus V_d must not contain an irrational pencil of V_{d-1}'s (when the stationary points fill up certain of the V_{d-1}'s)§. The theorem we prove is the following:

If on an algebraic V_d the set of stationary points of an everywhere finite simple integral is finite, then these points constitute a set of the canonical series $\{X_0(V_d)\}$ on V_d.

This has been proved for surfaces by Severi, in the paper last quoted, and is classical for curves. We therefore prove it by induction, assuming it to be true for varieties of dimension less than d.

We suppose V_d to be free from singularities in a suitable space S_r. Let u be the integral of the first kind under consideration. It defines a function of position on V_d whose value $u(P)$ at any point P of V_d is determined to within an integral combination of the R_1 periods of u, R_1 being the first Betti number of V_d. Consider the points P such that $u(P) \equiv a$ (mod.

† Todd, "The geometrical invariants of algebraic loci", *Proc. London Math. Soc.* (2), 43 (1937), 127–138; see also Severi, *Rend. Acc. Lincei* (6), 23 (1936), 921.

‡ Severi, *Comentarii Mathematici Helvetici*, 4 (1932), 268–326 (278); see also Eger, *Comptes rendus*, 204 (1937), 92, where the property in question is used as a definition of the canonical series.

§ Hence (as Prof. Hodge has pointed out to me) V_d possesses *at least two* such integrals.

periods). The aggregate of these points P forms a transcendental variety Σ_a, of $d-1$ dimensions, which is clearly analytic, so that it is algebroid in the neighbourhood of any one of its points. As a varies, this variety Σ_a moves in a transcendental pencil $\{\Sigma\}$. A certain number of the varieties of Σ will have double points, and the set x of these points is the Jacobian set of the integral u.

We first show that *the pencil $\{\Sigma\}$ is free from base-points.* The argument is exactly parallel to that used by Severi in the case $d=2$, and may be summarized briefly. Suppose that a point P is common to all the Σ_a; then some finite neighbourhood of P is met in *variable* points by the Σ_a, since the varieties Σ_a corresponding to different values of a define different analytic functions in the neighbourhood of P. A suitable birational transformation of V_d will make this neighbourhood appear as a (rational) exceptional V_{d-1} on the transformed locus $V_d{}^*$, met in variable V_{d-2}'s by a pencil $\{\Sigma^*\}$ of V_{d-1}'s which are defined as the varieties $u^* \equiv$ const., u^* being the everywhere finite integral on $V_d{}^*$ corresponding to u on V_d. Thus there is an effective everywhere finite simple integral on the rational exceptional V_{d-1}, which is impossible. Hence $\{\Sigma\}$ can have no base points.

We now take on V_d a general pencil $|C|$ of V_{d-1}'s, whose base V_{d-2}, (C^2), is irreducible. Let δ denote the finite set of double points of V_{d-1}'s belonging to the pencil $|C|$. On a fixed C_1 of $|C|$ the integral u attached to V_d gives rise to an everywhere finite integral u_1 with the same periods as u (not constant if $|C|$ is sufficiently general), and the set of V_{d-2}'s, $u_1 \equiv$ const., on C_1 is cut out by $\{\Sigma\}$. The Jacobian set δ_1 of this integral u_1 is evidently the aggregate of points of C_1 where it is touched by a Σ, and by the inductive hypothesis this is a set of the canonical series $\{X_0(C_1)\}$ on C_1. The integral u defines also an everywhere finite integral u_2 on the base (C^2) of the pencil, and the loci $u_2 \equiv$ const. are the intersections of (C^2) with the Σ. At a double point of one of these loci a Σ touches one of the C's of the pencil. These points form the Jacobian set δ_2 of the integral u_2 on (C^2), which, by hypothesis, is a set of the canonical series $\{X_0(C^2)\}$ on (C^2).

Let J denote the curve of contacts of a C and a Σ, so that J is the locus of the set δ_1 as C_1 varies in the pencil. Since the set δ_1 is the Jacobian set of the integral u_1 derived from u, it is determined rationally on each C_1 of the pencil $|C|$. Hence J is an algebraic curve. From what has been said it evidently passes through the set δ_2 on the base (C^2), and meets C_1 further in the set δ_1. Hence we have the relation of equivalence on V_d,

$$(JC) \equiv \delta_1 + \delta_2. \tag{1}$$

The curve J passes through the set δ of double points of the pencil $|C|$ and through the set x of double points of $\{\Sigma\}$. Consider the linear series cut out

on J by $|C|$. The Jacobian set of this linear series is equivalent to the set $2(JC)+X_0(J)$, where $X_0(J)$ is a canonical set on J. The double points δ of the pencil $|C|$ evidently belong to the Jacobian set, as does the set δ_2 in which J meets (C^2). The set δ_2 counts doubly, since its points are fixed points of the linear series. At any other point of the Jacobian set the curve J touches the unique C which passes through the point. The point is thus one of the points of stationary contact of a Σ and a C. If τ denotes the set of such points, then we have the equivalence

$$2(JC)+X_0(J)\equiv\delta+2\delta_2+\tau. \tag{2}$$

The integral u determines an Abelian integral of the first kind on J, whose Jacobian set is a set $X_0(J)$. This set consists of those points of J such that the locus Σ passing through the point meets J in two coincident points. Thus it consists of the set x of double points of the pencil Σ, the set δ_2 of the points of contact of (C^2) with Σ, and the set τ of stationary contacts, where J, since it touches C, touches Σ as well. Thus

$$X_0(J)\equiv x+\delta_2+\tau. \tag{3}$$

From (1), (2), and (3) it follows that

$$x\equiv\delta-2\delta_1-\delta_2.$$

But $\delta-2\delta_1-\delta_2$ is by definition a set of the canonical series on V_d. Hence

$$x\equiv X_0(V_d),$$

which proves the theorem.

I should like to express my thanks to Dr. P. Du Val for drawing my attention to this problem, and for helpful criticism of the proof.

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NOTE ON A RESULT IN THE ADDITIVE THEORY OF NUMBERS

By H. DAVENPORT *and* H. HEILBRONN.

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1. It has been proved by Romanoff† that the positive integers which are representable as the sum of a prime and a k-th power have positive density. The object of this note is to give a proof that almost all‡ positive integers are representable as $p+x^k$.

The proof can easily be modified to show that, if k is odd, almost all positive integers are representable in one of the two forms $p+x^k$, $2p+x^k$, with $p \equiv 1 \pmod 4$. It follows from this that almost all positive integers are representable as $x^2+y^2+z^k$.

2. Let P be a large positive integer,

$$R = [\log P]^k, \quad X = [\tfrac{1}{4} \log P], \quad \xi = e^{-\sqrt{\log P}}.$$

Throughout the paper, all small Latin letters except c, e, i, s denote positive integers, and p always denotes a prime. $c_1, c_2, \ldots$ denote positive numbers depending only on k. The constants implied by the symbol O depend only on k and ϵ.

We divide the interval $0 \leqslant \alpha \leqslant 1$ in the usual way into Farey arcs, corresponding to all Farey fractions a/q with

$$q \leqslant P^{k-\frac{1}{2}}, \quad a \leqslant q, \quad (a, q) = 1.$$

We divide these arcs into major arcs $\mathfrak{M}\,(q \leqslant R)$ and minor arcs $\mathfrak{m}$ $(R < q \leqslant P^{k-\frac{1}{2}})$. In either case an arc has the form

$$\alpha = a/q+\beta, \quad -\vartheta_1 q^{-1} P^{-k+\frac{1}{2}} \leqslant \beta \leqslant \vartheta_2 q^{-1} P^{-k+\frac{1}{2}},$$

where $\frac{1}{2} \leqslant \vartheta_1 \leqslant 1$, $\frac{1}{2} \leqslant \vartheta_2 \leqslant 1$.

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† *Math. Annalen*, 109 (1934), 668–678.

‡ *I.e.*, the number of non-representable numbers not exceeding η is $o(\eta)$ as $\eta \to \infty$.

It is convenient to divide each major arc into two parts, as follows:

$\mathfrak{M}_1$: $$|\beta| \leqslant P^{-k} R^2,$$

$\mathfrak{M}_2$: $$|\beta| \geqslant P^{-k} R^2.$$

We use the abbreviations

$$e(\eta) = e^{2\pi i \eta}, \quad e_q(\eta) = e(\eta/q).$$

Let $$T(\alpha) = \sum_{x=1}^{P} e(\alpha x^k), \quad U(\alpha) = \sum_{p \leqslant P^k} e(\alpha p),$$

$$T(\alpha)\, U(\alpha) = \sum_n r(n)\, e(n\alpha),$$

so that, for $n \leqslant P^k$, $r(n)$ is the number of representations of n as $p + x^k$. Let

$$S_{a,q} = \sum_{x=1}^{q} e_q(ax^k),$$

$$I^{(k)}(\beta) = \frac{1}{k} \sum_{n=1}^{P^k} n^{-1+(1/k)} e(\beta n), \qquad I(\beta) = \sum_{n=2}^{P^k} \frac{1}{\log n}\, e(\beta n),$$

$$T^*(\alpha, a, q) = q^{-1} S_{a,q} I^{(k)}(\alpha - a/q), \quad U^*(\alpha, a, q) = \frac{\mu(q)}{\phi(q)} I(\alpha - a/q),$$

$$A(n, q) = \frac{\mu(q)}{q\phi(q)} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} S_{a,q} e_q(-na),$$

$$\mathfrak{S}(n) = \sum_{q=1}^{R} A(n, q),$$

$$\mathfrak{S}'(n) = \prod_{p \leqslant X} \big(1 + A(n, p)\big),$$

$$\pi(x) = \sum_{p \leqslant x} 1, \qquad \pi(x;\, q, l) = \sum_{\substack{p \leqslant x \\ p \equiv l \pmod q}} 1,$$

$$\mathrm{li}(x) = \int_2^x \frac{d\eta}{\log \eta}.$$

3. LEMMA 1. *If* $q \leqslant R$, $(l, q) = 1$, $n \leqslant P^k$, *then*

$$\pi(n;\, q, l) = \frac{1}{\phi(q)} \mathrm{li}(n) + O(P^k \xi^{c_1}).$$

Proof. Let $\sigma(q)$ be the greatest real zero of any Dirichlet L-function to modulus q. It was proved by Page† that

$$\pi(n;\ q,\ l) = \frac{1}{\phi(q)}\,\mathrm{li}(n) + O(ne^{-c_2\sqrt{\log n}}) + O\left(\frac{n^{\sigma(q)}}{\phi(q)\log n}\right).$$

Hence it suffices to prove that

$$\sigma(q) \leqslant 1 - C(\epsilon)\,q^{-\epsilon} \quad [C(\epsilon) > 0], \tag{1}$$

for then we have, taking $\epsilon = (2k)^{-1}$,

$$n^{\sigma(q)} \leqslant P^{k(1-C(\epsilon)q^{-\epsilon})} \leqslant P^k\,P^{-kC(\epsilon)R^{-\epsilon}} \leqslant P^k e^{-kC(\epsilon)(\log P)} = P^k\,\xi^{c_3}.$$

For L-functions formed with complex characters, (1) was proved by Gronwall‡.

For L-functions formed with real non-principal characters, it is sufficient to consider those arising from primitive characters mod q. If χ is such a character, there exists an integer $\Delta = \pm q$ such that $\chi(n) = (\Delta/n)$ (Kronecker symbol). The fact that the greatest real zero of $L(s,\ \chi) = L_\Delta(s)$ satisfies (1) is an easy consequence of Siegel's result§

$$L_\Delta(1) = \sum_{n=1}^{\infty}\left(\frac{\Delta}{n}\right)n^{-1} > C_1(\epsilon)\,|\Delta|^{-\epsilon} \quad [C_1(\epsilon) > 0],$$

and the elementary inequality

$$L_\Delta'(s) = O(\log^2|\Delta|) \quad \text{for} \quad s \geqslant 1 - (\log|\Delta|)^{-1}.$$

LEMMA 2. *If* $q \leqslant R$, $(a,\ q) = 1$, $\alpha = a/q + \beta$, *then*

$$U(\alpha) - U^*(\alpha,\ a,\ q) = O\Big(P^k\,\xi^{c_4}(1 + P^k|\beta|)\Big).$$

Proof. We have, for $n \leqslant P^k$,

$$S_n = \sum_{p \leqslant n} e_q(ap) = \sum_{\substack{l=1\\(l,\ q)=1}}^{q} \pi(n;\ q,\ l)\,e_q(al) + O\Big(d(q)\Big)$$

$$= \frac{1}{\phi(q)}\,\mathrm{li}(n) \sum_{\substack{l=1\\(l,\ q)=1}}^{q} e_q(al) + O(q\,P^k\,\xi^{c_1})$$

$$= \frac{\mu(q)}{\phi(q)}\,\mathrm{li}(n) + O(P^k\,\xi^{c_4}),$$

† *Proc. London Math. Soc.* (2), 39 (1935), 116–141, Theorem 1.
‡ *Rend. di Palermo*, 35 (1913), 145–159, end of § 1.
§ *Acta Arithmetica*, 1 (1936), 83–86.

by Lemma 1. Hence

$$U(\alpha) = \sum_{n=2}^{P^k} (S_n - S_{n-1})\, e(\beta n)$$

$$= \sum_{n=2}^{P^k} S_n \Big(e(\beta n) - e\big(\beta(n+1)\big) \Big) + S_{P^k} e\big(\beta(P^k+1)\big)$$

$$= \frac{\mu(q)}{\phi(q)} \Big\{ \sum_{n=2}^{P^k} \mathrm{li}(n) \Big(e(\beta n) - e\big(\beta(n+1)\big) \Big) + \mathrm{li}(P^k)\, e\big(\beta(P^k+1)\big) \Big\}$$

$$+ O\Big(P^k \xi^{c_4} \Big\{ 1 + \sum_{n=2}^{P^k} |1 - e(\beta)| \Big\} \Big)$$

$$= \frac{\mu(q)}{\phi(q)} \sum_{n=3}^{P^k} \{\mathrm{li}(n) - \mathrm{li}(n-1)\}\, e(\beta n) + O\big(P^k \xi^{c_4} (1 + P^k |\beta|) \big),$$

whence the result, since

$$\mathrm{li}(n) - \mathrm{li}(n-1) = \frac{1}{\log n} + O\Big(\frac{1}{n \log n}\Big).$$

LEMMA 3. *If* $(a, q) = 1$, *and* $\alpha = a/q + \beta$, *then*

$$T(\alpha) - T^*(a, a, q) = O\big(q(1 + P^k |\beta|) \big).$$

Proof. Let

$$S_n = \sum_{x^k \leqslant n} e_q(a x^k).$$

Clearly

$$S_n = \frac{n^{1/k}}{q} S_{a,q} + O(q).$$

The result now follows by the usual partial summation, as in the proof of the previous lemma.

LEMMA 4. *If* $(a, q) = 1$, $\alpha = a/q + \beta$, $|\beta| \leqslant \frac{1}{2}$, *then*

$$U^*(a, a, q) = O\Big(\frac{1}{\phi(q)} \min\Big(\frac{P^k}{\log P}, |\beta|^{-1} \Big) \Big), \tag{2}$$

$$T^*(a, a, q) = O\big(q^{-1/k} \min(P, |\beta|^{-1/k}) \big). \tag{3}$$

Proof. Since†

$$S_{a,q} = O(q^{1-(1/k)}),$$

† Landau, *Vorlesungen über Zahlentheorie*, 1, Satz 315.

and

$$I(\beta) = O\Big(\frac{P^k}{\log P}\Big), \quad I^{(k)}(\beta) = O(P),$$

it suffices to prove that

$$I(\beta) = O(|\beta|^{-1}), \quad I^{(k)}(\beta) = O(|\beta|^{-1/k}).$$

These inequalities follow at once on dividing the sums into $n \leqslant |\beta|^{-1}$ and $n > |\beta|^{-1}$ and applying Abel's lemma to the second parts.

4. LEMMA 5. *On* $\mathfrak{m}$, *if* $q \geqslant P^{\frac{1}{2}}$, $T(\alpha) = O(P^{1-c_5})$.

Proof. Let S_n be as in the proof of Lemma 3. Writing $K = 2^{k-1}$ we have†

$$|S_n|^K = O\Big(n^\epsilon q^\epsilon (n^{K-1} + n^K q^{-k} + n^{K-k} q^k)\Big).$$

Hence, for $n \leqslant P^k$,

$$S_n = O(P^{1-c_5}).$$

The result now follows by the usual partial summation (as in the proof of Lemma 2) when we observe that

$$P^k |\beta| \leqslant P^k q^{-1} P^{-k+\frac{1}{2}} \leqslant 1.$$

LEMMA 6. *On* $\mathfrak{m}$, *if* $q < P^{\frac{1}{2}}$, *then* $T(\alpha) = O(PR^{-1/k})$.

Proof. By Lemma 3 and (3),

$$T(\alpha) = O(P^{\frac{1}{2}}) + O(PR^{-1/k}) = O(PR^{-1/k}).$$

LEMMA 7. *On* $\mathfrak{M}_2$, $T(\alpha) = O(PR^{-2/k})$.

Proof. By Lemma 3 and (3),

$$T(\alpha) = O\Big(R(1+P^{\frac{1}{2}})\Big) + O\Big(P(R^2)^{-1/k}\Big) = O(PR^{-2/k}).$$

LEMMA 8. $\Big(\sum_{\mathfrak{m}} \int_{\mathfrak{m}} + \sum_{\mathfrak{M}_2} \int_{\mathfrak{M}_2}\Big) |T(\alpha)\, U(\alpha)|^2 d\alpha = O\Big(\frac{P^{k+2}}{\log P}\, R^{-2/k}\Big)$.

† Landau, *loc. cit.*, Satz 267.

Proof. By Lemmas 5, 6, 7 we have, on $\mathfrak{m}$ and $\mathfrak{M}_2$,

$$T(\alpha) = O(PR^{-1/k}).$$

Also
$$\int_0^1 |U(\alpha)|^2 d\alpha = \pi(P^k) = O\left(\frac{P^k}{\log P}\right).$$

Lemma 9. *On* $\mathfrak{M}_1$,

$$T(\alpha)\,U(\alpha) - T^*(\alpha, a, q)\,U^*(\alpha, a, q) = O(P^{k+1}\,\xi^{c_6}).$$

Proof. By Lemmas 2, 3, 4,

$$\begin{aligned} |TU - T^*U^*| &\leqslant |U - U^*||T| + |T - T^*||U^*| \\ &= O(P^k\,\xi^{c_4}\,R^2\,P) + O(qR^2\,P^k) \\ &= O(P^{k+1}\,\xi^{c_5}). \end{aligned}$$

Lemma 10.

$$\sum_{\mathfrak{M}_1} \int_{\mathfrak{M}_1} |T(\alpha)\,U(\alpha) - T^*(\alpha, a, q)\,U^*(\alpha, a, q)|^2 d\alpha = O(P^{k+2}\,\xi^{c_7}).$$

Proof. By Lemma 9, the expression on the left is

$$\begin{aligned} O\left(\sum_{q=1}^{R} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} P^{-k}\,R^2\,P^{2k+2}\,\xi^{2c_6}\right) &= O(P^{k+2}\,R^4\,\xi^{2c_6}) \\ &= O(P^{k+2}\,\xi^{c_7}). \end{aligned}$$

Lemma 11.

$$\int_0^1 \left| \sum_{q=1}^{R} \sum_{\substack{a=1 \\ (a,q)=1}}^{q}{}' T^*(\alpha, a, q)\,U^*(\alpha, a, q) \right|^2 d\alpha = O(P^{k+2}\,R^{-4/k}),$$

where the dash denotes that, if α *is on one of the arcs* $\mathfrak{M}_1$, *the term* a, q *corresponding to this arc is to be omitted from the sum.*

Proof. By Cauchy's inequality,

$$\left| \sum_q \sum_a{}' T^*\,U^* \right|^2 \leqslant \left(\sum_q \sum_a{}' 1\right)\left(\sum_q \sum_a{}' |T^*\,U^*|^2\right).$$

Hence the integral in question does not exceed

$$R^2 \sum_{q=1}^{R} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \left(\int_0^1 - \int_{\mathfrak{M}_1}\right) |T^*(\alpha, a, q)\, U^*(\alpha, a, q)|^2\, d\alpha$$

$$= O\left(R^2 \sum_{q=1}^{R} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \int_{P^{-k}R^2}^{\infty} \frac{1}{\phi^2(q)}\, q^{-2/k}\, \beta^{-2-(2/k)}\, d\beta\right)$$

$$= O\left(R^2 \sum_{q=1}^{R} \frac{1}{\phi(q)}\, q^{-2/k}\, P^{k+2} (R^2)^{-1-(2/k)}\right)$$

$$= O(P^{k+2} R^{-4/k}).$$

LEMMA 12.

$$\int_0^1 \left| T(\alpha)\, U(\alpha) - \sum_{q=1}^{R} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} T^*(\alpha, a, q)\, U^*(\alpha, a, q) \right|^2 d\alpha = O\left(P^{k+2} (\log P)^{-3}\right).$$

Proof. Lemmas 8, 10, 11.

LEMMA 13.

$$\sum_{q=1}^{R} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} T^*(\alpha, a, q)\, U^*(\alpha, a, q) = \sum_n \mathfrak{S}(n)\, \psi(n)\, e(n\alpha),$$

where, for $3 \leqslant n \leqslant P^k$,

$$\frac{c_8 n^{1/k}}{\log n} < \psi(n) < \frac{c_9 n^{1/k}}{\log n}. \tag{4}$$

Proof. The sum on the left is

$$\frac{1}{k} \sum_{q=1}^{R} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} q^{-1} S_{a,q} \frac{\mu(q)}{\phi(q)} \sum_{n_1=1}^{P^k} n_1^{-1+(1/k)} \sum_{n_2=2}^{P^k} \frac{1}{\log n_2}\, e\left((n_1+n_2)(\alpha - a/q)\right)$$

$$= \sum_n \mathfrak{S}(n)\, \psi(n)\, e(n\alpha),$$

where

$$\psi(n) = \frac{1}{k} \sum_{\substack{n_1=1 \\ n_1+n_2=n}}^{P^k} \sum_{n_2=2}^{P^k} n_1^{-1+(1/k)} (\log n_2)^{-1}.$$

For $3 \leqslant n \leqslant P^k$, $\psi(n)$ obviously satisfies the inequalities (4).

7. LEMMA 14.

$$-\frac{(k, p-1)-1}{p-1} \leqslant A(n, p) \leqslant \frac{1}{p-1},$$

and
$$|A(n, q)| \leqslant \frac{k^{\nu(q)}}{\phi(q)},$$

where $\nu(q)$ denotes the number of different prime factors of q.

Proof. We have

$$\begin{aligned} A(n, p) &= \frac{\mu(p)}{\phi(p)}\, p^{-1} \sum_{a=1}^{p-1} S_{a,p}\, e_p(-na) \\ &= -\frac{1}{p(p-1)} \sum_{a=1}^{p-1} \sum_{x=1}^{p} e_p\Big(a(x^k-n)\Big) \\ &= -\frac{1}{p(p-1)}\,(-p+pN), \end{aligned}$$

where N is the number of solutions of the congruence $x^k \equiv n \pmod{p}$. Since $0 \leqslant N \leqslant (k, p-1)$, the first part of the lemma is established.

It is easily proved that, if $(q_1, q_2) = 1$, then

$$A(n, q_1 q_2) = A(n, q_1)\, A(n, q_2),$$

and this gives the second part of the lemma, if we note that $A(n, q) = 0$ unless q is quadratfrei.

Lemma 15.

$$\sum_{n=1}^{P^k} \Big(\mathfrak{S}(n)-\mathfrak{S}'(n)\Big)^2 = O\Big(P^k (\log P)^{-1+\epsilon}\Big).$$

Proof. We have

$$\prod_{p \leqslant X} p < X^{\pi(X)} < e^{2X} \leqslant P^{\frac{1}{2}}.$$

Hence

$$\mathfrak{S}(n)-\mathfrak{S}'(n) = \sum_{X<q<P^{\frac{1}{2}}} \theta(q)\, A(n, q),$$

where $\theta(q)$ is 0, 1, or -1, and is independent of n. Therefore

$$\begin{aligned} \sum_{n=1}^{P^k} \Big(\mathfrak{S}(n)-\mathfrak{S}'(n)\Big)^2 &= \sum_{n=1}^{P^k} \sum_{X<q<P^{\frac{1}{2}}} \theta(q)^2 A(n, q)^2 \\ &\quad + 2 \sum_{X<q_1<q_2<P^{\frac{1}{2}}} \theta(q_1)\,\theta(q_2) \sum_{n=1}^{P^k} A(n, q_1)\, A(n, q_2) \\ &= \Sigma_1 + \Sigma_2, \end{aligned}$$

say. By Lemma 14,

$$\begin{aligned} \sum_{X<q<P^{\frac{1}{2}}} \theta(q)^2 A(n, q)^2 &\leqslant \sum_{q=X}^{\infty} \frac{k^{2\nu(q)}}{\phi^2(q)} \\ &= O(X^{-1+\epsilon}). \end{aligned}$$

Hence

$$\Sigma_1 = O\Big(P^k(\log P)^{-1+\epsilon}\Big).$$

To deal with Σ_2 we observe that, if $q_1 \neq q_2$ and $(a_1, q_1) = (a_2, q_2) = 1$, then $a_1 q_2 + a_2 q_1 \not\equiv 0 \pmod{q_1 q_2}$. Hence $A(n, q_1) A(n, q_2)$ is periodic in n with period $q_1 q_2$ and average value zero.

Hence, using Lemma 14, we have

$$\Big|\sum_{n=1}^{P^k} A(n, q_1) A(n, q_2)\Big| \leqslant \sum_{n=1}^{q_1 q_2} |A(n, q_1)\, A(n, q_2)|$$

$$\leqslant \frac{q_1 q_2 k^{\nu(q_1)} k^{\nu(q_2)}}{\phi(q_1)\phi(q_2)} = O(q_1^{\epsilon} q_2^{\epsilon}).$$

Thus

$$\Sigma_2 = O\Big(\sum_{X < q_1 < q_2 < P^k} q_1^{\epsilon} q_2^{\epsilon}\Big) = O(P^{1+\epsilon}),$$

and this completes the proof of Lemma 15.

LEMMA 16. $\mathfrak{S}'(n) > c_{10} (\log\log P)^{-k}$.

Proof. By Lemma 14,

$$\mathfrak{S}'(n) = \prod_{p \leqslant 2k} \Big(1 + A(n, p)\Big) \prod_{2k < p \leqslant X} \Big(1 + A(n, p)\Big)$$

$$\geqslant \prod_{p \leqslant 2k} \frac{1}{p-1} \prod_{2k < p \leqslant X} \Big(1 - \frac{k}{p-1}\Big)$$

$$> c_{11} \prod_{p \leqslant X} (1 - p^{-1})^k$$

$$> c_{10} (\log\log P)^{-k}.$$

8. THEOREM. *Almost all positive integers are representable in the form* $p + x^k$.

Proof. By Lemmas 12, 13,

$$\sum_{n=3}^{P^k} \Big(r(n) - \mathfrak{S}(n)\psi(n)\Big)^2 = O\Big(P^{k+2}(\log P)^{-3}\Big).$$

By Lemma 15 and (4),

$$\sum_{n=3}^{P^k} \Big(\mathfrak{S}(n)\psi(n) - \mathfrak{S}'(n)\psi(n)\Big)^2 = O\Big(P^{k+2}(\log P)^{-3+\epsilon}\Big).$$

Hence

$$\sum_{n=3}^{P^k} \Big(r(n)-\mathfrak{S}'(n)\psi(n)\Big)^2 = O\Big(P^{k+2}(\log P)^{-3+\epsilon}\Big).$$

Thus, by Lemma 16 and (4),

$$\sum_{\substack{n=3\\ r(n)=0}}^{P^k} \frac{n^{2/k}}{(\log n)^2}(\log\log P)^{-2k} = O\Big(P^{k+2}(\log P)^{-3+\epsilon}\Big),$$

and so

$$\sum_{\substack{n=3\\ r(n)=0}}^{P^k} n^{2/k} = O\Big(P^{k+2}(\log P)^{-1+2\epsilon}\Big).$$

Let N denote the number of numbers $n \leqslant P^k$ for which $r(n)=0$. Then

$$\frac{1}{1+2/k} N^{1+2/k} \leqslant \sum_{n=1}^{N} n^{2/k} \leqslant \sum_{\substack{n=3\\ r(n)=0}}^{P^k} n^{2/k} = O\Big(P^{k+2}(\log P)^{-1+2\epsilon}\Big),$$

$$N = O\Big(P^k(\log P)^{-\frac{(1-2\epsilon)k}{k+2}}\Big) = o(P^k).$$

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ON THE STRESS EQUATIONS IN AN ELASTIC SOLID

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1. It is known that, in an elastic solid in equilibrium under no body-force, the stress-components satisfy, in addition to the three dynamical, or so-called "body-stress", equations

$$\partial p\widehat{x}/\partial x+\partial p\widehat{y}/\partial y+\partial p\widehat{z}/\partial z=0 \quad (p=x,\ y,\ z), \tag{1}$$

six equations of the form

$$(1+\eta)\,\nabla^2\widehat{pq}+\partial^2(\widehat{xx}+\widehat{yy}+\widehat{zz})/\partial p\,\partial q=0, \tag{2}$$

where η is Poisson's ratio.

The equations (2) are sometimes referred to as the "consistency" equations for the stress, but it is important to note that they already involve the use of the dynamical equations (1) and are by no means identical with the six "compatibility" equations derived from the conditions of compatibility of strain. These compatibility equations are, if we write $\widehat{xx}+\widehat{yy}+\widehat{zz}=\Theta$,

$$2\,\frac{\partial^2\widehat{yz}}{\partial y\,\partial z}=\frac{\partial^2}{\partial y^2}\left(\widehat{zz}-\frac{\eta}{1+\eta}\,\Theta\right)+\frac{\partial^2}{\partial z^2}\left(\widehat{yy}-\frac{\eta}{1+\eta}\,\Theta\right), \tag{3.1}$$

$$\frac{\partial^2}{\partial y\,\partial z}\left(\widehat{xx}-\frac{\eta}{1+\eta}\,\Theta\right)=\frac{\partial}{\partial x}\left(\frac{\partial\widehat{xy}}{\partial z}+\frac{\partial\widehat{xz}}{\partial y}-\frac{\partial\widehat{yz}}{\partial x}\right), \tag{3.2}$$

and by themselves ensure that the stresses are derived from a displacement, but they do not involve that the dynamical equations are satisfied and are in fact independent of them.

In what follows equations (2) will be referred to as the "necessary" stress-equations, to avoid possible confusion between "consistency" and "compatibility" equations. Also the term "body-stress" equations will

not be used, since all the equations (1), (2), and (3) equally involve body-stresses. The terms "dynamical", "necessary", and "compatibility" equations will be used consistently to denote equations of types (1), (2), (3) respectively.

A set of stress-components which satisfy the necessary, but not the dynamical, equations will, in general, be found to be incompatible.

So far as the writer is aware, no necessary equations corresponding to (2) for the case where there is body-force, and the elastic solid is in motion, have ever been published, nor has a proof been given that (1) and (2) taken together give a *sufficient*, as well as a necessary, condition that a set of stress-components should be a possible solution for the equilibrium of an elastic solid.

The object of the present note is (i) to generalize equations (2) so as to include body-force and kinetic terms; (ii) to examine under what conditions solutions of the necessary equations are possible stresses in an elastic solid.

2. It is well known that, if we substitute into the equations of motion from the stress-strain relations

$$\widehat{xx} = \lambda\delta + 2\mu\,\partial u/\partial x, \tag{4.1}$$

$$\widehat{yz} = \mu(\partial v/\partial z + \partial w/\partial y), \tag{4.2}$$

where $\delta = \operatorname{div}(u, v, w)$ and λ, μ are Lamé's constants, we obtain the differential equations for the displacements u, v, w,

$$(\lambda+\mu)(\partial/\partial x, \partial/\partial y, \partial/\partial z)\,\delta + \mu\nabla^2(u, v, w) = \rho(\partial^2/\partial t^2)(u, v, w) + \rho(\partial/\partial x, \partial/\partial y, \partial/\partial z)\,\Omega, \tag{5}$$

Ω being the body-force potential.

We have also

$$\Theta = \widehat{xx} + \widehat{yy} + \widehat{zz}, \tag{5.1}$$

$$\delta = \Theta/(3\lambda + 2\mu). \tag{5.2}$$

Operating on the second and third of equations (5) with $\partial/\partial z$, $\partial/\partial y$ and adding, we find, on rearranging:

$$(1+\eta)\{\nabla^2 - (\rho/\mu)\,\partial^2/\partial t^2\}\,\widehat{yz} + (\partial^2/\partial y\,\partial z)\{\Theta - 2\rho(1+\eta)\Omega\} = 0, \tag{6.1}$$

and two similar equations by cyclic interchange.

Again, operating on the first of (5) with $2\partial/\partial x$ and using the stress-strain relation

$$2\mu(\partial u/\partial x) = \widehat{xx} - \eta\Theta/(1+\eta),$$

we have

$$(1+\eta)\{\nabla^2 - (\rho/\mu)\partial^2/\partial t^2\}\{\widehat{xx} - \eta\Theta/(1+\eta)\} + (\partial^2/\partial x^2)\{\Theta - 2\rho(1+\eta)\Omega\} = 0. \tag{6.2}$$

(6.1) and (6.2) give the necessary equations satisfied by the stresses in the most general case.

When there is no body-force, or when the second differential coefficients of Ω all vanish, which includes the important case of weight at the earth's surface, equations (6.2) take the form

$$\{\nabla^2 - (\rho/\mu)\partial^2/\partial t^2\}\{(1+\eta)\widehat{xx} - \eta\Theta\} + \partial^2\Theta/\partial x^2 = 0. \tag{7}$$

These can be put into an alternative form.

For, adding the three equations of type (7), we obtain

$$\nabla^2\Theta = \{\rho(1-2\eta)/2\mu(1-\eta)\}(\partial^2\Theta/\partial t^2),$$

the well-known equation of longitudinal vibrations, and, substituting from this into equations (7), we find

$$(1+\eta)\{\nabla^2 - (\rho/\mu)\partial^2/\partial t^2\}\widehat{xx} + \{\partial^2/\partial x^2 + [\rho\eta/2\mu(1-\eta)]\partial^2/\partial t^2\}\Theta = 0, \tag{8.2}$$

the equations (6.1) taking the form

$$(1+\eta)\{\nabla^2 - (\rho/\mu)\partial^2/\partial t^2\}\widehat{yz} + \partial^2\Theta/\partial y\,\partial z = 0. \tag{8.1}$$

If, on the other hand, there is body-force and the problem is *statical*, equations (6.1) and (6.2) reduce to

$$(1+\eta)\nabla^2\widehat{yz} + (\partial^2/\partial y\,\partial z)\{\Theta - 2\rho(1+\eta)\Omega\} = 0, \tag{9.1}$$

$$(1+\eta)\nabla^2\widehat{xx} - \eta\nabla^2\Theta + (\partial^2/\partial x^2)\{\Theta - 2\rho(1+\eta)\Omega\} = 0, \tag{9.2}$$

leading to $$\nabla^2\Theta = \{\rho(1+\eta)/(1-\eta)\}\nabla^2\Omega,$$

so that (9.2) can be written

$$(1+\eta)\nabla^2\{\widehat{xx} - \rho\eta\Omega/(1-\eta)\} + (\partial^2/\partial x^2)\{\Theta - 2\rho(1+\eta)\Omega\} = 0. \tag{9.3}$$

In this last problem the dynamical equations take the form

$$\partial\widehat{px}/\partial x + \partial\widehat{py}/\partial y + \partial\widehat{pz}/\partial z = \rho\,\partial\Omega/\partial p \tag{10}$$

and do not involve the displacements explicitly.

3. We will first of all investigate the conditions for the statical problem (with body-force). The necessary stress-equations will be taken in the form (9.1), (9.2). Clearly, since (9.2) are derivable from (9.3) by linear combination, either set may be used indifferently.

It will now be shown that any six functions $\widehat{pq}$ which satisfy (9.1) and (9.2), and which also satisfy the dynamical equations (10), are necessarily derivable from displacements u, v, w which satisfy the elastic equations.

Let u_1, v_1, w_1 be any particular integrals of the equations

$$\left.\begin{aligned} 2\mu(\partial u_1/\partial x) = \widehat{xx} - \eta\Theta/(1+\eta), \quad 2\mu(\partial v_1/\partial y) = \widehat{yy} - \eta\Theta/(1+\eta), \\ 2\mu(\partial w_1/\partial z) = \widehat{zz} - \eta\Theta/(1+\eta). \end{aligned}\right\} \quad (11)$$

The stress-strain relations (4) then show that, the suffix 1 denoting stresses derived from u_1, v_1, w_1,

$$\widehat{xx}_1 = \widehat{xx}, \quad \widehat{yy}_1 = \widehat{yy}, \quad \widehat{zz}_1 = \widehat{zz}, \quad \Theta_1 = \Theta, \qquad (12.1)$$

but, in general,

$$\widehat{yz} = \widehat{yz}_1 + \widehat{yz}_2, \quad \widehat{zx} = \widehat{zx}_1 + \widehat{zx}_2, \quad \widehat{xy} = \widehat{xy}_1 + \widehat{xy}_2, \qquad (12.2)$$

and $\widehat{yz}_2$, $\widehat{zx}_2$, $\widehat{xy}_2$ are not zero.

Write now

$$\left.\begin{aligned} \partial\widehat{xx}_1/\partial x + \partial\widehat{xy}_1/\partial y + \partial\widehat{xz}_1/\partial z - \rho\partial\Omega/\partial x \equiv A_1, \\ \partial\widehat{yx}_1/\partial x + \partial\widehat{yy}_1/\partial y + \partial\widehat{yz}_1/\partial z - \rho\partial\Omega/\partial y \equiv B_1, \\ \partial\widehat{zx}_1/\partial x + \partial\widehat{zy}_1/\partial y + \partial\widehat{zz}_1/dz - \rho\partial\Omega/\partial z \equiv C_1. \end{aligned}\right\} \quad (13)$$

Hence

$$\partial A_1/\partial x = \partial^2\widehat{xx}_1/\partial x^2 + \partial^2\widehat{xy}_1/\partial x\,\partial y + \partial^2\widehat{xz}_1/\partial x\,\partial z - \rho\partial^2\Omega/\partial x^2.$$

But

$$\partial^2\widehat{xy}_1/\partial x\,\partial y + \partial^2\widehat{xz}_1/\partial x\,\partial z$$

$$= \mu\{(\partial^2/\partial y^2)(\partial u_1/\partial x) + (\partial^2/\partial x^2)(\partial v_1/\partial y) + (\partial^2/\partial z^2)(\partial u_1/\partial x) + (\partial^2/\partial x^2)(\partial w_1/\partial z)\}$$

$$= \mu\{\nabla^2(\partial u_1/\partial x) + (\partial^2/\partial x^2)(\partial v_1/\partial y + \partial w_1/\partial z - \partial u_1/\partial x)\}$$

$$= \tfrac{1}{2}\{\nabla^2[\widehat{xx}_1 - \eta\Theta_1/(1+\eta)] + (\partial^2/\partial x^2)[\widehat{yy}_1 + \widehat{zz}_1 - \widehat{xx}_1 - \eta\Theta_1/(1+\eta)]\},$$

by (11).

Hence

$$2\partial A_1/\partial x = \nabla^2\{\widehat{xx}_1 - \eta\Theta_1/(1+\eta)\} + (\partial^2/\partial x^2)\{\Theta_1/(1+\eta) - 2\rho\Omega\}$$

$$= \nabla^2\{\widehat{xx} - \eta\Theta/(1+\eta)\} + (\partial^2/\partial x^2)\{\Theta/(1+\eta) - 2\rho\Omega\} = 0, \quad (14.1)$$

in virtue of the first of (9.2).

Similarly

$$\partial B_1/\partial y = 0, \quad (14.2)$$

$$\partial C_1/\partial z = 0. \quad (14.3)$$

Since the stresses $\widehat{pq}$ are given as satisfying equations (10), we have

$$\partial\widehat{xx}/\partial x + \partial\widehat{xy}/\partial y + \partial\widehat{xz}/\partial z - \rho\partial\Omega/\partial x = 0. \quad (15.1)$$

Subtracting from this the first of equations (13) and using (12), we have

$$\partial\widehat{xy}_2/\partial y + \partial\widehat{xz}_2/\partial z = -A_1, \quad (16.1)$$

and similarly

$$\partial\widehat{yz}_2/\partial z + \partial\widehat{yx}_2/\partial x = -B_1, \quad (16.2)$$

$$\partial\widehat{zx}_2/\partial x + \partial\widehat{zy}_2/\partial y = -C_1. \quad (16.3)$$

Further, returning to (13), operating on the second and third of these equations with $\partial/\partial z$, $\partial/\partial y$, and adding, we have

$$\partial B_1/\partial z + \partial C_1/\partial y = \partial^2\widehat{xy}_1/\partial x\partial z + \partial^2\widehat{xz}_1/\partial x\,\partial y + (\partial^2/\partial y^2 + \partial^2/\partial z^2)\widehat{yz}_1$$

$$+ (\partial^2/\partial y\,\partial z)(\widehat{yy}_1 + \widehat{zz}_1 - 2\rho\Omega).$$

Now

$$\partial^2\widehat{xy}_1/\partial x\,\partial z + \partial^2\widehat{xz}_1/\partial x\,\partial y = \mu\{2(\partial^2/\partial y\,\partial z)(\partial u_1/\partial x) + (\partial^2/\partial x^2)(\partial v_1/\partial z + \partial w_1/\partial y)\}$$

$$= (\partial^2/\partial y\partial z)\{\widehat{xx}_1 - \eta\Theta_1/(1+\eta)\} + (\partial^2/\partial x^2)\widehat{yz}_1,$$

so that

$$\partial B_1/\partial z + \partial C_1/\partial y = \nabla^2\widehat{yz}_1 + (\partial^2/\partial y\,\partial z)\{\Theta_1/(1+\eta) - 2\rho\Omega\} \quad (17.1)$$

and two similar equations.

But, from (16.2) and (16.3),

$$-\partial B_1/\partial z - \partial C_1/\partial y = (\partial/\partial x)\{\partial\widehat{xy}_2/\partial z + \partial\widehat{zx}_2/\partial y - \partial\widehat{yz}_2/\partial x\} + \nabla^2\widehat{yz}_2. \quad (18.1)$$

Adding this to (17.1) and using (12), we have

$$0 = (\partial/\partial x)(\partial\widehat{xy}_2/\partial z+\partial\widehat{zx}_2/\partial y-\partial\widehat{yz}_2/\partial x)+\nabla^2\widehat{yz}+(\partial^2/\partial y\,\partial z)\{\Theta/(1+\eta)-2\rho\Omega\},$$

so that
$$(\partial/\partial x)(\partial\widehat{xy}_2/\partial z+\partial\widehat{zx}_2/\partial y-\partial\widehat{yz}_2/\partial x) = 0, \tag{19.1}$$

in virtue of (9.1).

Similarly we obtain

$$(\partial/\partial y)(\partial\widehat{yz}_2/\partial x+\partial\widehat{xy}_2/\partial z-\partial\widehat{zx}_2/\partial y) = 0, \tag{19.2}$$

$$(\partial/\partial z)(\partial\widehat{zx}_2/\partial y+\partial\widehat{yz}_2/\partial x-\partial\widehat{xy}_2/\partial z) = 0. \tag{19.3}$$

Further, substituting from equations (16) into (14) we find easily

$$\partial^2\widehat{yz}_2/\partial y\,\partial z = \partial^2\widehat{zx}_2/\partial z\,\partial x = \partial^2\widehat{xy}_2/\partial x\,\partial y = 0, \tag{20}$$

leading to

$$\widehat{yz}_2 = h_1(x, y)+g_1(z, x), \tag{21.1}$$

$$\widehat{zx}_2 = f_2(y, z)+h_2(x, y), \tag{21.2}$$

$$\widehat{xy}_2 = g_3(z, x)+f_3(y, z). \tag{21.3}$$

If these are substituted into (19) and we write

$$\partial f_2/\partial y-\partial f_3/\partial z \equiv F, \quad \partial g_3/\partial z-\partial g_1/\partial x \equiv G, \quad \partial h_1/\partial x-\partial h_2/\partial y \equiv H, \tag{22}$$

we find

$$(\partial/\partial x)(G-H) = 0, \quad (\partial/\partial y)(H-F) = 0, \quad (\partial/\partial z)(F-G) = 0. \tag{23}$$

Since F is a function of y, z only and H is a function of x, y only, it follows from the second of (23) that F can only involve z in a term independent of y, and similarly from the third of (23) that it can only involve y in a term independent of z.

Thus we must have

$$F = Y_1+Z_1, \quad G = Z_2+X_2, \quad H = X_3+Y_3,$$

where X_n, Y_n, Z_n are functions of x only, y only, and z only respectively. When we substitute into equations (23) it now appears that X_2 and X_3, Y_3 and Y_1, Z_1 and Z_2 can differ only by constants, so that

$$F = Y_0+Z_0+A, \quad G = Z_0+X_0+B, \quad H = X_0+Y_0+C,$$

where A, B, C are constants.

Writing now

$$dX/dx = X_0 + \tfrac{1}{2}(B+C-A),$$

$$dY/dy = Y_0 + \tfrac{1}{2}(C+A-B),$$

$$dZ/dz = Z_0 + \tfrac{1}{2}(A+B-C),$$

we have

$$F = dY/dy + dZ/dz, \quad G = dZ/dz + dX/dx, \quad H = dX/dx + dY/dy. \quad (24)$$

Substituting these values in (22), we obtain

$$(\partial/\partial y)(f_2 - Y) = (\partial/\partial z)(f_3 + Z) \quad (25.1)$$

and two similar equations.

From (25.1) it follows that

$$f_2 = Y + \mu(\partial u_2/\partial z), \quad f_3 = -Z + \mu(\partial u_2/\partial y), \quad (26.1)$$

where u_2 is some function of y, z; and similarly

$$g_3 = Z + \mu(\partial v_2/\partial x), \quad g_1 = -X + \mu(\partial v_2/\partial z), \quad (26.2)$$

$$h_1 = X + \mu(\partial w_2/\partial y), \quad h_2 = -Y + \mu(\partial w_2/\partial x), \quad (26.3)$$

v_2 being some function of z, x, and w_2 of x, y.

Substituting from equations (26) into (21), we obtain

$$\widehat{yz}_2 = \mu(\partial v_2/\partial z + \partial w_2/\partial y) \quad (27.1)$$

and two similar equations.

Since $\partial u_2/\partial x = \partial v_2/\partial y = \partial w_2/\partial z = 0$, so that $\widehat{xx}_2$, $\widehat{yy}_2$, $\widehat{zz}_2$ all vanish, the stresses corresponding to the displacements u_1+u_2, v_1+v_2, w_1+w_2 are

$$\widehat{xx}_1, \quad \widehat{yy}_1, \quad \widehat{zz}_1, \quad \widehat{yz}_1+\widehat{yz}_2, \quad \widehat{zx}_1+\widehat{zx}_2, \quad \widehat{xy}_1+\widehat{xy}_2.$$

or $\widehat{xx}$, $\widehat{yy}$, $\widehat{zz}$, $\widehat{yz}$, $\widehat{zx}$, $\widehat{xy}$, which are the given stresses.

Provided therefore that a set of stresses satisfy simultaneously the equations (9) and (10), we know that they must be compatible and, since they satisfy the dynamical equations, they give a possible solution of the stresses in an elastic solid under the given conditions of body-force, and it is unnecessary to verify by solving for the displacements.

4. The kinetic case is more difficult, because here the dynamical equations involve the displacements explicitly, so that we have to look elsewhere for a test that the correct stresses have been found.

All that can be done here is to assume that the necessary stress-equations (6) and the compatibility equations (3) are simultaneously satisfied.

It is known that the satisfaction of the compatibility equations necessarily implies that the stresses are derivable from a displacement (u, v, w).

Let now

$$A \equiv \partial \widehat{xx}/\partial x + \partial \widehat{xy}/\partial y + \partial \widehat{zx}/\partial z - \rho \partial \Omega/\partial x - \rho \partial^2 u/\partial t^2,$$

with similar definitions of B and C.

Proceeding as before, we find that

$$2\partial A/\partial x = \{\nabla^2 - (\rho/\mu)\partial^2/\partial t^2\}\{\widehat{xx} - \eta\Theta/(1+\eta)\} + (\partial^2/\partial x^2)\{\Theta(1+\eta) - 2\rho\Omega\} \tag{28.1}$$

$$\partial B/\partial z + \partial C/\partial y = \{\nabla^2 - (\rho/\mu)\partial^2/\partial t^2\}\widehat{yz} + (\partial^2/\partial y\,\partial z)\{\Theta/(1+\eta) - 2\rho\Omega\}. \tag{28.2}$$

The right-hand sides of (28) vanish in virtue of the "necessary" equations (6). We have, therefore, the six equations

$$\partial A/\partial x = 0, \quad \partial B/\partial z + \partial C/\partial y = 0.$$

These require that the displacement (A, B, C) corresponds to zero strain. Accordingly it must reduce to a rigid body displacement. Let then (u_0, v_0, w_0) be a rigid body displacement varying with the time and given by

$$\rho(\partial^2 u_0/\partial t^2) = -A, \quad \rho(\partial^2 v_0/\partial t^2) = -B, \quad \rho(\partial^2 w_0/\partial t^2) = -C.$$

The displacement $(u+u_0, v+v_0, w+w_0)$ gives the same stresses as before and leads to

$$\partial \widehat{xx}/\partial x + \partial \widehat{xy}/\partial y + \partial \widehat{zx}/\partial z - \rho(\partial \Omega/\partial x) - \rho^2(\partial^2/\partial t^2)(u+u_0) = 0,$$

etc., which are the dynamical equations.

It follows that, in the most general case of motion, any set of stresses which satisfy both the necessary and the compatibility equations lead to displacements which satisfy the dynamical equations and so are a possible solution.

The same, of course, also holds of the statical case, but here, as we have seen, it is much simpler to combine the dynamical equations with the necessary equations, since this gives only nine equations to be satisfied instead of twelve.

Alternatively, the dynamical equations might be combined with the compatibility equations, a result which is well-known, and indeed obvious. The more symmetrical form of the "necessary" equations, however, makes them easier to deal with. Hence the relevance of the above investigation.

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ON A GENERALIZED WARING PROBLEM

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1. *Introduction.*

In the first part of this paper we wish to find a generalization for polynomials of the Hardy-Littlewood asymptotic formula in Waring's problem, that is to say a formula for the number of solutions of the Diophantine equation

$$N = P_1(h_1)+\ldots+P_s(h_s) \quad (h_\nu \geqslant 0), \tag{1}$$

where the P's are integral-valued polynomials of the k-th degree with positive highest coefficient. The method used is essentially due to Vinogradow (**3**)‡. For major arcs, Heilbronn's method in (**1**) is adopted.

Let $G\{P(h)\}$ be the least integer s such that (1) has a solution for $P_1(h) = P_2(h) = \ldots = P_s(h) = P(h)$§ when N is sufficiently large. Then $P(h)$ must satisfy the condition that, if q is an integer greater than 1, there exists no integer d such that

$$P(h) \equiv d \pmod{q}$$

identically.

The principal results for $G\{P(h)\}$ are as follows.

Theorem A. *If $P(h)$ is an odd integral-valued polynomial of the k-th degree, where $k > 20$, then*

$$G\{P(h)\} \leqslant \tfrac{1}{3}(2^{k+1}-1);$$

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‡ See the list of references at the end of the paper.

§ A study of the problem with different P's will be given later.

if, further, the highest coefficient of $P(h)$ *is* $a/k!$, *where* $(a, k!) = 1$, *then*

$$G\{P(h)\} = O(2^{\epsilon k}),$$

for any $\epsilon > 0$.

This is equivalent to Theorems 5 and 6 together.

THEOREM B. *If* $P(h)$ *is an integral-valued polynomial of the* k*-th degree, where* $k > 20$, *then*

$$G\{P(h)\} \leqslant 3^{\frac{1}{2}k}\, 2^{\frac{1}{2}(k-1)+7}\, k,$$

and

$$G\{P(h)\} = O(k^3\, 2^{k-1});$$

if, further, the highest coefficient of $P(h)$ *is* $a/k!$, *where* $(a, k!) = 1$, *then*

$$G\{P(h)\} = O(3^{k(\frac{1}{2}+\epsilon)}).$$

This is equivalent to Theorems 7, 8, and 9.

The existence of the upper bound of $G\{P(h)\}$ was first proved by Kamke (**1**), but no one has given a definite upper bound for $G\{P(h)\}$.

The main difficulties of the work are arithmetical and not analytical, and in some cases I have quoted analytic theorems of Vinogradow or other writers in slightly generalized form. The alterations necessary in their analysis are not difficult.

I wish to take this opportunity of expressing my gratitude to Dr. Heilbronn for his encouragement and advice.

2. *Notation.*

$r_{P_1, \ldots, P_s}(N) = r(N)$ is the number of solutions of the Diophantine equation

$$N = P_1(h_1) + P_2(h_2) + \ldots + P_s(h_s) \quad (h_\nu \geqslant 0,\ \nu = 1, 2, \ldots, s),$$

where $P_\nu(h)$ $(\nu = 1, 2, \ldots, s)$ are integral-valued polynomials of the k-th degree with positive coefficients and constant term zero†.

N_ν is the greatest positive root of $P_\nu(h) = N (> 0)$. It exists and is positive and tends to infinity with N.

k is an integer.

† The only essential restriction is that the highest coefficient of each polynomial is positive. For we can always choose an integer l large enough to make all the coefficients of $Q_\nu(h) = P_\nu(h+l)$ positive.

$b = k/(k+1)$.

q_ν^* denotes the connected number† of q with respect to $P_\nu(h)$.

We write

$$e^{2\pi i x} = e(x), \quad \rho = e(l/q), \quad (l, q) = 1,$$

$$S_{\rho\nu} = \sum_{h=0}^{q_\nu^*} \rho^{P_\nu(h)} \quad (\nu = 1, 2, \ldots, s),$$

$$A(q) = A_s(q, j) = \sum_\rho \left(\prod_{\nu=1}^{s} \frac{S_{\rho\nu}}{q_\nu^*} \right) \rho^{-j},$$

where ρ runs through all primitive q-th roots of unity. In what follows Π and Σ will mean $\prod_{\nu=1}^{s}$ and $\sum_{\nu=1}^{s}$ respectively, unless the contrary is stated.

$$\mathfrak{S}(n) = \sum_{q=1}^{\infty} A_s(q, n),$$

$$T_\nu = T_\nu(\alpha) = \sum_{1 \leqslant \mu \leqslant P} e\{P_\nu(\mu)\alpha\},$$

$$I_\nu = I_\nu(\alpha) = \int_0^P e(\beta a_\nu v^k)\, dv,$$

where a_ν is the highest coefficient in $P_\nu(h)$.

N is the integer to be represented. We can define a constant c such that

$$\max_{1 \leqslant \nu \leqslant s} N_\nu \leqslant cN^{1/k}.$$

P is a number satisfying the condition

$$cN^{1/k} \leqslant P = O(N^{1/k}).$$

3. *Lemmas for exponential sums.*

LEMMA 3.1. *If $Q(h)$ is a polynomial with integral coefficients, and $(l, q) = 1$, then*

$$\sum_{h=1}^{q} e\left(\frac{l}{q} Q(h)\right) = O(q^{1-1/k+\epsilon}),$$

where the O depends only on ϵ and the coefficients of $Q(h)$ [Hua (5)].

† Hua (3), (4).

LEMMA 3.2. *Let $k > 20$ and let N be a sufficiently large integer. Then we can choose P and H, depending only on k, so that*

$$cN^{1/k} \leqslant P = O(N^{1/k}), \quad 0 < H < 9k^3 \log k,$$

$$\int_0^1 |S|^H d\alpha = O(P^{H-k+\frac{1}{36}}),$$

where

$$S = \sum_{x=1}^{P} e\{Q(x)\}$$

and $Q(x)$ is an integral-valued polynomial of the k-th degree.

[This is a slight generalization of Lemma K of Vinogradow (**3**).]

LEMMA 3.3. *If $\alpha_0, \alpha_1, \ldots, \alpha_k$ are real numbers, $k > 20$, and*

$$\left|\alpha_0 - \frac{l}{q}\right| < \frac{1}{q^2}, \quad f(x) = \alpha_0 x^k + \ldots + \alpha_k \quad [(l, q) = 1],$$

then

$$\left|\sum_{x=R+1}^{R+P} e\{f(x)\}\right| = O(P^{1-\frac{1}{18}k^{-3}(\log k)^{-1}}),$$

provided that

$$P^b < q \leqslant P^{k-b}.$$

[Vinogradow (**3**)].

4. *Farey dissection.*

We divide the interval $0 \leqslant \alpha \leqslant 1$ in the usual way into Farey arcs belonging to all rational points l/q with $1 \leqslant q \leqslant P^{k-b}$, $0 \leqslant l < q$, $(l, q) = 1$. We divide these arcs into major arcs $\mathfrak{M}(1 \leqslant q \leqslant P^b)$ and minor arcs $\mathfrak{m}(P^b < q \leqslant P^{k-b})$. In each case the arc has the form

$$\alpha = \frac{l}{q} + \beta, \quad -\vartheta_1 q^{-1} P^{b-k} \leqslant \beta \leqslant \vartheta_2 q^{-1} P^{b-k}, \quad \tfrac{1}{2} \leqslant \vartheta_1 \leqslant 1, \quad \tfrac{1}{2} \leqslant \vartheta_2 \leqslant 1.$$

5. *Lemmas for major arcs.*

LEMMA 5.1. $$S_{\rho\nu} = O(q^{1-1/k}).$$

Proof. Let d be the least common denominator of the coefficients of $P_\nu(h)$. Then, by the definition of the connected number q_ν^*, $qd = q_\nu^* d_1$ for some integer d_1, and we have

$$S_{\rho\nu} = \sum_{h=1}^{q_\nu^*} e\left(\frac{dP_\nu(h)}{q_\nu^* d_1}\right) = \frac{1}{d_1} \sum_{h=1}^{q_\nu^* d_1} e\left(\frac{dP_\nu(h)}{q_\nu^* d_1}\right).$$

Since $dP_\nu(h)$ is a polynomial with integral coefficients and $d \leqslant k!$, we have

$$S_{\rho\nu} = O\{(q_\nu^* d_1)^{1-1/k} d_1^{-1}\} = O(q^{1-1/k}),$$

by Lemma 3.1.

LEMMA 5.2. *If* $s > 2k$, *then*

$$\mathfrak{S} = \mathfrak{S}(n)$$

converges absolutely.

Proof. By Lemma 5.1,

$$\left| \sum_\rho \left(\prod \frac{S_{\rho\nu}}{q_\nu^*} \right) \rho^{-n} \right| \leqslant q, \prod \left| \frac{S_{\rho\nu}}{q_\nu^*} \right| = O(q^{-1-1/k}),$$

which proves the lemma.

LEMMA 5.3.

$$I_\nu = O\{\min(P, |\beta|^{-1/k})\}$$

[Heilbronn (**1**), formula (13)].

LEMMA 5.4. *On* $\mathfrak{M}$ *we have*

$$T_\nu - \frac{S_{\rho\nu}}{q_\nu^*} I_\nu + O(P^b)$$

[Heilbronn (**1**), Lemma 4; Landau (**3**)].

LEMMA 5.5. *If* $|A_\nu - B_\nu| \leqslant C$ *and* $|B_\nu| \leqslant D$, *then*

$$\left| \prod_{\nu=1}^{t} A_\nu - \prod_{\nu=1}^{t} B_\nu \right| \leqslant K(t)(C^t + CD^{t-1}).$$

Proof. The lemma is evident for $t = 1$, and the general case follows by induction.

LEMMA 5.6. *If* $s > 2k+1$, *then*

$$\sum_{\mathfrak{M}} \int_{\mathfrak{M}} \left| \prod T_\nu - \prod \frac{S_{\rho\nu}}{q_\nu^*} I_\nu \right| d\alpha = O(P^{s-k+1+b})$$

[Lemma 5.5 and Heilbronn (**1**), Lemma 5].

LEMMA 5.7. *If* $0 < N \leqslant P^k$, *then*

$$\int_{-\infty}^{\infty} e^{-2\pi i N\beta} \prod I_\nu d\beta = \frac{\Gamma^s(1+1/k)}{\Gamma(s/k)} \prod a_\nu^{-1/k} N^{s/k-1}$$

[Landau (**3**)].

LEMMA 5.8. *If* $0 < N \leqslant P^k$, *then*

$$\int_{\mathfrak{M}} e^{-2\pi i N\beta} \prod I_\nu \, d\beta = \frac{\Gamma^s(1+1/k)}{\Gamma(s/k)} \prod a_\nu^{-1/k} N^{(s/k)-1} + O(P^{s-k-b\{(s/k)-1\}} q^{(s/k)-1})$$

[Heilbronn (1), Lemma 7].

LEMMA 5.9. *If* $0 < N \leqslant P^k$, *then*

$$\sum_{\mathfrak{M}} \int_{\mathfrak{M}} e^{-2\pi i N\alpha} \prod I_\nu \, d\alpha = \prod a_\nu^{-1/k} \frac{\Gamma^s(1+1/k)}{\Gamma(s/k)} \mathfrak{S}(N) N^{(s/k)-1} + O(P^{s-k-(1-b)}).$$

Proof. By Lemma 5.6, the left-hand side of the formula is

$$O(P^{s-k-1+b}) + \sum_{\mathfrak{M}} \left(\prod \frac{S_{\rho\nu}}{q_\nu^*} \right) \int_{\mathfrak{M}} e^{-2\pi i N\alpha} \prod I_\nu \, d\alpha$$

$$= O(P^{s-k-1+b}) + \frac{\Gamma^s(1+1/k)}{\Gamma(s/k)} \prod a_\nu^{-1/k} \sum_{\mathfrak{M}} \prod \frac{S_{\rho\nu}}{q_\nu^*} e^{-2\pi i N(a/q)}$$

$$+ O\left(\sum_{\mathfrak{M}} q^{-s/k} P^{-s-k-b\{(s/k)-1\}} q^{(s/k)-1} \right)$$

(Lemmas 5.8 and 5.9)

$$= O(P^{s-k-(1-b)}) + \frac{\Gamma^s(1+1/k)}{\Gamma(s/k)} \prod a_\nu^{-1/k} N^{(s/k)-1} \{ \mathfrak{S} + \sum_{q > P^b} O(q^{1-(s/k)}) \}$$

$$+ O(P^{s-k-b(s/k-1)}) \sum_{\mathfrak{M}} q^{-1}$$

$$= O(P^{s-k-(1-b)}) + \frac{\Gamma^s(1+1/k)}{\Gamma(s/k)} \mathfrak{S} N^{(s/k)-1} + O(P^{s-k-b\{(s/k)-2\}}).$$

6. *Lemmas for minor arcs.*

LEMMA 6.1. *If* $s > 10k^3 \log k$ *and* $k > 20$, *then*

$$\sum_{\mathfrak{m}} \int_{\mathfrak{m}} \prod_{\nu=1}^{s} T_\nu e^{-2\pi i N\alpha} \, d\alpha = O(P^{s-k-(1-b)}).$$

Proof. Choose H and P to satisfy the requirements of Lemma 3.2. Since the geometric mean is less than the arithmetic mean, we have

$$\int_0^1 \prod_{\nu=1}^{H} |T_\nu| \, d\alpha \leqslant \frac{1}{H} \sum_{\nu=1}^{H} \left(\int_0^1 |T_\nu|^H \, d\alpha \right) = O(P^{H-k+\frac{1}{10}}).$$

Thus, by Lemma 3.3,

$$\sum_{\mathfrak{m}} \int_{\mathfrak{m}} \prod_{\nu=1}^{s} |T_\nu| \, d\alpha = O\left(\max_{\text{on } \mathfrak{m}} \prod_{\nu=H+1}^{s} |T_\nu| \int_0^1 \prod_{\nu=1}^{H} |T_\nu| \, d\alpha\right)$$

$$= O(P^{\{1-\frac{1}{18}k^{-3}(\log k)^{-1}\}(s-H)+H-k+\frac{1}{20}})$$

$$= O(P^{s-k-(1-b)}).$$

Theorem 1. *Let $r(N)$ be the number of solutions of the Diophantine equation*

$$N = P_1(x_1)+\ldots+P_s(x_s), \quad x_\nu \geqslant 0.$$

Then, if $k > 20$ and $s > 10k^3 \log k$, we have

$$r(N) = \left(\prod_{\nu=1}^{s} a_\nu^{-1/k}\right) \frac{\Gamma^s(1+1/k)}{\Gamma(s/k)} \mathfrak{S}(N) N^{(s/k)-1} + O(P^{s-k-(1-b)}).$$

The theorem follows from the Lemmas 5.9 and 6.1.

7. *Lemmas concerning congruences and singular series.*

The following notation will be used throughout the remaining part of this paper:

d_ν is the least common denominator of the coefficients of $P_\nu(h)$.

$d_\nu P_\nu(h) = \phi_\nu(h)$.

θ_ν is the highest power of p such that

$$\phi_\nu'(h) \equiv 0 \pmod{p^{\theta_\nu}}$$

for all values of h.

$P_\nu^*(h) = p^{-\theta_\nu}\phi_\nu'(h)$.

$M(n) = M(P_1, \ldots, P_s, m, n)$ is the number of solutions of

$$(7.1) \quad n \equiv P_1(h_1)+\ldots+P_s(h_s) \pmod{m}, \quad 0 \leqslant h_\nu \leqslant m_\nu^*, \quad \nu = 1, 2, \ldots, s.$$

When $m = p^l$, $N(p^l, n) = N(P_1, \ldots, P_s; p^l, n)$ denotes the number of solutions of the congruence (7.1) in which at least one of the numbers $P_1^*(h_1), \ldots, P_s^*(h_s)$ is prime to p.

$$M(s.P, m, n) = M(P_1, \ldots, P_s; m, n)$$

and

$$N(s.P, p^l, n) = N(P_1, \ldots, P_s, p^l, n),$$

provided that $P_1, \ldots, P_s$ are the same polynomial.

δ is the greatest integer satisfying $p^\delta \leqslant k-1$.

If we define t_ν by $p_\nu^* = p^{1+t_\nu}$, then $d_\nu = p^{t_\nu} d_\nu'$ and $(d_\nu', p) = 1$.

$$\gamma_\nu = \begin{cases} \theta_\nu + 2 - t_\nu + \delta & \text{for } p = 2, \\ \theta_\nu + 1 - t_\nu + \delta & \text{for } p \neq 2. \end{cases}$$

$\gamma = \max(\gamma_1, \ldots, \gamma_s)$.

LEMMA 7.1. *If* $(m^{(1)}, m^{(2)}) = 1$, *then*

$$M(m^{(1)}) M(m^{(2)}) = M(m^{(1)} m^{(2)}).$$

LEMMA 7.2.

$$\sum_{q/m} A(q) = \frac{m}{\prod m_\nu^*} M(m).$$

LEMMA 7.3. *If* $(m^{(1)}, m^{(2)}) = 1$, *then*

$$\sum_{q/m^{(1)} m^{(2)}} A(q) = \sum_{q/m^{(1)}} A(q) \sum_{q/m^{(2)}} A(q).$$

These lemmas can be proved by the method used by Landau (**4**).

LEMMA 7.4. *If* $Q(h)$ *is a residual polynomial of the* k*-th degree* (mod p^α), *and* $Q'(h)$ *is a residual polynomial* (mod p^β), *then*

$$p^{\alpha-\beta} \leqslant k.$$

Proof. (1) First let us consider the special case

$$Q(h) = Q_r(h) = h(h-1) \ldots (h-r+1).$$

In this case, $\alpha = \sum_{l=1}^{\infty} \left[\frac{r}{p^l}\right]$. Since $|Q_r'(i)| = i!(r-i-1)!$, we have

$$\beta = \min_{0 \leqslant i \leqslant r} \left\{ \sum_{l=1}^{\infty} \left(\left[\frac{r-1-i}{p^l}\right] + \left[\frac{r}{p^l}\right] \right) \right\}.$$

If $p^\sigma | r$ and $p^{\sigma+1} \nmid r$, then

$$\alpha - \sum_{l=1}^{\infty} \left[\frac{r-1}{p^l}\right] = \sigma$$

and

$$\sum_{l=1}^{\infty} \left[\frac{r-1}{p^l}\right] - \sum_{l=1}^{\infty} \left(\left[\frac{r-1-i}{p^l}\right] + \left[\frac{i}{p^l}\right] \right) \leqslant \delta_1 - \sigma,$$

where δ_1 is the greatest integer satisfying $p^{\delta_1} \leqslant r$. Hence the lemma is true for $Q_r(h)$.

(2) Since $Q(h)$ can be put into the form

$$Q(h) = \sum_{r=1}^{k} a_r Q_r(h),$$

and $a_r Q_r(h)$ $(r = 1, 2, \ldots, k)$ are residual polynomials (mod p^α), the lemma follows from (1).

LEMMA 7.5. *If* $l \geqq \gamma_\nu + 1$ *and* $h = y + p^{l+t_\nu-\theta_\nu-1}z$, *then*

$$P_\nu(h) \equiv P_\nu(y) + zp^{l-1} P_\nu^*(y) \pmod{p^l}$$

and

$$P_\nu^*(h) \equiv P_\nu^*(y) \pmod{p}.$$

Proof. By Taylor's expansion, we have

$$\phi_\nu(h) \equiv \phi_\nu(y) + zp^{l+t_\nu-\theta_\nu-1}\phi_\nu'(y) + \frac{1}{2!} z^2 p^{2(l+t_\nu-\theta_\nu-1)}\phi_\nu''(y) + \frac{1}{3!} z^3 p^{3(l+t_\nu-\theta_\nu-1)}\phi_\nu'''(y) + \ldots.$$

(1) $p \neq 2$. Since $\phi_\nu''(y)$ is a residual polynomial (mod $p^{\theta_\nu-\delta}$), it follows that

$$\phi_\nu(h) \equiv \phi_\nu(y) + zp^{l+t_\nu-\theta_\nu-1}\phi_\nu'(y) \pmod{p^{l+i_\nu}}.$$

(Notice that

$$p^{i(l+t_\nu-\theta_\nu-1)} \frac{\phi_\nu^{(i)}(y)}{i!} \equiv 0 \pmod{p^{l+t_\nu}} \quad \text{for} \quad i \geqq z\Big).$$

(2) $p = 2$. In this case, $\frac{1}{2}\phi_\nu''(y)$ is a residual polynomial (mod $p^{\theta_\nu-\delta-1}$), and we have the lemma in the same way as in the case (1).

Using the same argument, we get the second conclusion.

LEMMA 7.6. *If* $l \geqq \gamma_\nu + 1$, *then*

$$N(p^l) = p^{s-1} N(p^{l-1}).$$

Proof. In (7.1) let $h_\nu = y_\nu + p^{l+t_\nu-\theta_\nu-1}z_\nu$, where $0 \leqq y_\nu < p^{l+t_\nu-\theta_\nu-1}$, $0 \leqq z_\nu < p^{\theta_\nu+1}$, and p does not divide every $P_\nu^*(y_\nu)$. We obtain

$$\sum_{\nu=1}^{s} P_\nu(y_\nu) + p^{l-1} \sum_{\nu=1}^{s} z_\nu P_\nu^*(y_\nu) \equiv n \pmod{p^l}, \quad \begin{cases} 0 \leqq y_\nu < p^{l+t_\nu-\theta_\nu-1}, \\ 0 \leqq z_\nu < p^{\theta_\nu+1}). \end{cases}$$

Then to each solution of this congruence there corresponds a solution of the two congruences

$$(7.6.1) \quad \sum_{\nu=1}^{s} P_\nu(y_\nu) \equiv n \pmod{p^{l-1}}, \quad 0 \leqq y_\nu < p^{l+t_\nu-\theta_\nu-1}, \quad p \nmid \text{every } P_\nu^*(y_\nu),$$

$$(7.6.2) \quad \sum_{\nu=1}^{s} z_\nu P_\nu^*(y_\nu) \equiv p^{-(l-1)}\Big(n - \sum_{\nu=1}^{s} P_\nu(y_\nu)\Big) \pmod{p}, \quad 0 \leqq z_\nu < p^{\theta_\nu+1};$$

and conversely.

We shall prove next that the former has $\Pi p^{-\theta_\nu} N(p^{l-1})$ and the latter $p^{-1}\Pi p^{\theta_\nu+1}$ solutions. It follows from Lemma 7.5 that, if $x_\nu \equiv y_\nu \pmod{p^{l+t_\nu-\theta_\nu-1}}$, then $P_\nu(x_\nu) \equiv P_\nu(y_\nu) \pmod{p^{l-1}}$.

Hence, each solution of (7.6.1) gives Πp^{θ_ν} solutions of

$$\sum_{\nu=1}^{s} P_\nu(y_\nu) \equiv n \pmod{p^{l-1}}, \quad 0 \leqslant y_\nu < p^{l+t_\nu-1}, \quad p \nmid \text{every } P_\nu^*(y_\nu).$$

Since p does not divide every $P_\nu^*(y_\nu)$, we may assume that $p \nmid P_1^*(y_1)$. Then $z_\nu (\nu = 2, 3, \ldots, s)$ may be chosen arbitrarily $(\bmod p^{\theta_\nu+1})$ in (7.6.2) and z_1 is determined uniquely $(\bmod p)$. Thus, for any set of z_ν $(\bmod p^{\theta_\nu+1})$ $(\nu = 2, 3, \ldots, s)$, there are p^{θ_1} solutions z_1 for which $0 \leqslant z_1 < p^{\theta_1+1}$. Hence (7.6.2) has $p^{-1} \prod_{\nu=1}^{s} p^{\theta_\nu+1}$ solutions.

As a consequence of the above lemma we have

LEMMA 7.7. *If $l \geqslant \gamma$, then*

$$N(p^l) = p^{(l-\gamma)(s-1)} N(p^\gamma).$$

LEMMA 7.8. *If $M(P_1, \ldots, P_{s-1}, p^l, n) > 0$ for all integers n, then*

$$N(P_1, \ldots, P_{s-1}, P_s, p^l, n) > 0$$

for all integers n.

Proof. Since $P_s^*(h)$ is not congruent to zero $(\bmod p)$ identically, there is an integer h_s such that $p \nmid P_s^*(h_s)$, $0 \leqslant h < p^{l+t_s}$. By hypothesis, we see that the equation

$$\sum_{\nu=1}^{s-1} P_\nu(h_\nu) = n - P_s(h_s) \pmod{p^l}$$

has at least one solution and hence the lemma is true.

Let H denote the hypothesis that

$$N(P_1, \ldots, P_{s_0}, n, p^\gamma) > 0$$

for all primes p and all integers n.

LEMMA 7.9. *If H is true, and if $s \geqslant \max(s_0, 2k)$, then*

$$\mathfrak{S}(n) \geqslant D_1 > 0,$$

where D_1 (and later $D_2, \ldots, D_4$) is a positive constant depending on the coefficients of $P_\nu(h)$ $(\nu = 1, 2, \ldots, s)$.

Proof. By Lemma 7.2, we have

$$\sum_{q/p^l} A(q) = \frac{p^l}{\Pi p^{l+t_\nu}} M(p^l).$$

By Lemma 7.7,

$$\sum_{q/p} A(q) \geqslant p^{-l(s-1)-\Sigma t_\nu} N(p^l) \tag{7.9.1}$$

$$\geqslant p^{-\gamma(s-1)-\Sigma t_\nu},$$

which is independent of l. By Lemma 5.1,

$$\sum_{q/p^l} A(q) = 1 + \sum_{\lambda=1}^{l} A(q^\lambda) > 1 - D_2 \sum_{\lambda=1}^{\infty} p^{-\lambda/b}$$

$$= 1 - D_2/(p^{1/b}-1).$$

If $p > D_3 = (D_2+1)^b$, then

$$\prod_{D_3<p\leqslant p^l} \sum_{q/p^l} A(q) > \prod_{D_3<p} \{1-D_2/(p^{1/b}-1)\},$$

where p_l is the l-th prime. The right-hand side converges. Further, by (7.9.1), we have

$$\prod_{p\leqslant D_3} \sum_{q/p^l} A(q) > D_4,$$

which is independent of l.

Let l tend to infinity. We have, by Lemma 7.3,

$$\mathfrak{S}(n) \geqslant D_1 > 0.$$

Thus we have

THEOREM 2. *If H is true, and $s > \max(s_0-1,\ 10k^3 \log k)$, then*

$$r_{P_1, \ldots, P_s}(n) > 0.$$

We know [Hardy-Littlewood, (1)] that hypothesis H is satisfied when $s_0 = 4k$ and $P_1 = P_2 = \ldots = P_{4k} = x^k$. *A fortiori*, it is satisfied when $P_1 = P_2 = \ldots = P_{4k} = x^k$ and $P_{4k+1}, \ldots, P_{s_0}$ are arbitrary. Consequently we have

THEOREM 3. *If $P_{4k+1}(h), \ldots, P_s(h)$ are any integral-valued polynomials of the k-th degree, and $s > 10k^3 \log k$, then, if N is a sufficiently large positive integer, we can choose positive integers $h_1, h_2, \ldots, h_s$ such that*

$$N = h_1^k + \ldots + h_{4k}^k + P_{4k+1}(h_{4k+1}) + \ldots + P_s(h_s) \quad (h_\nu \geqslant 0).$$

More generally, we have

THEOREM 4. *If $a_1, a_2, \ldots, a_t$ is an admissible set as defined by Huston* (1), *and $P_{t+1}(h), \ldots, P_s(h)$ are any integral-valued polynomials, then, if N is*

sufficiently large and $s > 10k^3 \log k$, *the Diophantine equation*

$$N = a_1 h_1^k + \ldots + a_t h_t^k + P_{t+1}(h_{t+1}) + \ldots + P_s(h_s)$$

has a solution in positive integers.

In the remaining part of this paper the indices ν of θ_ν, t_ν, ϕ_ν, ... are suppressed.

8. *Lemmas for* θ.

LEMMA 8.1. *If* $p > k$, *then* $\theta = 0$ *and* $t = 0$.

LEMMA 8.2. *If* $p \leqslant k$, *then*

$$p^\theta \leqslant k 2^{k-1}.$$

Proof. Let

$$\phi(h) = c_0 h^k + \ldots + c_{k-1} h.$$

Then $$p \nmid (c_0, \ldots, c_{k-1}).$$

Let δ_1 be the greatest power of p for which

$$p^{\delta_1} \mid \{kc_0, (k-1)c_1, \ldots, 2c_{k-2}, c_{k-1}\},$$

and let δ_2 be the greatest power of p for which there may exist a residual polynomial (mod p^{δ_2}) of the $(k-1)$-th degree of which at least one coefficient is not divisible by p. Then

$$\theta \leqslant \delta_1 + \delta_2.$$

Since $p \nmid (c_0, c_1, \ldots, c_{k-1})$, p^{δ_1} must divide one of the integers $1, 2, \ldots, k$. Therefore $p^{\delta_1} \leqslant k$.

Since $$\delta_2 \leqslant \sum_{l=1}^{\infty} \left[\frac{k-1}{p^l}\right] \leqslant \frac{k-1}{p-1},$$

we have $$p^{\delta_2} \leqslant p^{(k-1)/(p-1)} = e^{(k-1)\log p/(p-1)}.$$

When x takes all positive integral values greater than or equal to 2, the greatest value of $\log x/(x-1)$ is given by $x = 2$. Thus

$$p^\theta \leqslant p^{\delta_1+\delta_2} \leqslant k 2^{k-1}.$$

Consequently we have

LEMMA 8.3. *If* $p < k$ *and* $k \geqslant 4$, *then*

$$p^\gamma \leqslant k^3 2^{k-1}.$$

9. *The relation between Tarry's problem and Waring's problem for odd polynomials.*

Let $P(h)$ be an odd polynomial. Then

$$P(h) = b_0 Q_{2l+1}(h) + b_1 Q_{2l-1}(h) + \ldots + b_l Q_1(h),$$

where
$$Q_{2\mu+1}(h) = \frac{h(h^2-1)\ldots(h^2-\mu^2)}{(2\mu+1)!},$$

the b's are integers, and

$$(b_0, b_1, \ldots, b_l) = 1$$

[*cf.* Hua (**3**), Lemmas 2 and 4].

LEMMA 9.1. *If* $s \geqslant \frac{1}{3}(2^{k+1}-1)$, *then*

$$N(s.P, p^l, n) > 0$$

[Hua (**3**), Lemma 40].

In §§ 9-10 it will be shown that, if we have found a particular solution (non-trivial) of Tarry's problem for a definite $k = k_0$, then we can derive from it an upper bound of $G\{P(h)\}$ for all $k \geqslant k_0$.

Let us suppose that the particular solution is $a_1, \ldots, a_{j_0}; b_1, \ldots, b_{j_0}$, that is to say

$$(9.1) \qquad \begin{cases} a_1^h + \ldots + a_{j_0}^h = b_1^h + \ldots + b_{j_0}^h \quad (1 \leqslant h \leqslant k_0 - 2), \\ a_1^{k_0-1} + \ldots + a_{j_0}^{k_0-1} \neq b_1^{k_0-1} + \ldots + b_{j_0}^{k_0-1}. \end{cases}$$

Let
$$a_1^{k_0-1} + \ldots + a_{j_0}^{k_0-1} - b_1^{k_0-1} - \ldots - b_{j_0}^{k_0-1} = X(a, b, k_0),$$

and let the greatest prime factor of X be $\mathfrak{p}$. For every solution of (9.1) we have a corresponding number $q = \max(j_0, \mathfrak{p})$. The least of the q's is denoted by $T(k_0)$.

LEMMA 9.2. *Let* $k \geqslant k_0$,

$$k-1 = l(k_0-1)+m, \quad l > 0, \quad 0 \leqslant m < k_0 - 1,$$

$j \geqslant 2^{m+l-1} j_0^{\,l}$. *Then there is a set of integers* $x_1, \ldots, x_j; y_1, \ldots, y_j$ *such that*

$$x_1^h + \ldots + x_j^h = y_1^h + \ldots + y_j^h \quad (1 \leqslant h \leqslant k-2),$$

$$x_1^{k-1} + \ldots + x_j^{k-1} \neq y_1^{k-1} + \ldots + y_j^{k-1}$$

and the greatest prime factor of

$$X(k) + x_1^{k-1} + \ldots + x_j^{k-1} - y_1^{k-1} - \ldots - y_j^{k-1}$$

does not exceed the greater of k *and* $\mathfrak{p}$.

Proof. By the same argument as that used by Maitland Wright (3), we can construct a polynomial

$$f_0(y) = (y-1)^{k_0-1} h_0(y) = \sum_{i=1}^{j_0} y^{a_i} - \sum_{i=1}^{j_0} y^{b_i};$$

and it is easy to verify that

$$\left\{\left(y\frac{d}{dy}\right)^{k_0-1} f_0(y)\right\}_{y=1} = (k_0-1)!\, h_0(1) = X(k_0).$$

Construct

$$f(y) = (y-1)^m \{f_0(y)\}^l = (y-1)^{k-1}\{k_0(y)\}^l.$$

Then $f(y)$ contains at most $2^m(2j_0{}^l)$ terms, each of the type $\pm y^a$. Further,

$$X(k) = \left\{\left(y\frac{d}{dy}\right)^{k-1} f(y)\right\}_{y=1} = (k-1)!\,\{h_0(1)\}^l,$$

and the lemma is proved.

LEMMA 9.3. *If $p > k$ and $s \geqslant \max\{\mathfrak{p},\ 2^{m+l-1} j_0{}^l,\ \frac{1}{3}(2^{k_0+1}-1)\}$, then*

$$N(s\,.\,P,\ p^\gamma,\ n) > 0$$

(actually $\gamma = 1$) for all odd polynomials $P(h)$.

It is worth while to notice that when k increases, only $2^{m+l-1} j_0{}^l$ increases, the other terms remain unchanged.

Proof. (1) If $p \leqslant \max(\mathfrak{p}, k)$, the conclusion is evident.

(2) If $p > \max(k, \mathfrak{p})$ (here $\gamma = 1$), then $p + X(k)$. If $p + a$ [a, which is not necessarily an integer, is the first coefficient in $P(h)$ and its denominator is relatively prime to p], then, if n is any integer, we can choose x_i, y_j, and x so that $j \leqslant 2^{m+l-1} j_0{}^l$ and

$$\sum_{i=1}^{j} \{P(x+x_i) - P(x+y_i)\} = aX(k)x + Y(k) \equiv n \pmod{p}.$$

If $p \mid a$, then $P(h) \equiv Q(h) \pmod{p}$, where $Q(h)$ is a polynomial whose degree is less than that of $P(h)$, and if its degree exceeds k_0, we can apply the same argument to $Q(h)$. If $Q(h)$ is a polynomial of the k_0-th degree or less, then, by Lemma 9.1, if n is any integer and $s \geqslant \frac{1}{3}(2^{k_0+1}-1)$, the congruence

$$\sum_{\nu=1}^{s} Q(h_\nu) \equiv n \pmod{p}$$

has a solution. The lemma is therefore proved.

Lemma 9.4. *If* $\epsilon > 0$, *then*

$$2^{m+l-1} j_0{}^l = O(2^{\epsilon h}) \quad \text{as} \quad k \to \infty.$$

Proof. If k_0 is large, it follows from Maitland Wright's result (**4**) that

$$2^{m+l-1} j_0{}^l \leqslant 2^{m+l-1} \left(\frac{7k_0{}^2(k_0-11)^2}{72} + 56\right)^l \leqslant 2^{m+l-1} k_0^{4l}$$

$$\leqslant 2^{k_0-1-(k-1)/(k_0-1)-1} k_0^{4(k-1)/(k_0-1)}$$

$$\leqslant 2^{k_0-2-(k-1)(1+4\log k_0/\log 2)/(k_0-1)}.$$

Since $$\lim_{k_0 \to \infty} \frac{1}{k_0-1}(1+4\log k_0/\log 2) = 0,$$

for any given $\epsilon > 0$, there is a k_0 such that

$$\frac{1}{k_0-1}(1+4\log k_0/\log 2) < \epsilon,$$

and this proves our lemma.

Lemma 9.5. *If* $k \geqslant 10$, $p > k$, *and* $s \geqslant 2^{\frac{1}{2}(k-1)+7}$, *and* $P(h)$ *is an odd polynomial, then*

$$N(s\,.\,P,\ p^\gamma,\ n) > 0.$$

Proof. Since

$$1^h + 5^h + 10^h + 24^h + 28^h + 42^h + 47^h + 51^h$$

$$= 2^h + 3^h + 12^h + 21^h + 31^h + 40^h + 49^h + 50^h \quad (1 \leqslant h \leqslant 7),$$

and

$$1^8 + 5^8 + 10^8 + 24^8 - 28^8 + 42^8 + 47^8 + 51^8$$

$$- 2^8 - 3^8 - 12^8 - 21^8 - 31^8 - 40^8 - 49^8 - 50^8 = 2^{11}\,.\,3^3\,.\,5^2\,.\,7^2\,.\,11\,.\,13,$$

it follows that

$$k_0 = 9, \quad j_0 = 8, \quad \mathfrak{p} = 13.$$

Here $$\max\{\mathfrak{p},\ \tfrac{1}{3}(2^{k_0+1})\} = 341$$

and $$2^{m+l-1} j_0{}^l - 2^{m+4l-1} \leqslant 2^{k_0-1-4(k-1)/(k_0-1)-1} = 2^{\frac{1}{2}(k-1)+7}.$$

The conclusion follows from Lemma 9.3.

By Theorem 2 and Lemma 9.1 we have

THEOREM 5. *If $k > 20$, every sufficiently large integer can be expressed as the sum of $\frac{1}{3}(2^{k+1}-1)$ values, for $h \geqslant 0$, of an odd polynomial $P(h)$ of the k-th degree.*

THEOREM 6. *If the highest coefficient of $P(h)$ is $a/k!$, $(a, k!) = 1$, then*

$$G\{P(h)\} = O(2^{ek}).$$

This theorem follows from Lemma 9.4 and the fact that $\gamma \leqslant \delta + 2$.

10. *Lemmas for general polynomials.*

Let $K(2l)$ be the least s such that the system of congruences

$$x_1^{2\nu} + \ldots + x_s^{2\nu} \equiv 0 \pmod{p}, \quad p \nmid x_1, \quad \nu = 1, \ldots, l$$

has a solution. We assume that $p > k$ and $2l \leqslant k$.

LEMMA 10.1. *If*

$$x_1^{\mu} + \ldots + x_{s_1}^{\mu} \equiv 0 \pmod{p}, \quad p \nmid x_1$$

and

$$y_1^{\nu} + \ldots + y_{s_2}^{\nu} \equiv 0 \pmod{p}, \quad p \nmid y_1,$$

then

$$\sum_{t_2=1}^{s_2} \sum_{t_1=1}^{s_1} (x_{t_1} y_{t_2})^{\nu} \equiv 0 \pmod{p}$$

and

$$\sum_{t_2=1}^{s_2} \sum_{t_1=1}^{s_1} (x_{t_1} y_{t_2})^{\mu} \equiv 0 \pmod{p}.$$

LEMMA 10.2. *There is an integer $m \leqslant 2l$ such that $m \mid p-1$, and there is no integer $\nu \leqslant l$ for which*

$$m \mid (p-1, 2\nu) \quad \text{and} \quad \frac{(p-1, 2\nu)}{m} > 1.$$

These two lemmas are evident.

LEMMA 10.3. *If m is chosen as in Lemma 10.2 and $s \geqslant m(m+1)^{[2l/m]}$, then the system of congruences*

$$x_1^{2\nu} + \ldots + x_s^{2\nu} \equiv 0 \pmod{p}, \quad p \nmid x_1, \quad \nu = 1, \ldots, l \tag{10.3}$$

has a solution.

Proof. Since $m \mid (p-1)$, there exists an integer g belonging to $m \pmod p$. Thus

$$1^{2\nu}+g^{2\nu}+(g^2)^{2\nu}+\ldots+(g^{m-1})^{2\nu} \equiv 0 \pmod p \tag{10.3.1}$$

for all ν's satisfying $m+2\nu$. The other ν's which have not been discussed are those which satisfy

$$(2\nu,\ p-1) = m.$$

By Satz 301 of Landau (**1**), we can choose $x_1, \ldots, x_{m+1}$ so that

$$x_1^{2\nu}+\ldots+x_{m+1}^{2\nu} \equiv 0 \pmod p, \quad p+x_1.$$

There are at most $[2\nu/m]$ ν's having this property. Therefore, by Lemma 10.1, the lemma is proved.

LEMMA 10.4.

$$K(2l) \leqslant 2l \cdot 3^l.$$

If $l \geqslant 22$, *then*

$$K(2l) < 3^l.$$

Proof. (1) Here

$$(m+1)^{[2l/m]} \leqslant (m+1)^{2l/m} = e^{2l \log(m+1)/m}.$$

Since $\log(x+1)/x$ decreases as x increases for $x > e-1$, we have

$$(m+1)^{[2l/m]} \leqslant 3^l.$$

(2) If $m = 2$, since there is no congruence of the type (10.3.1), we can prove that

$$K(2l) < 3^l.$$

If $m \geqslant 3$,

$$K(2l) < m(m+1)^{[2l/m]} \leqslant 2l \cdot 4^{\frac{3}{4}l};$$

and if $l \geqslant 22$,

$$2l \cdot 4^{\frac{3}{4}l} \leqslant 3^l.$$

LEMMA 10.5. *If* $s > K(2l)$, *and* t *is any odd integer, then there are integers* $h_1, \ldots, h_s$ *such that*

$$h_1^{2r}+\ldots+h_s^{2r} \equiv 0 \pmod p, \quad l \geqslant r \geqslant 1, \tag{10.5.1}$$

and

$$h_1^t+\ldots+h_s^t \not\equiv 0 \pmod p. \tag{10.5.2}$$

Proof. Since, by Lemma 10.4, we can choose $h_1, \ldots, h_s$ satisfying (10.3) and

$$h_1^t+h_2^t+\ldots+h_s^t \not\equiv (-h_1)^t+h_2^t+\ldots+h_s^t \pmod{p},$$

the conclusion follows immediately.

Every integral-valued polynomial $R(h)$ with no constant term can be written in the form

$$R(h) = \tfrac{1}{2}\{h\,R_1(h)+R_2(h)\},$$

where $R_1(h)$ and $R_2(h)$ are two odd polynomials. It is evident that $hR_1(h)$, $R_2(h)$ are both integral-valued. Let the degrees of $hR_1(h)$ and $R_2(h)$ be $2t_1$ and $2t_2+1$ respectively. We can write

$$R_2(h) = \sum_{\nu=1}^{t_2} c_\nu Q_{2\nu+1}(h),$$

where the c's are integers.

LEMMA 10.6. *If there is no integer a such that $P(h) \equiv a \pmod{p}$, then there is an integer q such that the coefficients c_ν of $R_2(h)$ deduced from $R(h) = P(h+q)$ satisfy the condition*

$$p+(c_1, \ldots, c_{t_2}) \quad (p \neq 2).$$

Proof. Evidently

$$P(h)-R(-h) = R_2(h).$$

If the conclusion is false, we have, since $R_2(0) = 0$,

$$P(h+q)-P(-h+q) \equiv 0 \pmod{p}$$

for all values of h and q. In particular, we take $h = q$. Then

$$P(2h) \equiv 0 \pmod{p}$$

for all values of h. This contradicts our hypothesis.

Let $\Gamma\{P(h), p\}$ be the least integer s for which

$$\sum_{\nu=1}^{s} P(h_\nu) \equiv n \pmod{p^\gamma}$$

has a solution for all n.

LEMMA 10.7. *If $p > k$, there exists an odd polynomial $Q(h) \not\equiv 0 \pmod p$, of degree not greater than $2t_1+1$, such that, if*

$$s \geqslant K(2t_1)\,\Gamma\{Q(h), p\},$$

then the congruence

$$\sum_{\nu=1}^{s} P(h_\nu) \equiv n \pmod p$$

has a solution for every integer n.

Proof. By Lemma 10.6, there exists an integer q such that

$$R_2(h) \not\equiv 0 \pmod p.$$

Thus we can define an integer τ such that

$$p \mid (c_{t_2}, \ldots, c_{t_2-\tau+1}), \quad p \nmid c_{t_2-\tau}, \quad 0 \leqslant \tau < t_2.$$

Let $h_1, \ldots, h_{s_1}$ $\{s_1 = K(2t_1)\}$ be the integers satisfying

$$h_1^{2\nu} + \ldots + h_s^{2\nu} \equiv 0 \pmod p, \quad \nu = 1, \ldots, t_1,$$

$$h_1^{2(t_2-\tau)+1} + \ldots + h_s^{2(t_2-\tau)+1} \not\equiv 0 \pmod p. \tag{10.7.1}$$

Consider

$$\Psi(h) = \sum_{\nu=1}^{s_1} R(h_\nu h) \pmod p.$$

Here $\Psi(h)$ is an odd polynomial, and, by (10.7.1),

$$\Psi(h) \not\equiv 0 \pmod p.$$

The lemma follows immediately.

By Lemmas 8.3, 9.3, 9.4, 10.4, and 10.7, we obtain

LEMMA 10.8. *If $\epsilon > 0$, then*

$$\Gamma\{P(h), p\} = O(3^{k(\frac{1}{2}+\epsilon)}), \quad \textit{for all} \quad p > k.$$

THEOREM 7.

$$G\{P(h)\} = O(k^3\, 2^{k-1}).$$

By Lemmas 8.3, 9.5, 10.4, and 10.7, we obtain

LEMMA 10.9. *If $k \geqslant 15$, then*

$$\Gamma\{P(h), p\} \leqslant \max\,(k3^{[\frac{1}{3}k]}\, 2^{\frac{1}{2}(k-1)+7},\ k^3\, 2^{k-1}) \leqslant k3^{\frac{1}{3}k}\, 2^{\frac{1}{2}(k-1)+7}.$$

Consequently we have

THEOREM 8. *If $k > 20$, every sufficiently large integer is the sum of $[k3^{\frac{1}{2}k}2^{\frac{1}{2}(k-1)+7}]$ values, for non-negative values of h, of a polynomial $P(h)$ of the k-th degree.*

THEOREM 9. *If the coefficient of $P(h)$ is $a/k!$, $(a, k!) = 1$, then*

$$G\{P(h)\} = O(3^{k(\frac{1}{2}+\epsilon)}).$$

11. *Some further results.*

In this section we describe some methods which can be applied to special polynomials and indicate briefly the more interesting results obtained by them.

(1) We can prove by Heilbronn's method that, if $4k$ of the integers

$$a_1, a_2, \ldots, a_s$$

form an admissible set as defined by Huston, then

$$a_1 h_1^k + \ldots + a_s h_s^k = n$$

has a solution in non-negative integers for all large integers n, provided that

$$s \geqslant 6k \log k + \left\{4 + 3 \log\left(3 + \frac{2}{k}\right)\right\} k + 3.$$

This includes one of Dickson's results [Dickson (**1**)] on the generalized Waring problem.

(2) If we replace Lemmas 3.2 and 3.3 by Weyl's approximation and use Wright's method [Wright (**2**), (**3**)], we can show that a generalization of Wright's asymptotic formula holds for polynomials, since

$$s \geqslant 2^{k-1}(k-2)+5 \quad (k \geqslant 3).$$

We can deduce from this the following results for polynomials of lower degree.

Let $G^*\{P(h)\}$ be the best possible integer for which the Waring-Kamke problem is solved with proportional conditions. Here certainly $P(h)$ has the property that there exist no integers c and d $(d > 1)$ such that

$$P(h) \equiv c \pmod{d}.$$

The results can be summarized as follows.

1. If $P(h)$ is a cubic integral-valued polynomial, then

$$G^*\{P(h)\} \leqslant 9.$$

This is better than James's result [James (**1**)], since the restriction upon the coefficients is removed. *Cf.* also James (**2**), Hua (**1**).

2. $$G^*\{\tfrac{1}{12}A(h^4-h^2)+Bh^2\} \leqslant 26 \quad [(A, 6B)=1].$$

3. $$G^*\{\tfrac{1}{12}A(h^4-h^2)+Bh^2\} \leqslant 31 \quad [(A, B)=1].$$

4. If $P(h)$ is a quartic integral-valued polynomial, then

$$G^*\{P(h)\} \leqslant 65.$$

5. If $P(h)$ is a quintic integral-valued polynomial, then

$$G^*\{P(h)\} \leqslant 81.$$

6. If $P(h)$ is an even integral-valued polynomial of the 6-th degree, then

$$G^*\{P(h)\} \leqslant 2304.$$

7. If $P(h)$ is an integral-valued polynomial of the 6-th degree, then

$$G^*\{P(h)\} \leqslant 4627.$$

8. If $P(h)$ is an integral-valued polynomial of the 7-th degree, then

$$G^*\{P(h)\} \leqslant 4691.$$

(3) In the case when $P(h)$ is an odd polynomial of the k-th degree, it can be shown by the method used by Hardy and Littlewood (**2**) that

$$G\{P(h)\} \leqslant (k-2)\,2^{k-2}+k+5+\left[\frac{(k-2)\log 2-\log k+\log(k-2)}{\log k-\log(k-1)}\right].$$

Although this upper bound is not very good for large k, it gives us some interesting results for $k \leqslant 20$ [Hua (**3**)].

(4) The upper bound in Lemma 8.3 can be improved by choosing special polynomials. There are a large number of them, for example, for which $\phi(0)$ (defined on p. 167) is equal to 1. In particular, it is easy to see that

$$G\{\tfrac{1}{2}(x^h+x)\} \leqslant 10k^3 \log k.$$

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EIN MITTELWERT EPSTEINSCHER ZETAFUNKTIONEN (II)

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I. Es seien a, b, c ganze rationale Zahlen; $a>0$, $d=b^2-4ac<0$ eine Fundamentaldiscriminante, $Q(x, y)=ax^2+bxy+cy^2$; für $s=\sigma+it$, $\sigma>1$ sei

$$Z_Q(s)=\sum_{\substack{m,\ n=-\infty \\ m,\ n\neq 0,\ 0}}^{\infty} Q(m, n)^{-s}=\sum_{n=1}^{\infty}\frac{r_Q(n)}{n^s},$$

$r_Q(n)$ die Anzahl der Darstellungen von n durch die Form $Q(x, y)$.

In einer früheren Arbeit* bewies ich, dass für $T\to\infty$

$$\int_0^T |Z_Q(\tfrac{1}{2}+it)|^2\,dt\sim a_d\,T\log^2 T, \tag{1.1}$$

wenn die Klassenzahl $h=1$ ist, oder wenn $d=-15, -20, -24$ $(h=2)$ ist, wobei a_d eine bestimmte, nur von d abhängige, von Q sonst unabhängige Konstante ist. Die dort ausgesprochene Vermutung, dass (1.1) für jede negative Fundamentaldiscriminante gilt, lässt sich nun unter Heranziehung der Lehre von den Idealzahlen beweisen; somit tritt zu den bereits bekannten Invarianten gegenüber den verschiedenen Klassen inbezug auf eine negative Fundamentaldiscriminante der Mittelwert auf der kritischen Geraden als weitere Invariante hinzu.

LEHRSATZ. *Es durchlaufe q die τ Primteiler von $-d$, dann ist für $T\to\infty$*

$$\int_0^T |Z_Q(\tfrac{1}{2}+it)|^2\,dt\sim\frac{12}{|d|}\prod_{q\mid -d}\frac{2q}{q+1}\,T\log^2 T. \tag{1.2}$$

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* "Ein Mittelwert Epsteinscher Zetafunktionen", *Proc. London Math. Soc.* (2), 42 (1937), 128–141.

Im folgenden sei der Einfachheit halber stets $d < -4$ angenommen. Bekanntlich lassen sich dann die verschiedenen nicht äquivalenten Formen der Discriminante d den verschiedenen Idealklassen des Körpers $P(\sqrt{d})$ so zuordnen, dass die Anzahl der Darstellungen einer positiven ganzen Zahl n durch eine Form genau doppelt so gross wie die Anzahl ganzer Ideale mit der Norm n in der entsprechenden Klasse ist. Es seien Q_j und Q_l irgend zwei Formen der Discriminante d, $\mathfrak{K}_j$ und $\mathfrak{K}_l$ die entsprechenden Idealklassen, $Z_j(s)$ und $Z_l(s)$ die entsprechenden Zetafunktionen, $c_{j,l}$ gleich 1 oder 0, je nachdem $\mathfrak{K}_j\,\mathfrak{K}_l$ das Quadrat einer Idealklasse ist oder nicht, dann wird hier allgemeiner bewiesen werden: Für $\epsilon > 0$, $\epsilon \to 0$ ist

$$(1) \qquad \int_0^\infty Z_j(\tfrac{1}{2}+it)\,Z_l(\tfrac{1}{2}-it)\,e^{-\epsilon t}\,dt = \frac{12}{|d|}\,c_{j,l}\prod_{q|d}\frac{2q}{q+1}\,\frac{\log^2\epsilon}{\epsilon}+O\left(\frac{\log\epsilon^{-1}}{\epsilon}\right).$$

Diese Gleichung besagt in den Fällen der Klassenzahl 1 gegenüber (1.2) nichts Neues. Für beliebiges h folgt aber (1.2) aus (1) nach einem bekannten Tauberschen Satze, da dann $\mathfrak{K}_j\,\mathfrak{K}_l$ gleich $\mathfrak{K}_j{}^2$ ist. Besonders einfache Ergebnisse erhält man, wenn h eine ungerade Primzahl und wenn h gleich 2 oder 2^{r-1} ist.

Satz (6), mit etwas anderem Beweise, stammt von O. Taussky, Cambridge, ich spreche hier meinen besten Dank aus.

II. Es sei wieder $\rho = \sqrt{(-d)} > 0$, $0 < \epsilon < \frac{1}{2}\pi$, $\phi = \frac{1}{2}\pi - \epsilon$, die Gesamtmenge aller Primzahlen sei zerlegt in die drei Teilmengen der Primzahlen P, p, q, welche bestimmt sind durch

$$\left(\frac{d}{P}\right) = 1, \quad \left(\frac{d}{p}\right) = -1, \quad \left(\frac{d}{q}\right) = 0,$$

$$F_Q(y) = e^{\frac{1}{2}y}\left(\sum_{m,\,n=-\infty}^{\infty} e^{-(2\pi/\rho)Q(m,\,n)\,e^{\,+i\phi}} - 1 - e^{-y-i\phi}\right),$$

$$G_Q(t) = \frac{1}{2\pi}\,\rho^{\frac{1}{2}}\,Z_Q(\tfrac{1}{2}+it)\,\Gamma(\tfrac{1}{2}+it)\left(\frac{\rho}{2\pi}\right)^{it} e^{\phi t - i\phi/2}.$$

Dann ist*

$$F_Q(y) = (2\pi)^{-\frac{1}{2}}\int_{-\infty}^{\infty} G_Q(t)\,e^{-iyt}\,dt,$$

also nach der Parsevalschen Formel, wenn $F_j(y)$ die Bedeutung $F_{Q_j}(y)$ hat, usw.

$$\int_{-\infty}^{\infty} F_j(y)\,\overline{F}_l(y)\,dy = \int_{-\infty}^{\infty} G_j(t)\,\overline{G}_l(t)\,dt$$

$$(j = 1, 2, \ldots, h;\ l = 1, 2, \ldots, h),$$

* *Cf.* E. C. Titchmarsh, "The mean value of the zeta-function on the critical line" *Proc. London Math. Soc.* (2), 27 (1928), 137–150.

und hieraus folgt schliesslich für $\epsilon \to 0$, festes $\delta > 0$

$$(2)\quad \int_0^\infty Z_j(\tfrac{1}{2}+it)\, Z_l(\tfrac{1}{2}-it)\, e^{-2t\epsilon}\, dt = \frac{1}{\sin\epsilon} \sum_{n=1}^{\infty} \frac{r_j(n)\, r_l(n)}{n}\, e^{-(4\pi/\rho)\, n \sin\epsilon} + 2\Re \left\{ \sum_{\substack{m,\, n=1 \\ m \neq n}}^{\infty} r_j(m)\, r_l(n)\, \frac{e^{-(2\pi/\rho)(me^{i\phi}+ne^{-i\phi})}}{me^{i\phi}+ne^{-i\phi}} \right\} + O(\epsilon^{-\delta})$$

Es muss jetzt, entsprechend den Überlegungen der früheren Arbeit, gezeigt werden, dass jede der zunächst für $\sigma > 1$ definierten Funktionen

$$(2.11)\qquad \psi_{j,l}(s) = \sum_{n=1}^{\infty} r_j(n)\, r_l(n)\, n^{-s} \qquad (j = 1, 2, \ldots, h;\ l = 1, 2, \ldots, h)$$

auch in einem Streifen, der die Gerade $\sigma = 1$ in seinem Inneren enthält, bis auf höchstens einen Doppelpol im Punkte $s = 1$ regulär und für $|t| \to \infty$ gleich $O(|t|^A)$ ist, dass also auch für $s \to 1$

$$(2.1)\qquad \psi_{j,l}(s)\, (s-1)^2 \to a_{j,l} \quad (|a_{j,l}| < \infty)$$

ist; dann folgt für $\epsilon \to 0$

$$\int_0^\infty Z_j(\tfrac{1}{2}+it)\, Z_l(\tfrac{1}{2}-it)\, e^{-2\epsilon t}\, dt = \frac{a_{j,l}}{2}\, \frac{\log^2 \epsilon}{\epsilon} + O\left(\frac{\log \epsilon^{-1}}{\epsilon}\right),$$

$$(2.2)\qquad \int_0^\infty Z_j(\tfrac{1}{2}+it)\, Z_l(\tfrac{1}{2}-it)\, e^{-\epsilon t}\, dt = a_{j,l}\, \frac{\log^2 \epsilon}{\epsilon} + O\left(\frac{\log \epsilon^{-1}}{\epsilon}\right) *.$$

* Die Überlegungen entsprechen denen in Abschnitt V der früheren Arbeit. Es ist nur zu beachten, dass für $\epsilon \to 0$; $j, l = 1, 2, \ldots, h$ auf Grund der Eigenschaften von $\psi_{j,l}(s)$

$$\sum_{n=1}^{\infty} r_j(n)\, r_l(n)\, e^{-(4\pi/\rho)\, n \sin\epsilon} = O\left(\frac{\log \epsilon^{-1}}{\epsilon}\right),$$

ebenso natürlich $\sum \{r_j(n)+r_l(n)\}^2 \exp\{-(4\pi/\rho)\, n \sin\epsilon\}$, ist, ferner bei der Abschätzung von

$$\sum_{\substack{m,\, n=1 \\ m \neq n}}^{\infty} \frac{r_j(m)\, r_l(n)}{m-n} \sin\left\{\frac{2\pi}{\rho}\,(m-n)\cos\epsilon\right\} e^{-(2\pi/\rho)(m+n)\sin\epsilon} = \sum_{\substack{m,\, n=1 \\ m \neq n}}^{\infty} \phi_j(m)\, \phi_l(n)\, g(m-n),$$

wenn $\qquad \phi_k(m) = r_k(m) \exp\{-(2\pi/\rho)\, m \sin\epsilon\}, \quad g(x) = x^{-1} \sin\{(2\pi/\rho)\, x \cos\epsilon\}$

gesetzt wird, dass $g(x)$ eine gerade Funktion ist, und dass die Reihe umzuformen ist in

$$\sum_{1 \leqq n < m}^{\infty} \{\phi_j(m)+\phi_l(m)\}\{\phi_j(n)+\phi_l(n)\}\, g(m-n) - \sum_{1 \leqq n < m}^{\infty} \phi_j(m)\, \phi_j(n)\, g(m-n) - \sum_{1 \leqq n < m}^{\infty} \phi_l(m)\, \phi_l(n)\, g(m-n),$$

dass schliesslich jede dieser Summen einzeln wie der erste Teil von Σ_2, S. 135, zu behandeln ist.

III. Für das Folgende wird der Hilfssatz gebraucht:

Für $|x| < \min(|uv|, |uv^{-1}|, |u^{-1}v|, |uv|^{-1})$ ist

$$(3)\quad \sum_{k=0}^{\infty} \left(\frac{x}{uv}\right)^k (1+u^2+u^4+\ldots+u^{2k})(1+v^2+v^4+\ldots+v^{2k}) = \frac{1-x^2}{(1-uvx)(1-u^{-1}vx)(1-uv^{-1}x)(1-u^{-1}v^{-1}x)}.$$

Der Beweis ergibt sich unschwer durch mehrfache Anwendung der Summenformel der unendlichen geometrischen Reihe.

Für $|u| = |v| = 1$, $|x| < 1$ folgt

$$(3')\quad \sum_{k=0}^{\infty} \left(\frac{x}{uv}\right)^k \sum_{m=0}^{k} u^{2m} \sum_{m=0}^{k} v^{2m} = \frac{1-x^2}{(1-uvx)(1-\bar{u}vx)(1-u\bar{v}x)(1-\bar{u}\bar{v}x)}.$$

IV. Es sei $\mathfrak{K}^{-1}$ die zu $\mathfrak{K}$ inverse Klasse, χ durchlaufe alle Charaktere der Gruppe der Idealklassen. Dann ist

$$(4.1)\quad r_j(n) = 2 \sum_{\substack{N\mathfrak{a}=n \\ \mathfrak{a} \text{ in } \mathfrak{K}_j}} 1 = \frac{2}{h} \sum_{\chi} \chi(\mathfrak{K}_j^{-1}) \sum_{N\mathfrak{a}=n} \chi(\mathfrak{a}).$$

Die letzte Summe ist über alle Ideale erstreckt, deren Norm gleich n ist. Es sei nun n eine durch Q_j darstellbare Zahl, also

$$n = P_1^{a_1} P_2^{a_2} \ldots p_1^{2b_1} p_2^{2b_2} \ldots q_1^{c_1} q_2^{c_2} \ldots = \prod_{i=1}^{k} P_i^{a_i} \prod_{i=1}^{r} p_i^{2b_i} \prod_{i=1}^{w} q_i^{c_i}.$$

Jedes P ist das Produkt zweier konjugierter Primideale $\mathfrak{p}$ und $\mathfrak{p}'$, jedes p selbst Primideal und $\chi(p) = 1$ für alle χ, jedes q das Quadrat eines Primideales $\mathfrak{q}$. Es gibt daher $\Pi(a_i+1)$, $i = 1, 2, \ldots, k$, Ideale $\mathfrak{a}_1$, deren Norm gleich n ist:

$$(4.2)\quad \mathfrak{a} = \prod_{i=1}^{k} \mathfrak{p}_i^{\alpha_i} \mathfrak{p}_i'^{a_i-\alpha_i} \prod_{i=1}^{r} p_i^{b_i} \prod_{i=1}^{w} \mathfrak{q}_i^{c_i} \quad (\alpha_i = 0, 1, 2, \ldots, a_i).$$

Es ist $\chi(\mathfrak{p}_i)\chi(\mathfrak{p}_i') = \chi(P_i) = 1$, setzt man also

$$(4.3)\quad f(P, a, \chi) = \sum_{\alpha=0}^{a} \chi(\mathfrak{p})^{\alpha} \chi(\mathfrak{p}')^{a-\alpha} = \chi(\mathfrak{p})^{-a} \sum_{\alpha=0}^{a} \chi(\mathfrak{p})^{2\alpha},$$

$$[|f(P, a, \chi)| \leq a+1],$$

so folgt aus (4.2) und (4.3)

$$(4.4)\qquad \sum_{N\mathfrak{a}=n}\chi(\mathfrak{a}) = \prod_{i=1}^{w}\chi(\mathfrak{q}_i)^{c_i}\sum_{\substack{a_i=0\\ i=1,\dots,k}}^{a_i}\prod_{i=1}^{k}\chi(\mathfrak{p}_i)^{a_i}\chi(\mathfrak{p}_i')^{\alpha_i-a_i}$$

$$= \prod_{i=1}^{w}\chi(\mathfrak{q}_i)^{c_i}\prod_{i=1}^{k} f(P_i, a_i, \chi),$$

$$(4.5)\qquad \frac{h^2}{4}\, r_j(n)\, r_l(n)$$

$$= \sum_{\chi,\chi'}\chi(\mathfrak{R}_j^{-1})\chi'(\mathfrak{R}_l^{-1})\prod_{i=1}^{w}\chi\chi'(\mathfrak{q}_i)^{c_i}\prod_{i=1}^{k} f(P_i, a_i, \chi)\, f(P_i, a_i, \chi')$$

aus (4.1) und (4.4), wobei χ und χ' unabhängig voneinander alle Charaktere durchlaufen, daher für $\sigma > 1$

$$(4)\qquad \frac{h^2}{4}\,\psi_{j,l}(s) = \frac{h^2}{4}\sum_{n=1}^{\infty}\frac{r_j(n)\, r_l(n)}{n^s} = \sum_{\chi,\chi'}\chi(\mathfrak{R}_j^{-1})\chi'(\mathfrak{R}_l^{-1})\, X(s;\ \chi, \chi'),$$

$$(4.6)\quad X(s;\ \chi,\chi') = \sum_{a_i, b_i, c_i=0}^{\infty}\frac{f(P_1, a_1, \chi) f(P_1, a_1, \chi') f(P_2, a_2, \chi) f(P_2, a_2, \chi') \cdots \chi\chi'(q_1)^{c_1}\chi\chi'(q_2)^{c_2}\cdots}{P_1^{a_1 s} P_2^{a_2 s}\cdots p_1^{2b_1 s} p_2^{2b_2 s}\cdots q_1^{c_1 s} q_2^{c_2 s}\cdots}$$

$$= \prod_{P,\,p,\,q}\sum_{a=0}^{\infty}\frac{f(P, a, \chi) f(P, a, \chi')}{P^{as}}\sum_{b=0}^{\infty}\frac{1}{p^{2bs}}\sum_{c=0}^{\infty}\frac{\chi\chi'(\mathfrak{q})^c}{q^{cs}}.$$

Hier sind P_1, P_2, ... bezw. p_1, p_2, ... bezw. q_1, q_2 ... sämtliche P bezw. p bezw. q. Mit Rücksicht auf (3′) und (4.3) ergibt sich

$$X(s;\ \chi,\chi')$$

$$= \prod_{P,\,p,\,q}\frac{(1-P^{-2s})(1-p^{-2s})^{-1}\{1-\chi\chi'(\mathfrak{q})\,q^{-s}\}^{-1}}{\{1-\chi\chi'(\mathfrak{p})\,P^{-s}\}\{1-\bar\chi\chi'(\mathfrak{p})\,P^{-s}\}\{1-\chi\bar\chi'(\mathfrak{p})\,P^{-s}\}\{1-\bar\chi\bar\chi'(\mathfrak{p})\,P^{-s}\}}.$$

Nun ist für jedes χ sicher $\chi^2(\mathfrak{q}) = \chi(\mathfrak{q}^2) = \chi(q) = 1$, wenn ferner $\mathfrak{r}$ alle Primideale durchläuft,

$$(4.7)\quad L(s;\chi) = \sum_{\mathfrak{a}}\frac{\chi(\mathfrak{a})}{N\mathfrak{a}^s} = \prod_{\mathfrak{r}}\frac{1}{1-\chi(\mathfrak{r})\,N\mathfrak{r}^{-s}} = \prod_{P,\,p,\,q}\frac{(1-p^{-2s})^{-1}\{1-\chi(\mathfrak{q})\,q^{-s}\}^{-1}}{\{1-\chi(\mathfrak{p})\,P^{-s}\}\{1-\bar\chi(\mathfrak{p})\,P^{-s}\}},$$

$$(4')\qquad X(s;\ \chi,\chi') = \frac{1}{\zeta(2s)}\, L(s, \chi\chi')\, L(s, \bar\chi\chi')\prod_{\mathfrak{q}}\{1+\bar\chi\chi'(\mathfrak{q})\,q^{-s}\}^{-1}.$$

V. Bekanntlich ist $L(s, \chi)$ eine ganze Funktion von s, wenn χ nicht der Hauptcharakter χ_0 ist, und $L(s, \chi_0)$ in der ganzen Ebene bis auf einen

einfachen Pol in $s=1$ regulär*, ferner sind in jeder Halbebene $\sigma \geqslant \sigma_1$ für $|t| \to \infty$ alle Funktionen $L(s, \chi)$ gleich $O(|t|^A)$.

Daher ist für $\sigma \geqslant \frac{1}{2}+\delta > \frac{1}{2}$ jede Funktion $X(s; \chi, \chi')$ gleich $O(|t|^A)$ für $|t| \to \infty$, ferner

regulär für $\sigma > \frac{1}{2}$, falls weder $\chi\chi'$ noch $\bar{\chi}\chi'$ gleich χ_0 ist;

regulär für $\sigma > \frac{1}{2}$ bis auf einen einfachen Pol in $s=1$, falls entweder $\chi\chi'$ oder $\bar{\chi}\chi'$ gleich χ_0 ist;

regulär für $\sigma > \frac{1}{2}$ bis auf einen doppelten Pol in $s=1$, falls $\chi\chi' = \chi_0$ und $\bar{\chi}\chi' = \chi_0$ ist.

Der letzte Fall liegt dann und nur dann vor, wenn χ reell und gleich χ' ist; dann ist auch $\chi\chi'(\mathfrak{q}) = \chi^2(\mathfrak{q}) = 1$, $\chi(\mathfrak{K}^{-1}) = \chi(\mathfrak{K})$,

$$L(s, \chi\chi') = L(s, \bar{\chi}\chi') = L(s, \chi_0) = \zeta(s) \sum_{n=1}^{\infty} \left(\frac{d}{n}\right) n^{-s} = \zeta(s)\, L_d(s),$$

$$L(s, \chi\chi')\, L(s, \bar{\chi}\chi')(s-1)^2 \to L_d^2(1) = \pi^2 h^2 |d|^{-1} \quad (s \to 1),$$

$$X(s; \chi, \chi')(s-1)^2 \to 6h^2 |d|^{-1} \prod_{q|-d} \frac{q}{q+1} \quad (s \to 1). \tag{5}$$

Die Funktion $\psi_{j,l}(s)$ (Abschnitt II) hat mit Rücksicht auf (4) und (5) die verlangten Eigenschaften, und es ist in der Tat für $s \to 1$

$$\psi_{j,l}(s)(s-1)^2 \to a_{j,l} = \frac{24}{|d|} \prod_{q|-d} \frac{q}{q+1} \sum_{\chi \text{ reell}} \chi(\mathfrak{K}_j, \mathfrak{K}_l). \tag{2.1}$$

hieraus folgt sofort (1) und (1.2), mit Rücksicht auf (2.2) und auf die Ergebnisse des folgenden Abschnittes. Wegen (2.11) ist $a_{j,l} \geqslant 0$.

VI. Es sei $c_{j,l} 2^{r-1} = N_{j,l} = \Sigma \chi(\mathfrak{K}_j \mathfrak{K}_l)$, wobei χ die reellen Charaktere durchläuft. Zufolge (2.11) und (2.1) ist

$$\sum_{j=1}^{h} \sum_{l=1}^{h} N_{j,l} = \sum_{\chi \text{ reell}} \sum_{j=1}^{h} \chi(\mathfrak{K}_j) \sum_{l=1}^{h} \chi(\mathfrak{K}_l) = \sum_{\chi \text{ reell}} \left\{ \sum_{j=1}^{h} \chi(\mathfrak{K}_j) \right\}^2 = h^2, \tag{6.1}$$

da die innerste Summe für den Hauptcharakter gleich h, sonst Null ist.

* E. Hecke, "Über die L-Funktionen . . . für einen beliebigen Zahlkörper", *Göttinger Nachr., Math.-Phys. Kl.* (1917), 299–318, § 4.

E. Landau, *Einführung in die . . . Theorie . . . der Ideale* (Leipzig und Berlin, 1927), Satz 155 und Satz 171.

(*a*) Es sei h eine ungerade Primzahl, dann ist nur der Hauptcharakter reell, also sicher $N_{j,l}$ für alle j, l gleich 1, $\tau = 1$, $c_{j,l} = 1$.

(*b*) Es sei $h = 2^{\tau-1}$—hierher gehört bekanntlich auch der Fall $h = 2$—dann ist

$$N_{j,j} = \sum_{\chi \text{ reell}} \chi(\mathfrak{R}_j)^2 = \sum_{\chi \text{ reell}} 1 = 2^{\tau-1} = h; \quad c_{j,j} = h 2^{1-\tau} = 1;$$

denn die Gruppe der Charaktere ist isomorph mit der der Idealklassen, den reellen Charakteren entsprechen die Klassen, deren Quadrat die Klasse der Hauptideale ist, und nach einem Satze von Gauss ist deren Anzahl und daher die Anzahl der reellen Charaktere, also auch $N_{j,j}$, gleich $2^{\tau-1}$. Daher ist wegen $N_{j,l} \geqq 0$ zufolge (6.1) für $j \neq l$

$$N_{j,l} = 0; \quad c_{j,l} = 0 \quad (j \neq l).$$

Für $j \neq l$ ist auch $\psi_{j,l}(s)$ identisch gleich Null.

(*c*) Das allgemeine Ergebnis (1) folgt aus dem Hilfssatze (O. Taussky):

(6) Je nachdem $\mathfrak{R}_j \mathfrak{R}_l$ das Quadrat einer Idealklasse ist oder nicht, ist $N_{j,l}$ gleich $2^{\tau-1}$ oder gleich 0.

Es bedeute $N'_{j,l}$ bezw. $N''_{j,l}$ ein $N_{j,l}$ der ersten bezw. zweiten Art, $\mathfrak{R}'$ bezw. $\mathfrak{R}''$ eine Klasse, die als Quadrat einer Klasse darstellbar ist bezw. es nicht ist. Dann folgt ähnlich wie in (*b*), dass $N'_{j,l} = 2^{\tau-1}$ ist. Es sei $\mathfrak{G}$ die Gruppe der Klassen, deren Quadrat die Klasse der Hauptideale ist, ihr Grad ist $2^{\tau-1}$. Jede Nebengruppe von $\mathfrak{G}$ hat die Eigenschaft, dass die Quadrate aller Elemente einer Nebengruppe gleich einem und demselben $\mathfrak{R}'$ sind; die $\mathfrak{R}'$ sind für die verschiedenen Nebengruppen untereinander und von $\mathfrak{E}$ verschieden, die Anzahl aller $\mathfrak{R}'$ ist somit gleich der Anzahl der Reihen, gleich $h/2^{\tau-1} = h 2^{1-\tau}$.

Es durchlaufe jetzt $\mathfrak{R}_l$ bei festem j alle Klassen $\mathfrak{R}_j^{-1} \mathfrak{R}^{-1}$, so ist

$$\sum_l N'_{j,l} = h 2^{1-\tau} 2^{\tau-1} = h,$$

also folgt:

$$\sum_{j=1}^{h} \sum_l N'_{j,l} = h^2,$$

daher mit Rücksicht auf (6.1) und auf $N_{j,l} \geqq 0$:

$$N''_{j,l} = 0.$$

Breslau, Frobenstr. 9.

THE ARITHMETICAL INVARIANTS OF ALGEBRAIC LOCI

By J. A. Todd.

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Introduction.

In a previous paper* I have shown how to define on an algebraic variety V_d of d dimensions an invariant system $\{X_k\}$ of V_k's of each dimension k from 0 to $d-1$, and have suggested the name *canonical* V_k for the members of such a system. These systems (which may be virtual) are relative invariants under birational transformations of V_d, as are clearly the systems $\{(X_k X_l)\}$, $\{(X_k X_l X_m)\}$, ... defined by the intersection of several canonical varieties of the same or different dimensions. In particular, if $k_1, k_2, \ldots, k_s$ be a set of integers less than d, such that

$$k_1+k_2+\ldots+k_s=(s-1)d,$$

then the intersection $(X_{k_1} X_{k_2} \ldots X_{k_s})$ consists, in general, of a set of points. The number of points in this set is clearly a relative numerical invariant of V_d. It is the study of these invariants which forms the main object of the present paper.

If we write $k_i=d-l_i$, the condition that $(X_{k_1} X_{k_2} \ldots X_{k_s})$ has dimension zero is simply $\Sigma l_i = d$. Thus there is just one of these invariant numbers corresponding to each partition of the number d. We may denote these invariants by enclosing the symbols for the partitions in brackets thus: $\{a_1^{h_1} a_2^{h_2} \ldots a_r^{h_r}\}$, where $\Sigma h_i a_i = d$. To illustrate the notation, in the case $d=2$, $\{1^2\}$, corresponding to the partition (1, 1) of the number 2, represents the number of points of intersection of two canonical curves X_1 of the surface, while $\{2\}$, corresponding to the remaining partition (2), repre-

* Todd, 3.

sents the number of points in a set of the canonical series $\{X_0\}$. In the ordinary notation for the invariants of a surface, $\{1^2\} = \omega - 1$, $\{2\} = I + 4$.

There are thus $p(d)$ of these invariants attached to a V_d, where $p(d)$ is the number of partitions of d. The main result of this paper is that these invariants are independent, and that the arithmetic genus of V_d can be expressed as a linear function of them; this function is determined explicitly for $d \leqslant 6$. This provides a geometrical interpretation of the arithmetic genus and at the same time gives a proof of its relative invariance. The method of proof is briefly as follows. We show that the invariants in question, and the arithmetic genus, can be expressed linearly in terms of the projective characters of the V_d. We then calculate the invariants explicitly for a particular V_d which has the property that its projective characters are linearly independent; it appears that the intersection of d primals in $[2d]$ is such a locus. Next we show that for this particular variety a relation of the type in question exists; the same relation must then hold on an arbitrary V_d in view of the linearity of the functions involved and the independence of the projective characters of the particular V_d considered.

Previous enumerative work on the invariants of algebraic loci has, until recently, centred round the arithmetic genus and the invariants Ω_i of the canonical system of V_{d-1}'s on V_d; Ω_i being the virtual arithmetic genus of the intersection of $d-i$ canonical V_{d-1}'s (Ω_0 is the grade of the system)*. There are two essentially distinct ways in which the arithmetic genus may be defined for a V_d†. In the first place, we may define P_a as the virtual freedom of the canonical system of V_{d-1}'s, increased by unity. Alternatively, we may consider the postulation formula for primals of high order l which pass through V_d (supposed without singularities in some suitable space); this formula gives for the postulation $v(l)$ the expression

$$v(l) = \sum_{i=0}^{d} k_i \binom{l+d-i}{d-i}, \tag{1}$$

where the numbers k_i depend only on V_d and not on l, and we may define

$$p_d = (-)^d \left(\sum_{i=0}^{d} k_i - 1 \right). \tag{2}$$

Severi has shown that the equation $p_d = P_a$ is true when $d = 3$; I indicate the lines of a general proof for all values of d below. The absolute

* For a general account see Lefschetz, 1, and Rosenblatt, 1.

† These notions derive from the fundamental paper of Severi, 2.

invariance of p_d and P_a has been proved by Severi when $d=3$, and by Albanese* for $d=4$. The *relative* invariance of these numbers, for all values of d, follows from the results of this paper.

The arithmetic genus of a V_3 has been computed in terms of its projective characters by the present author†, and Roth has given similar formulae for the invariants Ω_i‡. The corresponding results for a V_4 are given below§. Some recent results of B. Segre‖ on the invariant systems of a V_3 are also particular cases of the results in the present work.

1. *Projective characters of a* V_d.

We consider a V_d, Φ say, lying in a space $[r]$, where $r \geqslant 2d$; we suppose Φ to be free from singularities, except that, if $r=2d$, it may have a finite number of improper double points, where two sheets of the variety intersect without having a common tangent¶. Thus Φ has the generality of the projection of a non-singular variety. Let Φ_k denote the section of Φ by a general $[r-d+k]$, and consider the aggregate of ∞^k tangent spaces to Φ at points of Φ_k. Let Σ denote this system of $[d]$'s. Consider any Schubert symbol $(a_0, a_1, \ldots, a_d)$, where $a_0 < a_1 < \ldots < a_d$, this symbol representing the condition imposed on a $[d]$ of $[r]$ to meet given spaces $[a_i]$ in $[i]$ for $i=0, \ldots, d$; the space $[a_j]$ containing $[a_i]$ if $j > i$. If we write

$$\beta_i = r-(d-i)-a_i,$$

the condition $(a_0, a_1, \ldots, a_d)$ imposes $\beta = \sum_{i=0}^{d} \beta_i$ conditions on the $[d]$, and is said to be of *power* β. The inequalities $a_0 < a_1 < \ldots < a_d$ imply

$$\beta_0 \geqslant \beta_1 \geqslant \ldots \geqslant \beta_d.$$

There are a finite number of spaces of the system Σ which satisfy any Schubert condition of power k. This number is called an *elementary*

* Albanese, **1**, **2**, where a proof that $p_d = P_a$ when $d=4$ is also given.

† Todd, **2**.

‡ Roth, **4**.

§ Roth has recently computed the arithmetic genus and the invariants Ω_i of a V_4. His paper **5** came to my notice after the present paper had been completed.

‖ Segre, **1**.

¶ The number of these double points is expressible in terms of the other characters of V_d (Severi, **1**).

projective character (briefly e.p.c.) of Φ, and may be denoted by the symbol $\nu_{\beta_1\beta_2\ldots\beta_d}$, where the β's are the numbers derived as above from the Schubert symbol defining the character, so that $\Sigma\beta_i = k$. There is one such character corresponding to each partition $(\beta_0\beta_1 \ldots \beta_d)$ of k. Hence the total number of e.p.c.'s of a V_d is $\sum_{k=0}^{d} p(k)$, where $p(k)$ is the number of partitions of k and $p(0) = 1$, the character ν_0 being simply the order of the variety.

After Severi*, to whom these ideas are due, a projective invariant of Φ will be called a *numerical character* of Φ if it can be expressed algebraically in terms of the e.p.c.'s.

The e.p.c.'s possess the important property of being *additive*, that is, for a V_d consisting of two or more components without common points the e.p.c.'s of the composite variety are obtained by adding together the corresponding e.p.c.'s of the components. Any projective invariant of V_d which possesses this property will be called *additive*. If a numerical character of Φ is additive, it follows that its expression in terms of the e.p.c.'s is *linear*.

2. *The polar varieties M_k and the classes of Φ.*

Of special importance among the e.p.c.'s are the classes† $\mu_0, \mu_1, \ldots, \mu_d$ of Φ. The k-th class, μ_k, is the number of tangent $[d]$'s of Φ at points of Φ_k which meet a given $[r-d+k-2]$ in a $[k-1]$; in the previous notation it is $\nu_{11\ldots1}$, with k unit suffixes ($\mu_0 = \nu_0$). The aggregate of all tangent $[d]$'s which meet a given $[r-d+k-2]$ in a $[k-1]$ thus touch Φ along a variety M_{d-k} of $d-k$ dimensions and order μ_k. We may refer to these varieties M as the *polar varieties* on Φ; they exist for each dimension from 0 to d. M_d is, of course, Φ itself.

The importance of the polar varieties lies in the fact that the e.p.c.'s can be expressed linearly, and in a unique way, in terms of orders of the varieties M and of their mutual intersections‡. This is equivalent to the statement that any Schubert symbol for the tangent $[d]$'s can be expressed

* Severi, 1, 2.

† Severi, 1.

‡ That, conversely, the orders of the intersections of sets of varieties M are expressible in terms of the e.p.c's is very easily seen. For at a point common to M_{d-h} and M_{d-k} the tangent $[d]$ to V_d satisfies an incidence condition which is the product of two conditions of respective powers h and k, and this product may be expressed as the sum of elementary conditions of power $h+k$ to which correspond e.p.c's. A similar result holds for the intersection of any number of the polar varieties.

in terms of the particular symbols

$$\tau_k = (r-d-1,\ r-d,\ \ldots,\ r-d+k-2,\ r-d+k,\ r-d+k+1,\ \ldots,\ r) \qquad (k = 1,\ \ldots,\ d)$$

which define the characters μ_k.

To see this, denote by σ_k the condition $(r-d-k,\ r-d+1,\ \ldots,\ r)$ of power k. It is known* that any Schubert condition can be expressed in terms of the σ_k by the formula

$$(a_0,\ a_1,\ \ldots,\ a_d) = \begin{vmatrix} \sigma_{\beta_0} & \sigma_{\beta_0+1} & \ldots & \sigma_{\beta_0+d} \\ \sigma_{\beta_1-1} & \sigma_{\beta_1} & \ldots & \sigma_{\beta_1+d-1} \\ \ldots & \ldots & \ldots & \ldots \\ \sigma_{\beta_d-d} & \sigma_{\beta_d-d+1} & \ldots & \sigma_{\beta_d} \end{vmatrix},$$

where $\sigma_0 = 1$, $\sigma_h = 0$ if $h < 0$, $\beta_i = r-(d-i)-a_i$. Hence, in particular,

$$\tau_k = \begin{vmatrix} \sigma_1 & \sigma_2 & \ldots & \sigma_k \\ 1 & \sigma_1 & \ldots & \sigma_{k-1} \\ 0 & 1 & \ldots & \sigma_{k-2} \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & \sigma_1 \end{vmatrix}$$

on reduction.

Thus $\tau_1 = \sigma_1$, $\tau_2 = \sigma_1^2 - \sigma_2$, and generally, $\tau_k = (-)^{k-1}\sigma_k + f(\sigma_1, \ldots, \sigma_{k-1})$, which shows that σ_k is expressible as a polynomial in $\tau_1, \ldots, \tau_k$ (of weight k); and since $(a_0, a_1, \ldots, a_n)$ is expressible in terms of the σ's it must therefore be expressible in terms of the τ's.

Hence the orders of the intersections of the various polar varieties constitute an alternative set of (evidently additive) defining characters of Φ. If $h_1, h_2, \ldots, h_s$ is a set of integers whose sum h does not exceed d, then the locus of intersection of the varieties $M_{d-h_1}, M_{d-h_2}, \ldots, M_{d-h_s}$ is of dimension $d-h$, and its order may be denoted by $\mu_{h_1 h_2 \ldots h_s}$ (agreeing with the previous notation when $s = 1$). The aggregate of the μ's, like the ν's, forms a set of additive characters of Φ, in total number $\sum_{k=0}^{d} p(k)$, and any numerical character of Φ which is additive must therefore be a linear function of the μ's.

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* Giambelli, **1**. See also Todd, **1**.

3. *The canonical systems* $\{X_k\}$ *on* Φ.

We now show that the canonical varieties $\{X_k\}$ of Φ are expressible in terms of the varieties M and their linear sections.

Consider first the *canonical series** $\{X_0\}$ of sets of points of Φ. A pencil $|S|$ of prime sections of Φ contains μ_d sections by tangent primes, whose points of contact form a polar set M_0. The sets of the canonical series are then defined by the equivalence

$$X_0 \equiv M_0 - 2X_0(S) - X_0(S^2),$$

where $X_0(S)$, $X_0(S^2)$ are canonical sets of S and of the base (S^2) of the pencil. It is easily seen by induction from this that

$$X_0 \equiv \sum_{i=0}^{d} (-)^i (i+1)(M_i S^i), \tag{3}$$

where S^i denotes the section of Φ by a $[r-i]$, and $S^0 \equiv \Phi$.

The canonical systems of higher dimension are defined inductively by means of the *adjoint systems*†, the adjoint system $\{A_k(S)\}$ of dimension k of the prime sections being that system of equivalence which cuts out the system $\{X_{k-1}(S)\}$ on a general S. The system $\{X_k\}$ is then given by

$$X_k = A_k(S) - X_k(S).$$

Now M_k is the Jacobian of the linear ∞^{k+1} system of prime sections cut out by primes through a $[r-k-2]$. Hence‡

$$M_k \equiv \sum_{i=0}^{k+2} \binom{k+2}{i} X_k(S^i).$$

It follows easily, by induction on k, that

$$X_k \equiv \sum_{i=0}^{d-k} (-)^i \binom{k+1+i}{k+1} (M_{k+i} S^i), \tag{4}$$

since on S^i the polar M_k is $(M_{k+i} S^i)$. We note that (4) reduces to (3) when $k = 0$.

Thus the canonical systems on Φ are expressible linearly in terms of the polar loci M and their linear sections. Hence the invariants $\{a_1^{h_1} \dots a_s^{h_s}\}$

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* Todd, **3**. *Cf.* Severi, **3**, for $d = 2$, and Segre, **1**, for $d = 3$.

† Todd, **3**.

‡ Todd, *loc. cit.*; Eger, **1**. Also (for $d = 3$) Segre, **1**.

of Φ referred to in the introduction, being the orders of intersections of sets of $\{X_k\}$ which meet in sets of points, are linear functions of the characters $\mu_{h_1 h_2 \dots h_s}$ and hence of the e.p.c.'s. Explicit forms are given below in the cases $d = 3, 4$.

4. *The locus of intersection of d non-singular primals in $[2d]$.*

Suppose now that Φ is the intersection of d arbitrary primals in $[2d]$ of respective orders $n_1, n_2, \dots, n_d$. Then it is known* that the classes of Φ are given by

$$\mu_0 = n_1 n_2 \dots n_d,$$

$$\mu_k = \mu_0 \xi_k,$$

where ξ_k denotes the sum of all homogeneous products of the numbers $n_1-1, n_2-1, \dots, n_d-1$ taken k at a time; *e.g.*

$$\xi_2 = \Sigma (n_i-1)^2 + \sum_{i<j} (n_i-1)(n_j-1).$$

Thus M_{d-k} is the intersection of Φ with a V_{2d-k} of order ξ_k.

It follows that, for this particular variety,

$$\mu_{h_1 h_2 \dots h_s} = \mu_0 \xi_{h_1} \xi_{h_2} \dots \xi_{h_s}.$$

Now it is easily proved, by induction on d, that the d quantities $\xi_1, \xi_2, \dots, \xi_d$ are independent symmetric functions of $n_1, n_2, \dots, n_d$. Since ξ_i is of degree i in $n_1, \dots, n_d$ it follows that any integral symmetric function of the n's of degree less than or equal to d is expressible *uniquely* as a polynomial in $\xi_1, \dots, \xi_d$. It follows that the expressions $\mu_{h_1 h_2 \dots h_s}$ are *linearly independent*. Since the $\mu_{h_1 h_2 \dots h_s}$ determine the e.p.c.'s and *vice versa*, it follows that, *for the particular primal considered, the e.p.c.'s are linearly independent.*

A consequence of this is that *any linear relation between numerical characters of this particular primal Φ implies the existence of the same relation between the characters of any V_d.* For otherwise we should have a linear function of the e.p.c.'s of V_d, not identically zero, but vanishing for Φ, and this contradicts the result which we have proved, that the characters of Φ are linearly independent.

* Severi, 1.

The canonical system $\{X_k\}$ on Φ is cut out by a V_{d+k} of order μ'_{d-k}, where, by (3) and (4),

$$\mu'_{d-k} = \sum_{i=0}^{d-k} (-)^i \binom{k+1+i}{k+1} \xi_{d-k-i}. \tag{5}$$

The invariant $\{a_1^{h_1} \dots a_s^{h_s}\}$ (with $\Sigma h_i a_i = d$) is thus given by

$$\{a_1^{h_1} \dots a_s^{h_s}\} = \mu_0 \mu'^{h_1}_{a_1} \dots \mu'^{h_s}_{a_s}, \tag{6}$$

and so $\{a_1^{h_1} \dots a_s^{h_s}\}$ is a symmetric, non-homogeneous function of $n_1, \dots, n_d$ of degree d. Since the e.p.c.'s are linearly independent, it follows that there is no linear relation connecting these invariants.

5. *The relation* $P_a = p_a$.

In this section we outline a proof that the two definitions given in the introduction for the arithmetic genus of a variety define the same number. The proof proceeds by induction on d, the theorem being certainly true when $d = 1$ or 2. Let us assume that it has been proved for varieties of $d-1$ dimensions.

Severi has shown* that the number p_d is a numerical character of V_d. Since the postulation of a composite variety whose components have no common points is the sum of the postulations of the components, it follows from (1) and (2) that $p_d + (-1)^d$ possesses the additive property; hence, since it is a numerical character of V_d, *it is a linear function of the e.p.c.'s.*

For a reducible curve whose components do not intersect, the *canonical series* has been defined by Severi† as the aggregate of sets obtained by uniting one canonical set on each component. The natural extension, on a V_d with several non-intersecting components $V^{(1)}, \dots, V^{(t)}$, is to define a canonical V_{d-1}, K as the sum $K^{(1)} + \dots + K^{(t)}$, where $K^{(i)}$ is canonical on $V^{(i)}$. It is easy to see that this definition implies that the canonical system on the reducible V_d fulfils the usual relations of adjunction, etc.

The invariant Ω_{d-1} of the reducible V_d is thus the virtual arithmetic genus of the reducible V_{d-1}, $K^{(1)} + \dots + K^{(t)}$; and, since we are assuming that for V_{d-1} the numbers P_a and p_{d-1} are equal, it follows from (1) and (2) that the character $\Omega_{d-1} + (-1)^{d-1}$ is additive. In the same way the grade Ω_0 of the canonical system and the expressions $\Omega_i + (-1)^i$ are additive.

Now Ω_{d-1}, the arithmetic genus of K, is by hypothesis a linear function of the e.p.c.'s of K. These e.p.c.'s are numerical characters of V_d, for, V_d

* Severi, 2.

† Severi, 4.

lying in a space $[r]$, K can be constructed on V_d by drawing $r-d$ primals of orders $m_1, \ldots, m_{r-d}$ through V_d to meet residually in a locus V_d'; and the canonical system on V_d is cut out by primals of order $\Sigma m_i - r - 1$ through V_d'. Thus the characters of K are functions of the characters of V_d. *Hence* Ω_{d-1}, *and similarly* Ω_i, *are numerical characters of* V_d. Since Ω_0, $\Omega_i + (-1)^i$ possess the additive property, it follows that these expressions are linear functions of the e.p.c.'s of V_d.

Now it is known that*

$$P_a + (-)^{d-1} p_d = \Omega_0 - \Omega_1 + \ldots + (-)^{d-1} \Omega_{d-1} + d + (-1)^{d-1}. \tag{7}$$

If we write this equation in the form

$$\left(P_a + (-1)^d\right) + (-)^{d-1}\left(p_d + (-1)^d\right) = \Omega_0 - (\Omega_1 - 1) + \ldots + (-)^{d-1}\left(\Omega_{d-1} + (-1)^{d-1}\right),$$

it follows that the expression $P_a + (-1)^d$ is an additive numerical character of V_d.

To prove that $P_a = p_d$ it is now sufficient to show that this equality holds for the particular V_d of the last section, whose e.p.c.'s are linearly independent. On this variety, the complete intersection of d primals in $[2d]$ of respective orders $n_1, \ldots, n_d$, the canonical V_{d-1}'s are cut out by primals of order $\Sigma n_i - 2d - 1$. Hence

$$P_a = \binom{N-1}{2d} - \sum_i \binom{N-n_i-1}{2d} + \sum_{i,j} \binom{N-n_i-n_j-1}{2d} - \ldots + (-)^{d-1} \sum_i \binom{n_i-1}{2d}, \tag{8}$$

where $N = \Sigma n_i$.

On the other hand, it is known that the effective postulation $v'(l)$ of V_d for primals of any order l is given† by

$$v'(l) = \binom{l+2d}{2d} - \sum_i \binom{l-n_i+2d}{2d} + \sum_{i,j} \binom{l-n_i-n_j+2d}{2d} - \ldots + (-)^d \binom{l-N+2d}{2d}, \tag{9}$$

where meaningless binomial coefficients are to be replaced by zero.

* Severi, 2.

† *Cf.* Bertini, 1, 312.

For sufficiently large values of l this effective postulation is equal to the virtual postulation $v(l)$ given by (1),

$$v(l) = \sum_{i=0}^{d} k_i \binom{l+d-i}{d-i},$$

and, since $v'(l)$ and $v(l)$ are polynomials in l (of degree certainly not larger than $2d$) equal for all large integral values of l, it follows that they are identically equal. It is to be understood now that, in evaluating the polynomial $v'(l)$ for small values of l, the "meaningless" binomial coefficients in (9) are to be evaluated by the rule

$$\binom{a}{b} = \frac{a(a-1)(a-2)\dots(a-b+1)}{b!},$$

so that, if $0 \leqslant a \leqslant b-1$, $\binom{a}{b} = 0$, and, for a negative value $-c$ of a,

$$\binom{-c}{b} = \frac{(-c)(-c-1)\dots(-c-b+1)}{b!} = (-)^b \binom{c+b-1}{b}.$$

With this interpretation, of course, the right-hand side of (9) need no longer represent the actual postulation if l is small, but this is irrelevant to our purpose.

From $v(l) \equiv v'(l)$ it follows, in particular, that

$$v(0) = v'(0).$$

Now $v(0) = \sum_{i=0}^{d} k_i = 1+(-)^d p_d$, by (2), and

$$\begin{aligned} v'(0) &= 1 - \sum_i \binom{2d-n_i}{2d} + \sum_{i,j} \binom{2d-n_i-n_j}{2d} - \dots + (-)^d \binom{2d-N}{2d} \\ &= 1 - \sum_i \binom{n_i-1}{2d} + \sum_{i,j} \binom{n_i+n_j-1}{2d} - \dots + (-)^d \binom{N-1}{2d} \\ &= 1+(-)^d P_a, \quad \text{by (8).} \end{aligned}$$

Thus $P_a = p_d$ for the particular V_d considered, and hence, by what has preceded, for all V_d. It follows that equation (7) may be written in the form

$$(1+(-)^{d-1}) P_a = \Omega_0 - \Omega_1 + \dots + (-)^{d-1} \Omega_{d-1} + d + (-1)^{d-1}. \qquad (10)$$

Our next task is to show that the expression $P_a+(-1)^d$ is a linear function of the invariants $\{a_1^{h_1}\ldots a_s^{h_s}\}$. Since all these expressions are numerical additive characters of V_d, it suffices to prove this for the particular V_d described in § 4. The necessary calculations involve some tedious algebra, depending on a number of algebraic lemmas which are proved in the next section. The reader may prefer, in the first instance, to take the proofs of these lemmas for granted, and pass directly to the proof of the theorem in § 7.

6. *Algebraical lemmas.*

LEMMA 1. *Let $h_s(a_1, a_2, \ldots, a_m)$ denote the sum of all homogeneous products of degree s of $a_1, a_2, \ldots, a_m$. Then*

$$\sum_{i=1}^{m} \frac{\partial}{\partial a_i} h_s(a_1, a_2, \ldots, a_m) = (m+s-1)h_{s-1}(a_1, a_2, \ldots, a_m).$$

For the coefficient of the term $a_1^{r_1}\ldots a_m^{r_m}$ (with $\Sigma r_i = s-1$) on the left is

$$(r_1+1)+(r_2+1)+\ldots+(r_m+1) = s-1+m.$$

[When $s=0$, we define $h_0(a_1, a_2, \ldots, a_m) = 1$.]

LEMMA 2. *With the notation of Lemma 1,*

$$h_s(a_1+\lambda, a_2+\lambda, \ldots, a_m+\lambda) = \sum_{i=0}^{s} \binom{m+s-1}{i} \lambda^i h_{s-i}(a_1, a_2, \ldots, a_m).$$

For, by Taylor's theorem,

$$h_s(a_1+\lambda, a_2+\lambda, \ldots, a_m+\lambda) = \sum_{i=0}^{s} \frac{\lambda^i}{i!} \Delta^i h_s(a_1, a_2, \ldots, a_m),$$

where Δ is the operator

$$\frac{\partial}{\partial a_1}+\frac{\partial}{\partial a_2}+\ldots+\frac{\partial}{\partial a_m}.$$

The result now follows from Lemma 1.

We shall hereafter write h_s instead of $h_s(a_1, a_2, \ldots, a_m)$ when no confusion can arise by not stating the variables a explicitly.

We now introduce a set of operators $\delta_1, \delta_2, \ldots, \delta_m$, where

$$\delta_\mu = \sum_{i=0}^{m-\mu} h_i \frac{\partial}{\partial h_{i+\mu}} \tag{11}$$

(with the convention $h_0 = 1$)*.

* These operators arise in combinatory analysis. See Macmahon, 1, 36.

The expressions $h_1, h_2, \ldots, h_m$ form a set of m independent symmetric functions of $\alpha_1, \ldots, \alpha_m$ and every integral symmetric function of the α's whose degree does not exceed m is expressible uniquely as a polynomial in the h's, whose weight is equal to the degree of the function in the α's. In what follows, a symmetric function of the α's will mean a symmetric function whose degree does not exceed m, and all symbols used as exponents will represent positive integers.

If f denote any symmetric function of the α's of degree r ($\leqslant m$), then $\delta_\mu(f)$ is a symmetric function of degree $r-\mu$; in particular, $\delta_\mu(f) = 0$ if $r < \mu$. Since δ_μ is a linear operator, it is clear that

$$\delta_\mu(f+g) = \delta_\mu(f)+\delta_\mu(g),$$

$$\delta_\mu(fg) = f\delta_\mu(g)+g\delta_\mu(f).$$

LEMMA 3. *If $r \leqslant m$, then*

$$\delta_\mu\left(\sum_{i=1}^{m} \alpha_i^r\right) = 0 \quad (r \neq \mu),$$

$$= \mu \quad (r = \mu).$$

Proof. We have

$$\sum_{r=1}^{\infty} \frac{1}{r}\left(\sum_{i=1}^{m} \alpha_i^r\right) = \log \prod_{i=1}^{m} (1-\alpha_i)^{-1}$$

$$= \log (1+h_1+h_2+\ldots),$$

and so

$$\sum_{r=1}^{\infty} \frac{1}{r}\,\delta_\mu\left(\sum_{i=1}^{m} \alpha_i^r\right) = \delta_\mu[\log (1+h_1+h_2+\ldots)]$$

$$= \frac{1}{1+h_1+h_2+\ldots}\,\delta_\mu(1+h_1+h_2+\ldots)$$

$$= 1+\text{terms of degree greater than } m-\mu.$$

Hence, equating coefficients of terms of like degree, we have, if $r \leqslant m$,

$$\delta_\mu\left(\sum_{i=1}^{m} \alpha_i^\mu\right) = \mu, \quad \delta_\mu\left(\sum_{i=1}^{m} \alpha_i^r\right) = 0 \quad (r \neq \mu).$$

It will be understood that here, and elsewhere, the formal manipulations with series are merely devices for abbreviating the work. All the expressions with which we are concerned are polynomials of bounded degree, and no questions of convergence arise.

In order to avoid the use of elaborate summation signs we now introduce the following notation. If $a_1^{k_1} a_2^{k_2} \dots a_p^{k_p}$ is a product of several of the a's, then $\left((a_1^{k_1} \dots a_p^{k_p})\right)$ represents the sum of the formally distinct expressions obtained by permuting the a's in all possible ways, and the symbol $\left((a_1^{k_1} \dots a_k^{k_p})\right)'$ denotes the sum of all the expressions, whether distinct or not, obtained by permuting the a's. Thus $\left((a_1^2 a_2 a_3)\right)$ is $\Sigma a_1^2 a_2 a_3$, while $\left((a_1^2 a_2 a_3)\right)'$ is $2\Sigma a_1^2 a_2 a_3$. In general, if the indices k fall into sets, β_1 being equal to k_1, β_2 to k_2, ..., β_q to k_q, with all the k's different, and $\Sigma\beta_i = p$, then $\left((a_1^{k_1} \dots a_p^{k_p})\right)' = \beta_1!\beta_2! \dots \beta_q!\left((a_1^{k_1} \dots a_p^{k_p})\right)$.

It is clear that

$$\left((a_1^{k_1} \dots a_p^{k_p})\right)'\left((a_1{}^k)\right)' = \sum_{i=1}^{p}\left((a_1^{k_1} \dots a_i^{k_i+k} \dots a_p^{k_p})\right)' + \left((a_1^{k_1} \dots a_p^{k_p} a_{p+1}^{k})\right)' \quad (12)$$

(the last term being omitted if $p = m$).

LEMMA 4. *Let* $r_1+r_2+\dots+r_p = \mu$ $(p \leqslant m,\ r_i > 0)$. *Then*

$$\delta_\mu\left((a_1^{r_1} \dots a_p^{r_p})\right)' = (-)^{p-1}(p-1)!\,\mu.$$

Proof. When $p = 1$ this is included in Lemma 3. We therefore assume that it is true for p. By (12), if $r+\Sigma r_i = \mu$,

$$\delta_\mu\left((a_1^{r_1} \dots a_p^{r_p} a_{p+1}^{r})\right)' = \delta_\mu\left[\left((a_1{}^r)\right)'\left((a_1^{r_1} \dots a_p^{r_p})\right)'\right] - \sum_{i=1}^{p}\delta_\mu\left((a_1^{r_1} \dots a_i^{r_i+r} \dots a_p^{r_p})\right)'.$$

The first term on the right vanishes, since the degree of each factor is less than μ. By the inductive hypothesis, each of the p terms in the last sum is equal to $(-)^{p-1}(p-1)!\,\mu$. Hence the whole expression is equal to

$$-p(-)^{p-1}(p-1)!\,\mu = (-)^p p!\,\mu.$$

The lemma is thus true for $p+1$ if it is true for p. But it is true when $p = 1$, and hence is true universally.

LEMMA 5. *Let* $r_1+r_2+\dots+r_p > \mu$ $(p \leqslant m,\ r_i > 0)$. *Then*

$$\delta_\mu\left((a_1^{r_1} \dots a_p^{r_p})\right)' = \sum_{(s_j)}\left((a_1^{t_1} \dots a_u^{t_u})\right)'\delta_\mu\left((a_1^{s_1} \dots a_q^{s_q})\right)',$$

where $s_1, \dots, s_q$ *is a subset of* $r_1, \dots, r_p$ *such that* $\Sigma s_j = \mu$, $t_1, \dots, t_u$ *are the remaining numbers* r_i, *and the summation extends over all such subsets.*

Proof. When $p = 1$ the result is included in Lemma 3. We suppose it true for p, and prove it for $p+1$. From (12) it follows that

$$\delta_\mu\Big((\alpha_1^{r_1}\dots\alpha_p^{r_p}\alpha_{p+1}^{r})\Big)' = \Big((\alpha_1^{r_1}\dots\alpha_p^{r_p})\Big)'\,\delta_\mu\Big((\alpha_1^{r})\Big)' + \Big((\alpha_1^{r})\Big)'\,\delta_\mu\Big((\alpha_1^{r_1}\dots\alpha_p^{r_p})\Big)' - \sum_{i=1}^{p}\delta_\mu\Big((\alpha_1^{r_1}\dots\alpha_i^{r_i+r}\dots\alpha_p^{r_p})\Big)' \quad (13)$$

(using the rule for operating with δ_μ on a product).

Let $s_1, \dots, s_q$ be a subset of $r_1, \dots, r_p$ whose sum is μ, and let $t_1, \dots, t_u$ be the remaining numbers r_i. Consider the coefficient of $\delta_\mu\Big((\alpha_1^{s_1}\dots\alpha_q^{s_q})\Big)'$ in the right-hand member of (13). To this coefficient the second term will give a contribution equal to $\Big((\alpha_1^{r})\Big)'\Big((\alpha_1^{t_1}\dots\alpha_u^{t_u})\Big)'$. Of the p summands in the last term each gives a contribution except the q which correspond to values of i for which r_i is one of $s_1, \dots, s_q$. Thus the total coefficient of $\delta_\mu\Big((\alpha_1^{s_1}\dots\alpha_q^{s_q})\Big)'$ is

$$\Big((\alpha_1^{r})\Big)'\Big((\alpha_1^{t_1}\dots\alpha_u^{t_u})\Big)' - \sum_{j=1}^{u}\Big((\alpha_1^{t_1}\dots\alpha_j^{t_j+r}\dots\alpha_u^{t_u})\Big)' = \Big((\alpha_1^{t_1}\dots\alpha_u^{t}\alpha_{u+1}^{r})\Big)'.$$

The remainder of the terms in the expansion of the right-hand member of (13), apart from the first term, correspond to subsets $s_1'\dots s_v'$ of the r_i such that $\Sigma s_j' = \mu - r$. If $t_1', \dots, t_w'$ denote the remaining numbers r_i, then, by the inductive hypothesis, the coefficient of $\delta_\mu\Big((\alpha_1^{s_1'}\dots\alpha_j^{s_j'+r}\dots\alpha_v^{s_v'})\Big)'$ (which comes only from the last term) is $-\Big((\alpha_1^{t_1'}\dots\alpha_w^{t_w'})\Big)'$. But, by Lemma 4,

$$\delta_\mu\Big((\alpha_1^{s_1'}\dots\alpha_j^{s_j'+r}\dots\alpha_v^{s_v'})\Big)' = (-)^{v-1}(v-1)!\,\mu = -\frac{1}{v}\,\delta_\mu\Big((\alpha_1^{s_1'}\dots\alpha_v^{s_v'}\alpha_{v+1}^{r})\Big)'.$$

Since j can take the values 1, 2, ..., v it follows that the total contribution to the right of (13) corresponding to the set $s_1', \dots, s_v'$ is

$$\Big((\alpha_1^{t_1'}\dots\alpha_w^{t_w'})\Big)'\,\delta_\mu\Big((\alpha_1^{s_1'}\dots\alpha_v^{s_v'}\alpha_{v+1}^{r})\Big)'.$$

Taking into account the first term of the expression, which vanishes unless $r = \mu$, we see that the right-hand member of (13) is equal to

$$\sum_{(s_j)}\Big((\alpha_1^{t_1}\dots\alpha_u^{t_u}\alpha_{u+1}^{r})\Big)'\,\delta_\mu\Big((\alpha_1^{s_1}\dots\alpha_q^{s_q})\Big)' + \sum_{(s_j')}\Big((\alpha_1^{t_1'}\dots\alpha_w^{t_w'})\Big)'\,\delta_\mu\Big((\alpha_1^{s_1'}\dots\alpha_v^{s_v'}\alpha_{v+1}^{r})\Big)' + \Big((\alpha_1^{r_1}\dots\alpha_p^{r_p})\Big)'\,\delta_\mu\Big((\alpha_1^{r})\Big)',$$

where $\Sigma s_j = \mu$, $\Sigma s_j' = \mu - r$; and this proves Lemma 5 for $p+1$. The proof of the lemma, by induction, is thus complete.

LEMMA 6. *Let*

$$G_r = \sum_{(h_i)} \frac{\left((a_1^{h_1} \dots a_m^{h_m})\right)}{(h_1+1)! \dots (h_m+1)!} \quad (\Sigma h_i = r \leqslant m).$$

Then $\delta_\mu(G_r) = G_{r-\mu}\,\delta_\mu(G_\mu)$.

Proof. Evidently $\delta_\mu(G_r)$ is of degree $r-\mu$ in the a's. By Lemma 5, a term in $\delta_\mu(G_r)$ of the type

$$\left((a_1^{r_1} \dots a_p^{r_p})\right) \quad (r_1 + \dots + r_p = r - \mu)$$

can only arise from a term in G_r of the form

$$\left((a_1^{r_1} \dots a_p^{r_p}\, a_{p+1}^{s_1} \dots a_{p+q}^{s_q})\right),$$

where $s_1 + \dots + s_q = \mu$. Suppose that of the set $r_1, \dots, r_p, s_1, \dots, s_q$, β_1 of the r's and β_1' of the s's are equal to k_1, β_2 of the r's and β_2' of the s's to k_2, and so on, the k's being all distinct. Then

$$\left((a_1^{r_1} \dots a_p^{r_p})\right)' = \Pi(\beta_i!)\left((a_1^{r_1} \dots a_p^{r_p})\right);$$

$$\left((a_1^{s_1} \dots a_q^{s_q})\right)' = \Pi(\beta_i'!)\left((a_1^{s_1} \dots a_q^{s_q})\right);$$

$$\left((a_1^{r_1} \dots a_p^{r_p}\, a_{p+1}^{s_1} \dots a_{p+q}^{s_q})\right)' = \Pi[(\beta_i+\beta_i')!]\left((a_1^{r_1} \dots a_p^{r_p}\, a_{p+1}^{s_1} \dots a_{p+q}^{s_q})\right).$$

We can pick out q numbers respectively equal to $s_1, \dots, s_q$ from the given r's and s's in $\Pi\binom{\beta_i+\beta_i'}{\beta_i}$ ways. Thus, in the expansion of $\delta_\mu\left((a_1^{r_1} \dots a_{p+q}^{s_q})\right)'$, the term involving $\delta_\mu\left((a_1^{s_1} \dots a_q^{s_q})\right)'$ arises in $\Pi\binom{\beta_i+\beta_i'}{\beta_i}$ ways. Hence, since

$$\binom{\beta_i+\beta_i'}{\beta_i}\,\beta_i!\,\beta_i'! = (\beta_i+\beta_i')!,$$

it follows that the coefficient of $\left((a_1^{r_1} \dots a_p^{r_p})\right)$ in $\delta_\mu\left((a_1^{r_1} \dots a_p^{r_p}\, a_{p+1}^{s_1} \dots a_{p+q}^{s_q})\right)$ is precisely $\delta_\mu\left((a_1^{s_1} \dots a_q^{s_q})\right)$.

Now the coefficient of $\left((a_1^{r_1} \dots a_p^{r_p}\, a_{p+1}^{s_1} \dots a_{p+q}^{s_q})\right)$ in G_r is

$$\left(\Pi(r_i+1)!\;\Pi(s_j+1)!\right)^{-1}.$$

Hence the coefficient of $\left((\alpha_1^{r_1}\dots\alpha_p^{r_p})\right)$ in $\delta_\mu(G_r)$ is

$$\sum_{(s_j)} \frac{\delta_\mu\left((\alpha_1^{s_1}\dots\alpha_q^{s_q})\right)}{\Pi(r_i+1)!\,\Pi(s_j+1)!} \quad (s_1+\dots+s_q=\mu)$$

$$= \frac{1}{\Pi(r_i+1)!}\,\delta_\mu \sum_{(s_j)} \frac{\left((\alpha_1^{s_1}\dots\alpha_q^{s_q})\right)}{\Pi(s_j+1)!} = \frac{1}{\Pi(r_i+1)!}\,\delta_\mu(G_\mu).$$

Thus $$\delta_\mu(G_r) = \sum_{(r_i)} \frac{\left((\alpha_1^{r_1}\dots\alpha_p^{r_p})\right)}{\Pi(r_i+1)!}\,\delta_\mu(G_\mu) = G_{r-\mu}\,\delta_\mu(G_\mu),$$

and Lemma 6 is established.

LEMMA 7.

$$\delta_1(G_1)=\tfrac{1}{2}, \quad \delta_{2\mu+1}(G_{2\mu+1})=0 \ \ (\mu>0), \quad \delta_{2\mu}(G_{2\mu}) = \frac{(-)^{\mu-1}B_\mu}{(2\mu)!},$$

where B_μ is the μ-th Bernoulli number.

Proof. Let $(r_1^{\lambda_1} r_2^{\lambda_2}\dots r_k^{\lambda_k})$ be a partition of μ $(r_i>0,\ r_i\neq r_j)$. Then

$$\delta_\mu\left((\alpha_{11}^{r_1}\dots\alpha_{1\lambda_1}^{r_1}\alpha_{21}^{r_2}\dots\alpha_{2\lambda_2}^{r_2}\dots\alpha_{k1}^{r_k}\dots\alpha_{k\lambda_k}^{r_k})\right)$$

$$= \frac{1}{\lambda_1!\,\lambda_2!\dots\lambda_k!}\,\delta_\mu\left((\alpha_{11}^{r_1}\dots\alpha_{1\lambda_1}^{r_1}\dots\alpha_{k1}^{r_k}\dots\alpha_{k\lambda_k}^{r_k})\right)'$$

$$= (-)^{\lambda_1+\dots+\lambda_k-1}\,\frac{(\lambda_1+\dots+\lambda_k-1)!}{\lambda_1!\dots\lambda_k!}\,\mu,$$

by Lemma 4. Hence

$$\frac{1}{\mu}\,\delta_\mu(G_\mu) = \sum_{(\lambda_i)} \frac{(-)^{\lambda-1}(\lambda-1)!}{\lambda_1!\dots\lambda_k!\left((r_1+1)!\right)^{\lambda_1}\dots\left((r_k+1)!\right)^{\lambda_k}}$$

$$= -\sum_{(\lambda_i)} \frac{\lambda!}{\lambda_1!\dots\lambda_k!}\,\frac{1}{\lambda}\prod_{i=1}^{k}\left(\frac{-1}{(r_i+1)!}\right)^{\lambda_i}, \tag{14}$$

where $\lambda=\lambda_1+\dots+\lambda_k$, $\mu=r_1\lambda_1+\dots+r_k\lambda_k$, $r_i\neq r_j$, and the summation is over all the partitions of μ.

Consider now the expansion, in ascending powers of y, of

$$\log\left(1+\frac{x_1y}{2!}+\frac{x_2y^2}{3!}+\dots+\frac{x_ny^n}{(n+1)!}\right) = \sum_{h=1}^{\infty}\frac{(-)^{h-1}}{h}\,\frac{x_1y}{2!}+\dots+\frac{x_ny^n}{(n+1)!}\Big)^h$$

(where $n \geqslant m$). The coefficient of y^μ in $\left(\frac{x_1 y}{2!}+\ldots+\frac{x_n y^n}{(n+1)!}\right)^h$ is

$$\sum_{(\rho_i)} \frac{h!}{\rho_1! \ldots \rho_s!} \frac{x_{\gamma_1}^{\rho_1} \ldots x_{\gamma_s}^{\rho_s}}{\left((\gamma_1+1)!\right)^{\rho_1} \ldots \left((\gamma_s+1)!\right)^{\rho_s}}$$

$$(\rho_1+\ldots+\rho_s = h, \ \gamma_1\rho_1+\ldots+\gamma_s\rho_s = \mu),$$

and so the coefficient of y^μ in the expansion is

$$\sum_{(h,\,\rho)} \frac{(-)^{h-1}}{h} \frac{h!}{\rho_1! \ldots \rho_s!} \prod_{i=1}^{s} \left(\frac{x_\gamma}{(\gamma_i+1)!}\right)^{\rho_i} \tag{15}$$

$$(\rho_1+\ldots+\rho_s = h; \ \gamma_1\rho_1+\ldots+\gamma_s\rho_s = \mu).$$

Now put $x_1 = x_2 = \ldots = x_n = 1$, and let n tend to infinity. In view of (14), the expression (15) becomes $(1/\mu)\,\delta_\mu(G_\mu)$, and hence $(1/\mu)\,\delta_\mu(G_\mu)$ is the coefficient of y^μ in the expansion of

$$\log\left(1+\frac{y}{2!}+\frac{y^2}{3!}+\ldots\right) = \log\left(\frac{e^y-1}{y}\right).$$

But
$$\frac{y}{e^y-1} = 1-\tfrac{1}{2}y+\sum_{n=1}^{\infty}\frac{(-)^{n-1}B_n}{(2n)!}\,y^{2n},$$

hence
$$1+\frac{1}{e^y-1}-\frac{1}{y} = \tfrac{1}{2}+\sum_{n=1}^{\infty}\frac{(-)^{n-1}B_n}{(2n)!}\,y^{2n-1},$$

and, on integrating from 0 to y, we get

$$\log\frac{e^y-1}{y} = \frac{y}{2}+\sum_{n=1}^{\infty}\frac{(-)^{n-1}B_n}{(2n)!}\,\frac{y^{2n}}{2n}.$$

Thus, comparing coefficients of powers of y^μ, we find that

$$\delta_1(G_1) = \tfrac{1}{2}, \quad \delta_{2\mu+1}(G_{2\mu+1}) = 0 \quad (\mu > 0), \quad \delta_{2\mu}(G_{2\mu}) = \frac{(-)^{\mu-1}B_\mu}{(2\mu)!},$$

which proves Lemma 7.

LEMMA 8. *Let*

$$h_s' = h_s-\binom{\lambda}{1}h_{s-1}+\binom{\lambda}{2}h_{s-2}\ldots+(-)^s\binom{\lambda}{s},$$

and let $f(h_1, \ldots, h_m)$ be an isobaric polynomial of weight m in the h's. Let

$$f(h_1', \ldots, h_m') = u_0+u_1+u_2+\ldots+u_m,$$

where u_r is isobaric of weight $m-r$ in the h's. Then

$$-ku_k = \lambda[\delta_1 u_{k-1}+\delta_2 u_{k-2}+\ldots+\delta_k u_0] \quad (k = 1, 2, \ldots, m).$$

Proof. It is obviously sufficient to consider the case in which f is a single term $h_1^{r_1} \dots h_m^{r_m}$. Then $u_0 = h_1^{r_1} \dots h_m^{r_m}$ and

$$u_k = \sum_{\alpha_1, \dots, \alpha_s = 1}^{k} (-)^k \binom{\lambda}{\alpha_1} \dots \binom{\lambda}{\alpha_s} \delta_{\alpha_1} \dots \delta_{\alpha_s}(u_0), \quad \left(\sum_1^k \alpha_i = k\right), \tag{16}$$

where $\delta_{\alpha_1} \dots \delta_{\alpha_s}$ is the *product* of the operators $\delta_{\alpha_1}, \dots, \delta_{\alpha_s}$ and is equal to

$$\sum_{\sigma=1}^{s} \sum_{i_\sigma=1}^{m-\alpha_\sigma} h_{i_1} \dots h_{i_s} \frac{\partial^s}{\partial h_{i_1+\alpha_1} \dots \partial h_{i_s+\alpha_s}}.$$

It is easily seen that

$$\delta_\alpha[\delta_{\alpha_1} \dots \delta_{\alpha_s}(u_0)] = \left[\delta_\alpha \delta_{\alpha_1} \dots \delta_{\alpha_s} + \sum_{i=1}^{s} \delta_{\alpha_1} \dots \delta_{\alpha_i+\alpha} \dots \delta_{\alpha_s}\right](u_0). \tag{17}$$

Consider now the expression

$$\delta_1 u_{k-1} + \delta_2 u_{k-2} + \dots + \delta_k u_0.$$

Using (16) and (17) we see that it can be written in the form $\theta_k(u_0)$, where θ_k is a polynomial of weight k in $\delta_1, \delta_2, \dots, \delta_k$. The coefficient of $\delta_{\alpha_1} \dots \delta_{\alpha_s}$ in θ_k is, by (16) and (17),

$$\sum_{j=1}^{s} \sum_{i=1}^{\alpha_j - 1} (-)^{k-i} \binom{\lambda}{\alpha_1} \dots \binom{\lambda}{\alpha_j - i} \dots \binom{\lambda}{\alpha_s} + \sum_{j=1}^{s} (-)^{k-\alpha_j} \prod_{t=1}^{s} \binom{\lambda}{\alpha_t} \Big/ \binom{\lambda}{\alpha_j}.$$

If we write $\Pi_j' = \prod_{t=1}^{s} \binom{\lambda}{\alpha_t} \Big/ \binom{\lambda}{\alpha_j}$, this expression becomes

$$\begin{aligned}
&(-)^k \sum_{j=1}^{s} \left[\Pi_j' \left(\sum_{i=1}^{\alpha_j - 1} (-)^i \binom{\lambda}{\alpha_j - i} + (-1)^{\alpha_j} \right) \right] \\
&\quad = (-)^k \sum_{j=1}^{s} \left[\Pi_j' \sum_{i=1}^{\alpha_j} (-)^i \binom{\lambda}{\alpha_j - i} \right] \\
&\quad = (-)^{k+1} \sum_{j=1}^{s} \binom{\lambda - 1}{\alpha_j - 1} \Pi_j' \qquad \left[\text{since } \sum_{i=1}^{\alpha_j} (-)^i \binom{\lambda}{\alpha_j - i} = -\binom{\lambda-1}{\alpha_j - 1} \right] \\
&\quad = (-)^{k+1} \prod_{t=1}^{s} \binom{\lambda}{\alpha_t} \sum_{j=1}^{s} \binom{\lambda-1}{\alpha_j - 1} \Big/ \binom{\lambda}{\alpha_j} = (-)^{k+1} \prod_{t=1}^{s} \binom{\lambda}{\alpha_t} \sum_{j=1}^{s} \frac{\alpha_j}{\lambda} \\
&\quad = (-)^{k+1} \frac{k}{\lambda} \prod_{t=1}^{s} \binom{\lambda}{\alpha_t},
\end{aligned}$$

since $\alpha_1 + \dots + \alpha_s = k$.

Hence

$$\delta_1 u_{k-1}+\ldots+\delta_k u_0 = (-)^{k+1}\frac{k}{\lambda}\sum_{(a)}\binom{\lambda}{a_1}\ldots\binom{\lambda}{a_s}\delta_{a_1}\ldots\delta_{a_s}(u_0)$$

$$(a_1+\ldots+a_s=k)$$

$$= -\frac{k}{\lambda}u_k \quad \text{by} \quad (16).$$

This proves the lemma.

LEMMA 9. *Let*

$$f(x) = (x+1)(x+2)\ldots(x+m) = \sum_{r=0}^{m} S_r x^{m-r}.$$

Then

$$S_r = \frac{m+1}{r}\left[\tfrac{1}{2}(m+1-r)\,S_{r-1}+\sum_{i=1}^{[\frac{1}{2}r]}(-)^i\binom{m+2i-r}{2i}B_i S_{r-2i}\right]. \qquad (18)$$

*Proof**. We have, by Maclaurin's theorem, $S_r = \dfrac{1}{(m-r)!}f^{(m-r)}(0)$, the bracketed exponent denoting the $(m-r)$-th derivative. Hence (18) is equivalent to

$$rf^{(m-r)}(0) = (m+1)\left[\tfrac{1}{2}f^{(m-r+1)}(0)+\sum_{(i)}\frac{(-)^i B_i}{(2i)!}f^{(m-r+2i)}(0)\right]. \qquad (19)$$

Now, by the Euler-Maclaurin summation formula, for any polynomial $\phi(x)$,

$$\phi^{(k)}(1)-\phi^{(k)}(0)-\phi^{(k+1)}(0)$$

$$= \tfrac{1}{2}\left(\phi^{(k+1)}(1)-\phi^{(k+1)}(0)\right)+\sum_{(i)}\frac{(-)^i B_i}{(2i)!}\left(\phi^{(k+2i)}(1)-\phi^{(k+2i)}(0)\right). \qquad (20)$$

If in this we put $k = m-r$, and $\phi(x) = xf(x)$, then

$$\phi(x+1)-\phi(x) = (x+1)f(x+1)-xf(x) = (m+1)f(x),$$

and so the right-hand sides of (19) and (20) are identical. But

$$S_r = \frac{1}{(m-r)!}f^{(m-r)}(0) = \frac{1}{(m-r+1)!}\phi^{(m-r+1)}(0),$$

since

$$\phi(x) = \Sigma\, S_r x^{m-r+1}.$$

* I am indebted to Mr. A. Eagle for this simple proof of Lemma 9.

Thus

$$\phi^{(m-r+1)}(0) = (m-r+1)f^{(m-r)}(0),$$

and

$$\phi^{(m-r)}(1) - \phi^{(m-r)}(0) = (m+1)f^{(m-r)}(0).$$

Hence the left-hand members of (19) and (20) are also identical, which proves the lemma.

7. *The expression of P_a in terms of the other invariants.*

Since $P_a + (-1)^d$ is an additive numerical character of V_d, in order to prove that it is expressible linearly in terms of the invariants $\{a_1^{h_1} \dots a_s^{h_s}\}$ it is sufficient to prove that this is the case for the particular V_d considered above, the intersection of d general primals in $[2d]$. This we now proceed to do.

We have defined a number ξ_k as the sum of homogeneous products of degree k of the numbers $n_1-1, n_2-1, \dots, n_d-1$, where $n_1, n_2, \dots, n_d$ are the orders of the d primals which intersect in the V_d, Φ. Let η_k denote the corresponding function of $n_1, n_2, \dots, n_d$. In the notation of the previous section,

$$\xi_k = h_k(n_1-1, n_2-1, \dots, n_d-1), \quad \eta_k = h_k(n_1, n_2, \dots, n_d),$$

and hence, by Lemma 2,

$$\xi_k = \sum_{j=0}^{k} (-)^j \binom{d+k-1}{j} \eta_{k-j}.$$

From (5) it now follows that

$$\mu'_{d-h} = \sum_{i=0}^{d-h} \sum_{j=0}^{d-h-i} (-)^{i+j} \binom{h+1+i}{h+1} \binom{2d-h-i-1}{j} \eta_{d-h-i-j}$$

$$= \sum_{l=0}^{d-h} \sum_{i=0}^{d-h} (-)^l \binom{h+1+i}{h+1} \binom{2d-h-i-1}{l-i} \eta_{d-h-l}, \quad \text{where} \quad l = i+j.$$

But

$$\sum_{i=0}^{d-h} \binom{h+1+i}{h+1} \binom{2d-h-i-1}{l-i} = \sum_{i=0}^{l} \binom{h+1+i}{h+1} \binom{2d-h-i-1}{2d-h-l-1}$$

(since $l \leqslant d-h$)

$$= \binom{2d+1}{2d-l+1} = \binom{2d+1}{l},$$

and so

$$\mu'_{d-h} = \sum_{l=0}^{d-h} (-)^l \binom{2d+1}{l} \eta_{d-h-l},$$

or, putting $h = d-a$,

$$\mu_a' = \sum_{l=0}^{a} (-)^l \binom{2d+1}{l} \eta_{a-l}. \tag{21}$$

It follows from (6) that $\{a_1^{h_1} \dots a_s^{h_s}\}/\mu_0$ is a polynomial in $\eta_1, \eta_2, \dots, \eta_d$. The term of highest weight in the η's is $\prod_{i=1}^{s} \eta_{a_i}^{h_i}$, of weight d. Further, since $\eta_1, \dots, \eta_d$ are independent symmetric functions of $n_1, \dots, n_d$, it follows that the various terms $\prod_{i=1}^{s} \eta_{a_i}^{h_i}$ which arise from the $p(d)$ partitions of d are linearly independent. All these terms are homogeneous symmetric functions of the n's, of degree d. Denote such a term by $T_{(a)}$, where (a) represents a particular partition $(a_1^{h_1} \dots a_s^{h_s})$ of d, and let $U_{(a)}$ be the expression, in terms of the η's, of $\{a_1^{h_1} \dots a_s^{h_s}\}/\mu_0$, corresponding to the same partition. Then it is clear from (21) that $U_{(a)}$ is obtained from $T_{(a)}$ by replacing every η_j by

$$\eta_j - \binom{2d+1}{1} \eta_{j-1} + \binom{2d+1}{2} \eta_{j-2} \dots + (-)^j \binom{2d+1}{j} \eta_0.$$

Now consider the expression (8) for the arithmetic genus of Φ. It is easily seen that the term independent of the n's on the right of (8) is $(-1)^{d-1}$, and that $P_a + (-1)^d$ admits $n_1 n_2 \dots n_d$ as a factor. The quotient $[P_a + (-1)^d]/\mu_0$ must arise entirely from the term $\binom{N-1}{2d}$ in (8), which is the only term in which the product $n_1 n_2 \dots n_d$ can occur. It follows easily that

$$\frac{P_a + (-1)^d}{\mu_0} = \frac{1}{(2d)!} \sum_{r=0}^{d} (-)^r (2d-r)!\, A_r\, G_{d-r}, \tag{22}$$

where A_r is the coefficient of x^{2d-r} in $(x+1)(x+2) \dots (x+2d)$, and

$$G_{d-r} = \sum_{\alpha_1 \dots \alpha_s} \frac{1}{(\alpha_1+1)! \dots (\alpha_s+1)!} \left(\sum_{(n_i)} n_1^{\alpha_1} \dots n_s^{\alpha_s} \right) \quad (\alpha_1 + \dots + \alpha_s = d-r).$$

The leading term in (22) is G_d, which is homogeneous and of degree d in $n_1, \dots, n_d$. We can therefore express this term linearly in terms of the $p(d)$ linearly independent functions $T_{(a)}$. Suppose that

$$f(T_{(a)}) = G_d,$$

and that

$$f(U_{(a)}) = u_0 + u_1 + \ldots + u_d,$$

where $u_0 = G_d$, and u_r is of degree $d-r$ in $n_1, \ldots, n_d$.

By Lemma 8, the u's are determined by the relation

$$-ru_r = (2d+1)[\delta_1 u_{r-1} + \delta_2 u_{r-2} + \ldots + \delta_r u_0], \tag{23}$$

where δ_μ is the operator $\sum_{i=0}^{d-\mu} \eta_i \frac{\partial}{\partial \eta_{i+\mu}}$. In order to prove that

$$\frac{P_a + (-1)^d}{\mu_0} = f(U_{(a)}),$$

we have then only to prove that

$$(-)^r \frac{(2d-r)!}{(2d)!} A_r G_{d-r} = u_r,$$

or, in view of (23), that

$$-r(2d-r)! A_r G_{d-r} = (2d+1) \sum_{i=1}^{r} \delta_i \left((2d-r+i)! (-)^i A_{r-i} G_{d-r+i} \right).$$

But, by Lemmas 6 and 7, the expression on the right is equal to

$$(2d+1) \left[-\tfrac{1}{2}(2d-r+1)! A_{r-1} + \sum_j (-)^{j-1} \frac{(2d-r+2j)!}{(2j)!} A_{r-2j} B_j \right] G_{d-r},$$

where B_j is the j-th Bernoulli number, and, by Lemma 9, this is equal to $-r(2d-r)! A_r G_{d-r}$. Thus our theorem is established, namely:

The expression $P_a + (-1)^d$ can be expressed as a linear function of the invariants $\{a_1^{h_1} \ldots a_s^{h_s}\}$ derived from the canonical systems of V_d. This furnishes incidentally a proof that P_a is a relative birational invariant of V_d.

We shall now illustrate this theorem by particular cases, in which d has small values, and determine explicitly the relation referred to for $d \leqslant 6$. The cases $d = 1, 2$ are quickly disposed of. A curve has two characters, its order μ_0 (or ν_0) and its rank μ_1 (or ν_1). The only invariant connected with the canonical system is $\{1\}$, the number of points in a canonical set, and the relation between this invariant and the (arithmetic) genus is, of course,

$$2(p-1) = \{1\}.$$

For a surface we have four characters, namely, in addition to ν_0, ν_1, which refer to the linear sections of the surface, the class ν_{11} and the number

ν_2 of apparent pinch-points; the characters μ_2, μ_{11} are expressed in terms of these by the formulae

$$\mu_2 = \nu_{11}, \quad \mu_{11} = \nu_{11} + \nu_2.$$

There are two invariants, $\{1^2\}$ and $\{2\}$, corresponding to the two partitions of the integer 2. The expressions for these in terms of the projective characters can be obtained by using the formula of § 3, and we find, without difficulty, that*

$$\{2\} = \mu_2 - 2\mu_1 + 3\mu_0 = \nu_{11} - 2\nu_1 + 3\nu_0,$$

$$\{1^2\} = \mu_{11} - 6\mu_1 + 9\mu_0 = \nu_{11} - 6\nu_1 + 9\nu_0 + \nu_2.$$

To express p_a in terms of these, we consider the intersection of two primals in [4], of orders n_1 and n_2. We then find that

$$\{2\} = n_1 n_2[(n_1^2 + n_1 n_2 + n_2^2) - 5(n_1 + n_2) + 10],$$

$$\{1^2\} = n_1 n_2[(n_1^2 + 2n_1 n_2 + n_2^2) - 10(n_1 + n_2) + 25],$$

$$p_a + 1 = \frac{n_1 n_2}{24}[(4n_1^2 + 6n_1 n_2 + 4n_2^2) - 30(n_1 + n_2) + 70].$$

The terms of the second degree inside the brackets in these expressions are clearly connected by the relation

$$\tfrac{1}{24}(4n_1^2 + 6n_1 n_2 + 4n_2^2) = \tfrac{1}{12}(n_1^2 + n_1 n_2 + n_2^2) + \tfrac{1}{12}(n_1^2 + 2n_1 n_2 + n_2^2),$$

and our theorem then asserts (what is in this case easily verified directly) that

$$12(p_a + 1) = \{2\} + \{1^2\}.$$

This is the relation connecting the arithmetic genus of a surface with the other invariants. Since $\{2\} = I + 4$, $\{1^2\} = \omega - 1$, this is equivalent to the well-known relation†

$$I + \omega = 12p_a + 9.$$

8. *The invariants of a V_3.*

We now consider, from the point of view of what precedes, the invariant theory of a non-singular V_3, which has been the subject of investigation by

* *Cf.*, *e.g.*, Baker, 1, 200.

† See, *e.g.*, Baker, 1, 185.

Segre* and Roth†. A V_3 has seven e.p.c.'s; in addition to the characters ν_0, ν_1, ν_{11}, ν_2 associated with a prime section we have three others, ν_{111}, ν_{21}, ν_3, which, if the V_3 lies in [6], are the numbers of its tangent spaces satisfying the respective conditions (2, 3, 4, 6), (1, 3, 5, 6), (0, 4, 5, 6). Thus ν_{111} is the class of V_3, and ν_3 is the number of its apparent pinch-points. The five characters ν_0, ν_1, ν_{11}, ν_{111}, ν_2 are those which Roth denotes respectively by n, a, m, w, j; in place of ν_{21}, ν_3 he uses two other additive characters j_0, j_1 which, if V_3 is projected into [4], are the classes of immersion of the pinch-curve in the double surface and in the V_3. The characters ν_{21} and ν_3 are expressible in terms of Roth's j, j_0, j_1, the actual relations being determined below.

The polar varieties M_3, M_2, M_1, M_0 have respective orders $\mu_0 = \nu_0$, $\mu_1 = \nu_1$, $\mu_2 = \nu_{11}$, $\mu_3 = \nu_{111}$. M_3 is, of course, the V_3 itself. The points of the curve of intersection of two varieties M_2 are those for which the corresponding tangent [3]'s to V_3 satisfy two conditions (2, 4, 5, 6). But it is easily seen that

$$(2,\ 4,\ 5,\ 6)^2 = (1,\ 4,\ 5,\ 6) + (2,\ 3,\ 5,\ 6),$$

and so the order μ_{11} of the curve $(M_2{}^2)$ is given by

$$\mu_{11} = \nu_{11} + \nu_2, \tag{24}$$

as might have been inferred from the corresponding relation for surfaces. Again, since

$$(2,\ 4,\ 5,\ 6)(2,\ 3,\ 5,\ 6) = (1,\ 3,\ 5,\ 6) + (2,\ 3,\ 4,\ 6),$$

it follows that the order μ_{21} of the intersection $(M_2 M_1)$ is given by

$$\mu_{21} = \nu_{111} + \nu_{21}, \tag{25}$$

and similarly, since

$$(2,\ 4,\ 5,\ 6)^3 = (0,\ 4,\ 5,\ 6) + 2(1,\ 3,\ 5,\ 6) + (2,\ 3,\ 4,\ 6),$$

the character μ_{111}, which is the order of $(M_2)^3$, is given by

$$\mu_{111} = \nu_{111} + 2\nu_{21} + \nu_3. \tag{26}$$

These equations express the μ's in terms of the ν's.

* Segre, 1.
† Roth, 1, 2, 3, 4.

For the canonical systems $\{X_i\}$ we have the expressions, derived from (3) and (4),

$$\{X_0\} \equiv M_0 - 2(M_1 S) + 3(M_2 S^2) - 4(M_3 S^3),$$

$$\{X_1\} \equiv M_1 - 3(M_2 S) + 6(M_3 S^2),$$

$$\{X_2\} \equiv M_2 - 4(M_3 S),$$

where S denotes an arbitrary prime. The invariants $\{3\}$, $\{21\}$, $\{1^3\}$ of V_3 are the numbers of points in the respective intersections (X_0), $(X_1 X_2)$, $(X_2{}^3)$. Hence we find the expressions for these invariants in terms of the characters, namely:

$$\begin{aligned}\{3\} &= \mu_3 - 2\mu_2 + 3\mu_1 - 4\mu_0 \\ &= \nu_{111} - 2\nu_{11} + 3\nu_1 - 4\nu_0, \qquad (27)\end{aligned}$$

$$\begin{aligned}\{21\} &= \mu_{21} - 4\mu_2 - 3\mu_{11} + 18\mu_1 - 24\mu_0 \\ &= \nu_{111} - 7\nu_{11} + 18\nu_1 - 24\nu_0 + \nu_{21} - 3\nu_2, \qquad (28)\end{aligned}$$

$$\begin{aligned}\{1^3\} &= \mu_{111} - 12\mu_{11} + 48\mu_1 - 64\mu_0 \\ &= \nu_{111} - 12\nu_{11} + 48\nu_1 - 64\nu_0 + \nu_3 + 2\nu_{21} - 12\nu_2. \qquad (29)\end{aligned}$$

The invariant $\{1^3\}$ is merely Ω_0, the grade of the system of canonical surfaces. The invariant $\{21\}$ has been shown by Segre* to be equal to $12(\Omega_2 - \Omega_1 + \Omega_0 + 2) = 24(P_a - 1)$. Thus, comparing (29) with Roth's† formula for Ω_0, and (28) with the author's formula for P_a‡, we find that

$$\nu_{21} = 5j_1 - 2j_0 - j, \quad \nu_3 = -2j_1 + j_0 + j, \qquad (30)$$

which expresses ν_{21}, ν_3 in terms of j, j_0, j_1. It follows that ν_{21} is the character z_j of Roth§. It is worth noticing that the equations (30) may be written in the simple form

$$j + j_0 = 2j_1 + \nu_3, \quad j + j_1 = 2\nu_3 + \nu_{21}.$$

* Segre, **1**, 505.
† Roth, **4**, 228.
‡ Todd, 2.
§ Roth, **2**, 335.

The relation which connects P_a with the other invariants is in this case $24(P_a-1)=\{21\}$, as may be verified by calculating these invariants for the V_3 which is the complete intersection of three primals in [6].

9. *The invariants of a V_4.*

For a V_4, in addition to the seven elementary projective characters $\nu_0, \nu_1, \nu_2, \nu_{11}, \nu_3, \nu_{21}, \nu_{111}$ of a prime section, we have five additional characters $\nu_4, \nu_{31}, \nu_{22}, \nu_{211}, \nu_{1111}$ which, if V_4 lies in [8], are the numbers of its tangent [4] satisfying respectively the conditions (0, 5, 6, 7, 8), (1, 4, 6, 7, 8), (2, 3, 6, 7, 8), (2, 4, 5, 7, 8), (3, 4, 5, 6, 8), and five corresponding characters μ defined as the numbers of points of intersection of sets of polar loci M_i $(i=0, \ldots, 4)$. The relations expressing the μ's of "weight" less than four in terms of the ν's are, of course, those obtaining for a V_3, namely those given in (24)–(26). Since by definition $\mu_4=\nu_{1111}$, it remains only to consider the four characters $\mu_{31}, \mu_{22}, \mu_{211}, \mu_{1111}$, the orders of the intersections (M_1M_3), $(M_2{}^2)$, $(M_2M_3{}^2)$, $(M_3{}^4)$ respectively. From the relations

$$(3, 5, 6, 7, 8)\,(3, 4, 5, 7, 8)=(2, 4, 5, 7, 8)+(3, 4, 5, 6, 8),$$

$$(3, 4, 6, 7, 8)^2=(3, 4, 5, 6, 8)+(2, 3, 6, 7, 8)+(2, 4, 5, 7, 8),$$

$$(3, 5, 6, 7, 8)^2\,(3, 4, 6, 7, 8)=(3, 4, 5, 6, 8)+(2, 3, 6, 7, 8)+2(2, 4, 5, 7, 8)+(1, 4, 6, 7, 8),$$

$$(3, 5, 6, 7, 8)^4=(3, 4, 5, 6, 8)+2(2, 3, 6, 7, 8)+3(2, 4, 5, 7, 8)+3(1, 4, 6, 7, 8)+(0, 5, 6, 7, 8)$$

(which may easily be verified), it follows that

$$\begin{aligned}
&\mu_{31}=\nu_{211}+\nu_{1111}, \quad \mu_{22}=\nu_{22}+\nu_{211}+\nu_{1111},\\
&\mu_{211}=\nu_{31}+\nu_{22}+2\nu_{211}+\nu_{1111},\\
&\mu_{1111}=\nu_4+3\nu_{31}+2\nu_{22}+3\nu_{211}+\nu_{1111}.
\end{aligned} \tag{31}$$

The expressions for the canonical systems $\{X_i\}$ are given by (3) and (4), and the invariants $\{4\}$, $\{31\}$, $\{2^2\}$, $\{21^2\}$, $\{1^4\}$ of V_4 are the numbers of points in the sets (X_0), (X_1X_3), $(X_2{}^2)$, $(X_2X_3{}^2)$, $(X_3{}^4)$ respectively. Hence these invariants are expressed in terms of the projective characters by the

equations

$$\{4\} = \mu_4 - 2\mu_3 + 3\mu_2 - 4\mu_1 + 5\mu_0$$
$$= \nu_{1111} - 2\nu_{111} + 3\nu_{11} - 4\nu_1 + 5\nu_0, \tag{32}$$
$$\{31\} = \mu_{31} - 5\mu_3 - 3\mu_{21} + 15\mu_2 + 6\mu_{11} - 40\mu_1 + 50\mu_0$$
$$= \nu_{1111} - 8\nu_{111} + 21\nu_{11} - 40\nu_1 + 50\nu_0 + \nu_{211} - 3\nu_{21} + 6\nu_2, \tag{33}$$
$$\{2^2\} = \mu_{22} - 8\mu_{21} + 20\mu_2 + 16\mu_{11} - 80\mu_1 + 100\mu_0$$
$$= \nu_{1111} - 8\nu_{111} + 36\nu_{11} - 80\nu_1 + 100\nu_0 + \nu_{22} + \nu_{211} - 8\nu_{21} + 16\nu_2, \tag{34}$$
$$\{21^2\} = \mu_{211} - 10\mu_{21} - 4\mu_{111} + 25\mu_2 + 50\mu_{11} - 200\mu_1 + 250\mu_0$$
$$= \nu_{1111} - 14\nu_{111} + 75\nu_{11} - 200\nu_1 + 250\nu_0 + \nu_{31} + \nu_{22}$$
$$+ 2\nu_{211} - 4\nu_3 - 18\nu_{21} + 50\nu_2, \tag{35}$$
$$\{1^4\} = \mu_{1111} - 20\mu_{111} + 150\mu_{11} - 500\mu_1 + 625\mu_0$$
$$= \nu_{1111} - 20\nu_{111} + 150\nu_{11} - 500\nu_1 + 625\nu_0 + \nu_4 + 3\nu_{31}$$
$$+ 2\nu_{22} + 3\nu_{211} - 20\nu_3 - 40\nu_{21} + 150\nu_2. \tag{36}$$

To obtain the expression of P_a in terms of these characters we consider the intersection of four primals in [8], of orders $n_1, \ldots, n_4$. The results of §7 show that the relation which we require may be obtained by considering the terms of highest degree in the n's in the expressions for the invariants of this particular variety. These terms are in fact:

for $\{4\}$,

$$n_1 n_2 n_3 n_4 [\Sigma n_1^4 + \Sigma n_1^3 n_2 + \Sigma n_1^2 n_2^2 + \Sigma n_1^2 n_2 n_3 + n_1 n_2 n_3 n_4],$$

for $\{31\}$,

$$n_1 n_2 n_3 n_4 [\Sigma n_1^4 + 2\Sigma n_1^3 n_2 + 2\Sigma n_1^2 n_2^2 + 3\Sigma n_1^2 n_2 n_3 + 4 n_1 n_2 n_3 n_4],$$

for $\{2^2\}$,

$$n_1 n_2 n_3 n_4 [\Sigma n_1^4 + 2\Sigma n_1^3 n_2 + 3\Sigma n_1^2 n_2^2 + 4\Sigma n_1^2 n_2 n_3 + 6 n_1 n_2 n_3 n_4],$$

for $\{21^2\}$,

$$n_1 n_2 n_3 n_4 [\Sigma n_1^4 + 3\Sigma n_1^3 n_2 + 4\Sigma n_1^2 n_2^2 + 7\Sigma n_1^2 n_2 n_3 + 12 n_1 n_2 n_3 n_4],$$

for $\{1^4\}$,

$$n_1 n_2 n_3 n_4 [\Sigma n_1^4 + 4\Sigma n_1^3 n_2 + 6\Sigma n_1^2 n_2^2 + 12\Sigma n_1^2 n_2 n_3 + 24 n_1 n_2 n_3 n_4],$$

and, for P_a+1,

$$n_1n_2n_3n_4\left[\frac{1}{5!}\Sigma n_1^4+\frac{1}{4!\,2!}\Sigma n_1^3n_2+\frac{1}{(3!)^2}\Sigma n_1^2n_2^2\right.$$
$$\left.+\frac{1}{3!\,(2!)^2}\Sigma n_1^2n_2n_3+\frac{1}{(2!)^4}n_1n_2n_3n_4\right].$$

From these expressions we deduce the linear relation connecting P_a with the other invariants, namely

$$720(P_a+1)=-\{4\}+\{31\}+3\{2^2\}+4\{21^2\}-\{1^4\}. \tag{37}$$

From (32)–(37) we can now write down the expression for P_a in terms of the ν's, namely*

$$720(P_a+1)=6\nu_{1111}-66\nu_{111}+276\nu_{11}-576\nu_1+720\nu_0-\nu_4+\nu_{31}$$
$$+5\nu_{22}+9\nu_{211}+4\nu_3-59\nu_{21}+104\nu_2, \tag{38}$$

which generalizes the results already known for a V_3.

We now consider how to express the invariants Ω_0, Ω_1, Ω_2, Ω_3 of the canonical system on V_4 in terms of the five invariants $\{4\}$, $\{31\}$, $\{2^2\}$, $\{21^2\}$, $\{1^4\}$. Let $|K|$ denote this system of V_3's. The canonical system of surfaces on K is cut out by $|2K|$. Hence, if ω_0, ω_1, ω_2 are the invariants of the canonical system of K, and if p_a is the arithmetic genus of K, we have

$$\omega_2=2\Omega_2+\Omega_1,\quad \omega_1=4\Omega_1+4\Omega_0-3,\quad \omega_0=8\Omega_0,\quad p_a=\Omega_3, \tag{39}$$

from which, in virtue of the known relations of Pannelli† and Severi‡,

$$2(\omega_1-1)=3\omega_0,\quad \omega_2-\omega_1+\omega_0=2p_a-4,$$

connecting the invariants of a V_3, we deduce

$$\Omega_1-1=2\Omega_0,\quad \Omega_3-\Omega_2+\Omega_1-\Omega_0-3=0, \tag{40}$$

results due, in the first instance, to Albanese§.

The invariant Ω_0 is the grade of $|K|$, hence

$$\Omega_0=\{1^4\}. \tag{41}$$

* This is equivalent to the formula obtained by Roth, 5, whose choice of characters differs slightly from that adopted here. The other formulae given by Roth, for Ω_0 and Ω_2, correspond in the same way to those which would be obtained from (40), (41), and (44) below, if the expressions given in (32)–(36) are substituted for the appropriate invariants.

† Pannelli, 1.

‡ Severi, 2.

§ Albanese, 1.

The canonical curves $X_1(K)$ on K are cut out on K by the surfaces adjoint to K, $A_2(K)$ say. But*

$$A_2(K) \equiv X_2 + X_2(K),$$

and so $$X_1(K) \equiv \left(K \,.\, X_2 + X_2(K)\right) \equiv (K \,.\, X_2 + 2K^2),$$

or, since $K \equiv X_3$,

$$X_1(K) \equiv (X_2 X_3) + 2(X_3{}^3). \tag{42}$$

In particular, the invariant $\{21\}_K$ of K is the order of the intersection $\left(X_1(K)\, X_2(K)\right)$, where $X_2(K)$ is a canonical surface on K, and so

$$\{21\}_K = 2\{21^2\} + 4(1^4\}. \tag{43}$$

Hence, since on a V_3, $\{21\} = 24(P_a - 1)$, we have

$$12(\Omega_3 - 1) = \{21^2\} + 2\{1^4\}. \tag{44}$$

Thus, in view of (40), (41) and (44) express the Ω's in terms of the other invariants.

For the canonical series $\{X_0(K)\}$ on K we have, in the same way,

$$X_0(K) \equiv \left(K \,.\, A_1(K)\right) \equiv \left(K \,.\, X_1(K) + X_1\right),$$

where $A_1(K)$ is the system of curves adjoint to K, and hence, by (42),

$$X_0(K) \equiv (X_2 X_3{}^2) + 2(X_3{}^4) + (X_1 X_3), \tag{45}$$

whence $$\{3\}_K = \{31\} + \{21^2\} + 2\{1^4\}. \tag{46}$$

Finally, the invariant $\{1^3\}_K$ of K is the grade of the system cut on K by $|2K|$, and hence $\{1^3\}_K = 8\{1^4\}$. Thus all the invariants of the canonical system are expressed in terms of the five invariants $\{4\}$, $\{31\}$, $\{2^2\}$, $\{21^2\}$, $\{1^4\}$ of V_4.

10. *The canonical systems on V_5 and V_6.*

The number of projective characters required to define a V_d increases so rapidly with the dimension that the actual expressions of the invariants of V_d in terms of these characters ceases to be of much significance when $d \geqslant 4$. We can, however, consider how the invariants of the canonical system of V_{d-1}'s may be expressed in terms of those of V_d,.

* Todd, 3.

Consider a V_5, with canonical systems $\{X_0\}$, ..., $\{X_4\}$. If $|K|$ is the canonical system $\{X_4\}$ we denote, as usual, by Ω_0, Ω_1, ..., Ω_4 the (virtual) grade, linear genus, ... of $|K|$ and its characteristic systems. We refer to the corresponding invariants of K itself by the symbols $\{X_i(K)\}$, ω_0, ω_1, ω_2, ω_3.

Since the canonical system of V_3 on K is cut out by $|2K|$, we have the relations

$$\omega_3 = 2\Omega_3+\Omega_2, \quad \omega_2 = 4\Omega_2+4\Omega_1+\Omega_0-1,$$

$$\omega_1 = 8\Omega_1+12\Omega_0-7, \quad \omega_0 = 16\Omega_0, \tag{47}$$

which express the ω's in terms of Ω_0, ..., Ω_4. From the relations (40) applied to K we deduce without difficulty that*

$$2(\Omega_1-1) = 5\Omega_0 \tag{48}$$

and

$$2\Omega_3-3\Omega_2+4\Omega_1-5\Omega_0-9 = 0,$$

which can be written

$$2(\Omega_3-1)-3(\Omega_2+1)+4(\Omega_1-1)-5\Omega_0 = 0. \tag{49}$$

The invariant systems $\{X_i(K)\}$ are easily expressed in terms of the canonical systems $\{X_i\}$ of V_5. In fact, if $\{A_i(K)\}$ is the adjoint system of dimension i of K, so that

$$A_i(K) \equiv X_i(K)+X_i,$$

then

$$X_i(K) \equiv \big(K\, A_{i+1}(K)\big) \equiv \big(K\, X_{i+1}(K)\big)+(K\, X_{i+1}).$$

Hence, putting $i = 3, 2, 1, 0$ in turn, we find, since $K \equiv X_4$,

$$\left.\begin{aligned} X_3(K) &\equiv 2(X_4^2),\\ X_2(K) &\equiv 2(X_4^3)+(X_3 X_4),\\ X_1(K) &\equiv 2(X_4^4)+(X_3 X_4^2)+(X_2 X_4),\\ X_0(K) &\equiv 2(X_4^5)+(X_3 X_4^3)+(X_2 X_4^2)+(X_1 X_4). \end{aligned}\right\} \tag{50}$$

which lead to the following expressions for the invariants of K:

$$\left.\begin{aligned} \{1^4\}_K &= 16\{1^5\},\\ \{21^2\}_K &= 8\{1^5\}+4\{21^3\},\\ \{2^2\}_K &= 4\{1^5\}+4\{21^3\}+\{2^2 1\},\\ \{31\}_K &= 4\{1^5\}+2\{21^3\}+2\{31^2\},\\ \{4\}_K &= 2\{1^5\}+\{21^3\}+\{31^2\}+\{41\}. \end{aligned}\right\} \tag{51}$$

* Segre, 2.

The expression of P_a in terms of the invariants may in this case be obtained directly from (10) (as in any other case for which d is odd). In fact, in this case, (10) becomes

$$2P_a = \Omega_0 - \Omega_1 + \Omega_2 - \Omega_3 + \Omega_4 + 6,$$

which, in view of (48) and (49), may be written

$$4(P_a - 1) = 2\Omega_0 - (\Omega_2 + 1) + 2(\Omega_4 + 1). \tag{52}$$

Now, evidently,

$$\Omega_0 = \{1^5\}. \tag{53}$$

Applying (44) to the canonical system $|K|$, we have

$$12(\omega_3 - 1) = \{21^2\}_K + 2\{1^4\}_K.$$

When we substitute the values found in (47) and (51), and make use of (48) and (49), this becomes

$$12(\Omega_2 + 1) = 25\{1^5\} + \{21^3\}. \tag{54}$$

Again, by (37), we have

$$720(\Omega_4 + 1) = -\{4\}_K + \{31\}_K + 3\{2^2\}_K + 4\{21^2\}_K - \{1^4\}_K$$

which, in view of (51), gives

$$720(\Omega_4 + 1) = 30\{1^5\} + 29\{21^3\} + 3\{2^2 1\} + \{31^2\} - \{41\}. \tag{55}$$

In view of (48) and (49), this expresses all the Ω's in terms of the invariants $\{1^5\}$, $\{21^3\}$, $\{2^2 1\}$, $\{31^2\}$, $\{41\}$. Substituting in (52), we get

$$1440(P_a - 1) = 3\{2^2 1\} - \{21^3\} + \{31^2\} - \{41\}, \tag{56}$$

which can also be verified, as for a V_4, by calculating the leading terms on both sides when V_5 is a complete intersection of five primals in [10].

For a V_6 there are six invariants Ω_i ($i = 0, 1, \ldots, 5$). If K is a canonical V_5 on V_6, and ω_i ($i = 0, \ldots, 4$) are the invariants associated with it, it is easy to deduce the equations, analagous to (47),

$$\omega_4 = 2\Omega_4 + \Omega_3, \quad \omega_3 = 4\Omega_3 + 4\Omega_2 + \Omega_1, \quad \omega_2 = 8\Omega_2 + 12\Omega_1 + 6\Omega_0 - 5,$$

$$\omega_1 = 16\Omega_1 + 32\Omega_0 - 15, \quad \omega_0 = 32\Omega_0. \tag{57}$$

Applying to these (48), (49), and (52), we find, without difficulty, that

$$\Omega_1 - 1 = 3\Omega_0,^* \tag{58}$$

$$(\Omega_3 - 1) - 2(\Omega_2 + 1) + 3(\Omega_1 - 1) - 4\Omega_0 = 0, \tag{59}$$

$$\Omega_5 - \Omega_4 + \Omega_3 - \Omega_2 + \Omega_1 - \Omega_0 = 5, \tag{60}$$

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* Segre, 2.

of which the last is included in (10). The relation connecting P_a with the other invariants must in this case be deduced by the method of § 7, since equation (10) does not involve P_a when d is even. The calculations involved are laborious, owing to the fact that the canonical systems on V_6 define by their intersections no fewer than eleven invariants. It may, however, be verified that

$$60480(P_a+1) = 2\{6\}-2\{51\}-9\{42\}-5\{41^2\}-\{3^2\}+11\{321\}$$
$$+5\{31^3\}+10\{2^3\}+11\{2^2\,1^2\}-12\{21^4\}+2\{1^6\}. \quad (61)$$

We may remark that, in general, for odd values of d the relation between P_a and the invariants may be deduced from (10), but when d is even no such method is open, and the discovery of the relation entails much more labour. For $d = 8$, for example, the method of § 7 would entail the solution of a set of $p(8) = 22$ linear equations in order to determine the coefficients in the expression for P_a.

11. *Illustration: the variety representing pairs of points of a V_p and a V_q.*

In illustration of the foregoing results we consider the case when V_d is the locus representing the pairs of points of two non-singular varieties V_p and V_q of dimensions p, q respectively. Thus $d = p+q$, and V_d is the *topological product* of V_p and V_q. We express this symbolically in the form $V_d = V_p \times V_q$.

It is easy to prove that the canonical system of dimension i on V_d is given by

$$X_i \equiv \sum_{\nu=0}^{p} X_\nu(V_p) \times X_{i-\nu}(V_q), \quad (62)$$

where, on the right, meaningless symbols are to be replaced by zero. We prove (62) by induction, considering first the case in which $i = p+q-1$, so that (62) becomes

$$X_{p+q-1} \equiv X_{p-1}(V_p) \times V_q + V_p \times X_{q-1}(V_q). \quad (63)$$

This is true when $q = 0$. Suppose it to be true for $q-1$, and let H_{q-1} be a variety of dimension $q-1$ on V_q, varying in a linear system $|H|$. Then the variety $S = V_p \times H_{q-1}$ varies in a linear system $|S|$ on V_d. Since

(63) holds with q replaced by $q-1$,

$$X_{p+q-2}(S) \equiv X_{p-1}(V_p) \times H_{q-1} + V_p \times X_{q-2}(H_{q-1}).$$

But $X_{p+q-2}(S)$ is cut out on S by the adjoint system $A_{p+q-1}(S)$. Hence

$$\begin{aligned} A_{p+q-1}(S) &\equiv X_{p-1}(V_p) \times V_q + V_p \times A_{q-1}(H_{q-1}) \\ &\equiv X_{p-1}(V_p) \times V_q + V_p \times \left(X_{q-1}(V_q) + H_{q-1} \right) \end{aligned}$$

by the property of adjoint systems. Thus

$$X_{p+q-1}(S) \equiv A_{p+q-1}(S) - S \equiv X_{p-1}(V_p) \times V_q + V_p \times X_{q-1}(V_q),$$

which proves (63).

Suppose now that (62) is proved for $i = p+q-(r-1)$ and any q, and for $i = p+q-r$ with values of $q \leqslant q-1$. This means that

$$X_{p+q-r-1}(S) \equiv \sum_{\nu=0}^{p} X_{p-\nu}(V_p) \times X_{q-r+\nu-1}(H_{q-1}).$$

Thus*

$$\begin{aligned} A_{p+q-r}(S) &\equiv \sum_{\nu=0}^{p} X_{p-\nu}(V_p) \times A_{q-r+\nu}(H_{q-1}) \\ &\quad \sum_{\nu-0}^{p} X_{p-\nu}(V_p) \times \left(X_{q\ r+\nu}(V_q) + X_{q-r+\nu}(H_{q-1}) \right) \\ &\quad \sum_{\nu=0}^{p} X_{p-\nu}(V_p) \times X_{q-r+\nu}(V_q) + X_{p+q-r}(V_p \times H_{q-1}), \end{aligned}$$

and so, by the definition of $X_{p+q-r}(V_d)$,

$$X_{p+q-r}(V_d) \equiv A_{p+q-r}(S) - X_{p+q-r}(S) = \sum_{\nu=0}^{p} X_{p-\nu}(V_p) \times X_{q-r+\nu}(V_q),$$

which completes the proof of (62) by induction.

The canonical systems of $V_p \times V_q$ are thus expressed in terms of those of V_p and V_q. The process can evidently be extended to the product of any finite number of varieties.

* $A_q(H_{q-1}) = V_q$.

Suppose, for instance, that a V_3 represents the pairs of points of a surface S and a curve C. Then (62) gives

$$X_2(V_3) \equiv S \times X_0(C) + X_1(S) \times C,$$

$$X_1(V_3) \equiv X_1(S) \times X_0(C) + X_0(S) \times C,$$

$$X_0(V_3) \equiv X_0(S) \times X_0(C).$$

Thus (with an obvious notation) the invariants of V_3 are given by

$$\{3\} = \{2\}_S\{1\}_C, \quad \{21\} = \{2\}_S\{1\}_C + \{1^2\}_S\{1\}_C, \quad \{1^3\} = 3\{1^2\}_S\{1\}_C,$$

from the second of which it follows that

$$24(P_a - 1) = 12(p_a + 1) \cdot 2(p-1),$$

and hence
$$P_a - 1 = (p_a + 1)(p-1), \tag{64}$$

where p_a is the arithmetic genus of S, and p is the genus of C.

Again, if a V_4 represents the pairs of points of two surfaces S and S', of arithmetic genera p_a, p_a' respectively, we have

$$X_3(V) \equiv S \times X_1(S') + X_1(S) \times S',$$

$$X_2(V) \equiv S \times X_0(S') + X_1(S) \times X_1(S') + X_0(S) \times S',$$

$$X_1(V) \equiv X_1(S) \times X_0(S') + X_0(S) \times X_1(S'),$$

$$X_0(V) \equiv X_0(S) \times X_0(S'),$$

whence

$$\{4\} = \{2\}_S\{2\}_{S'}, \quad \{31\} = \{2\}_S\{1^2\}_{S'} + \{1^2\}_S\{2\}_{S'}, \quad \{2^2\} = \{1^2\}_S\{1^2\}_{S'} + 2\{2\}_S\{2\}_{S'},$$

$$\{21^2\} = 2\{1^2\}_S\{1^2\}_{S'} + \{2\}_S\{1^2\}_{S'} + \{1^2\}_S\{2\}_{S'}, \quad \{1^4\} = 6\{1^2\}_S\{1^2\}_{S'}.$$

Hence, by (37),

$$\begin{aligned} 720(P_a + 1) &= -\{4\} + \{31\} + 3\{2^2\} + 4\{21^2\} - \{1^4\} \\ &= 5[\{2\}_S + \{1^2\}_S][\{2\}_{S'} + \{1^2\}_{S'}] \\ &= 5 \cdot 144\,(p_a + 1)(p_a' + 1), \end{aligned}$$

and so
$$P_a + 1 = (p_a + 1)(p_a' + 1). \tag{65}$$

Consider finally a V_6 which is the topological product of two V_3's, S, S', of respective arithmetic genera p_a, p_a'. To economise the notation, denote the invariants $\{1^3\}$, $\{21\}$, $\{3\}$ of S by ξ, η, ζ respectively, and let ξ', η', ζ' denote the corresponding invariants of S'. (This is, in fact, Segre's

notation for the invariants of a V_3.) Using (62), we easily find that

$$\{6\} = \zeta\zeta', \quad \{51\} = \zeta\eta' + \eta\zeta', \quad \{42\} = \zeta\eta' + \eta\zeta' + \eta\eta', \quad \{41^2\} = \zeta\xi' + \xi\zeta' + 2\eta\eta',$$

$$\{3^2\} = 2(\zeta\zeta' + \eta\eta'), \quad \{321\} = \zeta\eta' + \eta\zeta' + 2\eta\eta' + \eta\xi' + \xi\eta',$$

$$\{31^3\} = \zeta\xi' + \xi\zeta' + 3(\eta\xi' + \xi\eta'), \quad \{2^3\} = 6\eta\eta' + \xi\xi',$$

$$\{2^2 1^2\} = 4\eta\eta' + 2(\eta\xi' + \xi\eta') + 2\xi\xi', \quad \{21^4\} = 4(\eta\xi' + \xi\eta') + 6\xi\xi',$$

$$\{1^6\} = 20\xi\xi',$$

and so, by (61),

$$60480(P_a + 1) = 105\eta\eta' = 105 \cdot 24 \cdot 24(p_a - 1)(p_a' - 1),$$

or

$$P_a + 1 = (p_a - 1)(p_a' - 1). \tag{66}$$

The formulae (64), (65), (66) inevitably suggest the empirical theorem* that if V_d represents the pairs of points of two varieties V_p, V_q whose arithmetic genera are respectively p_a, p_a', then the arithmetic genus P_a of V_d is given by

$$P_a + (-1)^d = [p_a + (-1)^p][p_a' + (-1)^q]. \tag{67}$$

This may be compared with the corresponding result for the *geometric* genera

$$P_g = p_g\, p_g'. \tag{68}$$

It is easy to prove (68) by transcendental methods, by calculating explicitly the form of the everywhere finite d-ple integrals on V_d. But no simple general proof of (67) suggests itself, and the increasingly complicated nature of the relations connecting the arithmetic genus with the other invariants suggests that a solution is not to be sought along the lines of this paper.

It seems however clear, from the results of this paper, that the set of invariants $\{a_1^{h_1} \ldots a_s^{h_s}\}$ are in a sense "complete" in that, at any rate in the cases considered, the other arithmetical invariants are expressible in terms of them. In particular, the arithmetic genus is so expressible, and hence can be defined in geometric terms. This has been possible hitherto only for odd values of d (and for $d = 2$), since the formula (10) only involves P_a if d is odd. Thus the present paper affords for the first time a proof that P_a is a relative invariant for all values of d. It would be desirable to know whether this invariance is in general absolute, as it is known to be for $d \leqslant 4$. To this end a study of the behaviour of the invariants $\{a_1^{h_1} \ldots a_s^{h_s}\}$ under birational transformations with fundamental elements is much to be desired.

* It is easy to verify this theorem in the remaining cases for which $p + q \leqslant 6$.

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THE CONSTRUCTION OF INVARIANT MATRICES

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Let $A = [a_{st}]$ be a matrix of order m^2, and let $P(A)$ be a matrix whose elements are polynomials in the elements of A. Then, if

$$P(A) \, . \, P(B) = P(AB), \tag{1}$$

$P(A)$ is said to be an invariant matrix of A.

If P is an invariant matrix, so also clearly is any transform of P, $T^{-1}PT$, where T is a constant matrix. Schur* has shown that any invariant matrix is expressible by transformation as the direct sum of *irreducible* invariant matrices. These irreducible invariant matrices are homogeneous in the elements of A, and the number of non-equivalent irreducible invariant matrices of degree m is equal to the number of characters of the symmetric group of order $m!$, each invariant matrix being associated with a character, and a partition of m.

The connection between the theory of invariant matrices and the theory of group characters is very intimate. The theory of invariant matrices is a generalization of the theory of matrix representations of a group to the continuous group of matrices of order m^2. An independent connection with group characters, in this case especially with the symmetric group, is shown by the fact that the spurs of the invariant matrices of A are the S-functions of the characteristic roots of A.

The theory of what is common to all equivalent matrix representations of a group leads to the theory of group characters. In the case of the symmetric group, however, Young† has constructed one specific matrix

* I. Schur, "Ueber eine Classe von Matrizen . . . ", *Diss. Berlin* (1901).

† A. Young, *Proc. London Math. Soc.* (2), 31 (1930), 253.

representation corresponding to each character, this leading to the theory of quantitative substitutional analysis. It is the purpose of this paper to do the same thing for invariant matrices, namely to construct one specific invariant matrix corresponding to each partition. We make use of Young's representation of the symmetric group, with a slight modification.

Let $(\lambda) = (\lambda_1, \lambda_2, \ldots, \lambda_p)$ be a partition of n, and let $a_1, a_2, \ldots, a_n$ be the n symbols permuted by the symmetric group of order $n!$ Young* has shown that there are $f^{(\lambda)} = \chi_0^{(\lambda)}$ standard tableaux

$$\begin{pmatrix} a_{11}, & a_{12}, & \ldots, & \ldots, & a_{1\lambda_1} \\ a_{21}, & a_{22}, & \ldots, & a_{2\lambda_2} & \\ \ldots & \ldots & \ldots & & \\ a_{p1}, & \ldots, & a_{p\lambda_p} & & \end{pmatrix}$$

such that the a_{ij} consist of the symbols $a_1, \ldots, a_n$ in some order, and the order of the symbols in each row and in each column follows the natural order. These $f^{(\lambda)}$ tableaux are arranged according to the "dictionary" order of the sequence

$$a_{11}, \ a_{12}, \ \ldots, \ a_{1\lambda_1}, \ a_{21}, \ \ldots, \ a_{2\lambda_2}, \ \ldots, \ a_{p1}, \ \ldots, \ a_{p\lambda_p},$$

and are denoted in this order by $T_1, T_2, \ldots, T_f$.

Let P_i be the product of the symmetric groups operating on the symbols in each row of T_i, and let N_i be the product of the negative symmetric groups, *i.e.*, the symmetric groups modified so that a minus sign is attached to each negative permutation, operating on the symbols in each column of T_i. Let σ_{ij} be the permutation which transforms the tableau T_i into the tableau T_j.

For $i < j$ there must be at least one pair of symbols in the same row in T and in the same column in T_j, and it follows that $N_j P_i = 0$. In most cases it happens also that there is a pair of symbols in the same row in T_j and in the same column in T_i. In this case $N_i P_j = 0$ also.

If such is true for every i and j, Young's matrix representation of the group is given by the equations

$$e_{ij} = \frac{f^{(\lambda)}}{m!} P_i \sigma_{ij} N_j. \tag{2}$$

Then $e_{ij} e_{jk} = e_{ik}$ and $e_{ij} e_{kr} = 0$ $(j \neq k)$, and if e_{ij} denotes a matrix with unity in the i-th row, j-th column, and zero elsewhere, we obtain the matrix representation of the group.

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* A. Young, *loc. cit.*

For certain pairs of tableaux corresponding to certain partitions it happens that there is no pair of symbols in the same row in T_j and in the same column in T_i. Then we have $N_i P_j \neq 0$. The matrix representation defined by (2) breaks down, for we should have $e_{ii} e_{jj} \neq 0$.

Such is the case if σ_{ij} can be expressed in the form $\sigma_{ij} = n_i p_j$, where n_i is a substitution belonging to N_i, and p_j a substitution belonging to P_j. It is convenient to put $\theta_{ij} = \pm 1$ according as n_i is a positive or a negative substitution.

Young introduces an extra factor M_i,

$$M_i = 1 - \sum_j \theta_{ij} \sigma_{ij} + \sum_{jk} \theta_{ij} \theta_{jk} \sigma_{ik} - \sum_{jkr} \theta_{ij} \theta_{jk} \theta_{kr} \sigma_{ir} + \ldots,$$

in which the summation is with respect to every j $(i<j)$ such that $N_i P_j \neq 0$, and for each j, for every k $(j<k)$ such that $N_j P_k \neq 0$, and so on. Equation (2) is replaced by

$$e_{ij} = \frac{f^{(\lambda)}}{m!} P_i \sigma_{ij} N_j M_j. \tag{3}$$

It is more convenient for the purpose of this paper to put the extra factor on the left instead of on the right. Hence we define M_i by the equation

$$M_i = 1 - \sum_j \theta_{ji} \sigma_{ji} + \sum_{jk} \theta_{kj} \theta_{ji} \sigma_{ki} - \sum_{jkr} \theta_{rk} \theta_{kj} \theta_{ji} \sigma_{ri} + \ldots,$$

summed for all j $(j<i)$ such that $N_j P_i \neq 0$, and for each j, for every k $(k<j)$ such that $N_k P_j \neq 0$, and so on. Then we have

$$e_{ij} = \frac{f^{(\lambda)}}{m!} M_i P_i \sigma_{ij} N_j. \tag{4}$$

This matrix representation, in preference to Young's, greatly simplifies the invariant matrices in such cases.

On substituting for M_i, P_i, σ_{ij}, etc., in (4), let this equation become

$$e_{ij} = \frac{f^{(\lambda)}}{m!} \sum_S \phi_{ij}^{(\lambda)}(S) \, . \, S, \tag{5}$$

where S is an arbitrary element of the symmetric group, and $\phi_{ij}^{(\lambda)}(S)$ is a scalar coefficient. Then Young proves that

$$S = \Sigma \phi_{ij}^{(\lambda)}(S^{-1}) \, . \, e_{ij}. \tag{6}$$

We now proceed to construct an invariant matrix corresponding to the partition (λ) of n, of the matrix $[a_{st}]$.

We shall use the term *minor* of a matrix in a more extensive sense than is usual, in that repetitions of the same row or of the same column are allowed. Thus the number of r-rowed minors of a matrix of order n^2 is $[n(n+1)\dots(n+r-1)/r!]^2$ rather than $[n(n-1)\dots(n-r+1)/r!]^2$. The extra minors thus introduced each have determinant zero, and for this reason are usually ignored in determinant theory.

There is a further convention that if a *row* is repeated p times in a minor, then if any product P_S of r symbols from different rows and columns is taken, as in the definitions of determinants, permanents, and immanants, the factor $1/p!$ is introduced. There is such a factor for every repeated row. Thus the permanent of the three-rowed minor

$$\begin{bmatrix} a_1, & a_2, & a_3 \\ a_1, & a_2, & a_3 \\ b_1, & b_2, & b_3 \end{bmatrix}$$

of the matrix

$$\begin{bmatrix} a_1, & a_2, & a_3 \\ b_1, & b_2, & b_3 \\ c_1, & c_2 & c_3 \end{bmatrix}$$

is $a_1a_2b_3+a_1a_3b_2+a_2a_3b_1$.

Let $(\sigma)=(\sigma_1, \sigma_2, \dots, \sigma_n)$ denote any n of the numbers 1, 2, ..., m, with possible repetitions, arranged in non-descending order.

$(\tau)=(\tau_1, \dots, \tau_n)$ is any other such set, or the same one. $A_{\sigma\tau}$ is the n-rowed minor of the matrix $[a_{st}]$ in which the rows are defined by the set of numbers (σ), and the columns by the set (τ).

Then, if $A_{\sigma\tau}=[q_{st}]$ and S is the permutation $e_1, e_2, \dots, e_n$ of the numbers 1, 2, ..., n, we define $P_{\sigma\tau}(S)$ by the equation

$$P_{\sigma\tau}(S)=q_{1e_1}q_{2e_2}\cdots q_{ne_n}/r_\sigma, \tag{7}$$

where $1/r_\sigma$ is the conventional factor introduced if any rows of $[a_{st}]$ are repeated in $A_{\sigma\tau}$.

THEOREM I. *The matrix*

$$T(A)=[\Sigma\phi^{(\lambda)}_{ts}(S^{-1})P_{\sigma\tau}(S)],$$

where s and (σ) define the rows and t and (τ) the columns, is an invariant matrix of the matrix $[a_{st}]$ corresponding to the partition (λ).

Let $[a_{st}]$, $[b_{st}]$, and $[c_{st}]$ be three matrices of order m^2 such that

$$[c_{st}] = [a_{st}][b_{st}],$$

so that

$$c_{st} = \sum_r a_{sr} b_{rt}.$$

Let the products obtained from the matrices $[b_{st}]$ and $[c_{st}]$ corresponding to the product $P_{\sigma\tau}(S)$ from $[a_{st}]$ be $Q_{\sigma\tau}(S)$ and $R_{\sigma\tau}(S)$ respectively. It is required to prove that

$$\sum_{r\mu ST} \phi^{(\lambda)}_{rs}(S^{-1})\, P_{\sigma\mu}(S)\, \phi^{(\lambda)}_{tr}(T^{-1})\, Q_{\mu\tau}(T) = \sum_S \phi^{(\lambda)}_{ts}(S^{-1})\, R_{\sigma\tau}(S). \tag{8}$$

On substituting $c_{st} = \sum a_{sr} b_{rt}$, it is easily seen that

$$R_{\sigma\tau}(S) = \sum_{\mu\tau} P_{\sigma\mu}(T)\, Q_{\mu\tau}(T^{-1} S).$$

Hence it is sufficient to prove that

$$\sum \phi^{(\lambda)}_{rs}(S^{-1})\, P_{\sigma\mu}(S)\, \phi^{(\lambda)}_{tr}(T^{-1})\, Q_{\mu\tau}(T) = \sum \phi^{(\lambda)}_{ts}(S^{-1})\, P_{\sigma\mu}(T)\, Q_{\mu\tau}(T^{-1} S).$$

That is, on picking out the coefficient of $P_{\sigma\mu}(S)\, Q_{\mu\tau}(T)$, it is sufficient to prove that

$$\sum \phi^{(\lambda)}_{rs}(S^{-1})\, \phi^{(\lambda)}_{tr}(T^{-1}) = \sum \phi^{(\lambda)}_{ts}(T^{-1} S^{-1}).$$

Substituting for S, T, and ST from (4) in the equation

$$S \,.\, T = ST,$$

we obtain

$$\sum \phi^{(\lambda)}_{ji}(S^{-1})\, e_{ij}\, \phi^{(\lambda)}_{rk}(T^{-1})\, e_{kr} = \sum \phi^{(\lambda)}_{ji}(T^{-1} S^{-1})\, e_{ij}.$$

Picking out the coefficient of e_{st}, we obtain

$$\sum \phi^{(\lambda)}_{rs}(S^{-1})\, \phi^{(\lambda)}_{tr}(T^{-1}) = \phi^{(\lambda)}_{ts}(T^{-1} S^{-1}),$$

which proves the theorem.

Example. We construct the invariant matrix of the matrix

$$A = \begin{bmatrix} a_1, & a_2, & a_3 \\ b_1 & b_2, & b_3 \\ c_1, & c_2, & c_3 \end{bmatrix}$$

corresponding to the partition (21) of 3.

Consider the symmetric group of permutations on the three symbols α, β, γ. There are two standard tableaux corresponding to (21), namely

$$T_1 = \begin{pmatrix} \alpha, & \beta \\ \gamma \end{pmatrix}, \qquad T_2 = \begin{pmatrix} \alpha, & \gamma \\ \beta \end{pmatrix},$$

from which we obtain

$$\begin{aligned} 3e_{11} &= 1+(\alpha\beta)-(\alpha\gamma)-(\alpha\beta\gamma), \\ 3e_{22} &= 1+(\alpha\gamma)-(\alpha\beta)-(\alpha\gamma\beta), \\ 3e_{12} &= (\beta\gamma)+(\alpha\gamma\beta)-(\alpha\beta\gamma)-(\alpha\gamma), \\ 3e_{21} &= (\beta\gamma)+(\alpha\beta\gamma)-(\alpha\gamma\beta)-(\alpha\beta). \end{aligned}$$

Hence the invariant matrix is given by

$T(A)$

$$= \begin{bmatrix}
a_1b_2c_3+a_2b_1c_3 & a_1b_3c_2+a_3b_1c_2 & 2a_1b_1c_2 & 0, & 2a_1b_1c_3 & 0, \\
-a_3b_2c_1-a_2b_3c_1, & -a_2b_3c_1-a_3b_2c_1, & -a_2b_1c_1-a_1b_2c_1, & & -a_3b_1c_1-a_1b_3c_1, & \\
a_1b_3c_2+a_2b_3c_1 & a_1b_2c_3+a_3b_2c_1 & 2a_1b_2c_1 & 0, & 2a_1b_3c_1 & 0, \\
-a_3b_1c_2-a_2b_1c_3, & -a_2b_1c_3-a_3b_1c_2, & -a_2b_1c_1-a_1b_1c_2, & & -a_3b_1c_1-a_1b_1c_3, & \\
a_1a_2b_3-a_2a_3b_1, & a_1a_3b_2-a_2a_3b_1, & a_1^2b_2-a_1a_2b_1, & 0, & a_1^2b_3-a_1a_3b_1, & 0, \\
-\tfrac{1}{2}(a_1a_2b_3 & -\tfrac{1}{2}(a_1a_3b_2 & -\tfrac{1}{2}(a_1^2b_2 & 0, & -\tfrac{1}{2}(a_1^2b_3 & 0, \\
\quad -a_2a_3b_1), & \quad -a_2a_3b_1), & \quad -a_1a_2b_1), & & \quad -a_1a_3b_1), & \\
a_1a_2c_3-a_2a_3c_1, & a_1a_3c_2-a_2a_3c_1, & a_1^2c_2-a_1a_2c_1, & 0, & a_1^2c_3-a_1a_3c_1, & 0, \\
-\tfrac{1}{2}(a_1a_2c_3 & -\tfrac{1}{2}(a_1a_3c_2 & -\tfrac{1}{2}(a_1^2c_2 & 0, & -\tfrac{1}{2}(a_1^2c_3 & 0, \\
\quad -a_2a_3c_1), & \quad -a_2a_3c_1), & \quad -a_1a_2c_1), & & \quad -a_1a_3c_1), & \\
b_1b_2c_3-b_2b_3c_1, & b_1b_3c_2-b_2b_3c_1, & b_1^2c_2-b_1b_2c_1, & 0, & b_1^2c_3-b_1b_3c_1, & 0, \\
-\tfrac{1}{2}(b_1b_2c_3 & -\tfrac{1}{2}(b_1b_3c_2 & -\tfrac{1}{2}(b_1^2c_2 & 0, & -\tfrac{1}{2}(b_1^2c_3 & 0, \\
\quad -b_2b_3c_1), & \quad -b_2b_3c_1), & \quad -b_1b_2c_1), & & \quad -b_1b_3c_1), & \\
\tfrac{1}{2}(a_1b_2b_3 & \tfrac{1}{2}(a_1b_2b_3 & \tfrac{1}{2}(a_1b_1b_2-a_2b_1^2), & 0, & \tfrac{1}{2}(a_1b_1b_3-a_3b_1^2), & 0, \\
\quad -a_3b_1b_2), & \quad -a_2b_1b_3), & & & & \\
\tfrac{1}{2}(a_1b_2b_3 & \tfrac{1}{2}(a_1b_2b_3 & \tfrac{1}{2}(a_1b_1b_2-a_2b_1^2), & 0, & \tfrac{1}{2}(a_1b_1b_3-a_3b_1^2), & 0, \\
\quad -a_3b_1b_2), & \quad -a_2b_1b_3), & & & & \\
\tfrac{1}{2}(a_1c_2c_3-a_3c_1c_2), & \tfrac{1}{2}(a_1c_2c_3-a_2c_1c_3), & \tfrac{1}{2}(a_1c_1c_2-a_2c_1^2), & 0, & \tfrac{1}{2}(a_1c_1c_3-a_3c_1^2), & 0, \\
\tfrac{1}{2}(a_1c_2c_3-a_3c_1c_2), & \tfrac{1}{2}(a_1c_2c_3-a_2c_1c_3), & \tfrac{1}{2}(a_1c_1c_2-a_2c_1^2), & 0, & \tfrac{1}{2}(a_1c_1c_3-a_3c_1^2), & 0, \\
\tfrac{1}{2}(b_1c_2c_3-b_3c_1c_2), & \tfrac{1}{2}(b_1c_2c_3-b_2c_1c_3), & \tfrac{1}{2}(b_1c_1c_2-b_2c_1^2), & 0, & \tfrac{1}{2}(b_1c_1c_3-b_3c_1^2), & 0, \\
\tfrac{1}{2}(b_1c_2c_3-b_3c_1c_2), & \tfrac{1}{2}(b_1c_2c_3-b_2c_1c_3), & \tfrac{1}{2}(b_1c_1c_2-b_2c_1^2), & 0, & \tfrac{1}{2}(b_1c_1c_3-b_3c_1^2), & 0,
\end{bmatrix}$$

$$\begin{array}{llllllll}
\ldots, & 2a_2b_2c_3 & a_1b_2c_2+a_2b_1c_2 & a_1b_3c_3+a_3b_1c_3 & a_2b_3c_3+a_3b_2c_3 \\
& -a_3b_2c_2-a_2b_3c_2, & 0, & -2a_2b_2c_1, & *, & -2a_3b_3c_1, & *, & -2a_3b_2c_1, \\
\ldots, & 2a_2b_3c_2 & 0, & a_1b_2c_2+a_2b_2c_1 & a_1b_3c_3+a_3b_3c_1 & a_2b_3c_3+a_3b_3c_2 \\
& -a_3b_2c_2-a_2b_2c_3, & & -2a_2b_1c_2, & *, & -2a_3b_1c_3, & *, & -2a_3b_2c_3 \\
\ldots, & a_2^2b_3-a_2a_3b_2, & 0, & a_1a_2b_2-a_2^2b_1, & *, & a_1a_3b_3-a_3^2b_1, & *, & a_2a_3b_3-a_3^2b_2, \\
\ldots, & -\frac{1}{2}(a_2^2b_3-a_2a_3b_2), & 0, & -\frac{1}{2}(a_1a_2b_2-a_2^2b_1), & *, & -\frac{1}{2}(a_1a_3b_3-a_3^2b_1), & *, & -\frac{1}{2}(a_2a_3b_3-a_3^2b_2), \\
\ldots, & a_2^2c_3-a_2a_3c_2, & 0, & a_1a_2c_2-a_2^2c_1, & *, & a_1a_3c_3-a_3^2c_1, & *, & a_2a_3c_3-a_3^2c_2, \\
\ldots, & -\frac{1}{2}(a_2^2c_3-a_2a_3c_2), & 0, & -\frac{1}{2}(a_1a_2c_2-a_2^2c_1), & *, & -\frac{1}{2}(a_1a_3c_3-a_3^2c_1), & *, & -\frac{1}{2}(a_2a_3c_3-a_3^2c_2), \\
\ldots, & b_2^2c_3-b_2b_3c_2, & 0, & b_1b_2c_2-b_2^2c_1, & *, & b_1b_3c_3-b_3^2c_1, & *, & b_2b_3c_3-b_3^2c_2, \\
\ldots, & -\frac{1}{2}(b_2^2c-b_2b_3c_2), & 0, & -\frac{1}{2}(b_1b_2c_2-b_2^2c_1), & *, & -\frac{1}{2}(b_1b_3c_3-b_3^2c_1), & *, & -\frac{1}{2}(b_2b_3c_3-b_3^2c_2), \\
\ldots, & \frac{1}{2}(a_2b_2b_3-a_3b_2^2), & 0, & \frac{1}{2}(a_1b_2^2-a_2b_1b_2), & *, & \frac{1}{2}(a_1b_3^2-a_3b_1b_3), & *, & \frac{1}{2}(a_2b_3^2-a_3b_2b_3), \\
\ldots, & \frac{1}{2}(a_2b_2b_3-a_3b_2^2), & 0, & \frac{1}{2}(a_1b_2^2-a_2b_1b_2), & *, & \frac{1}{2}(a_1b_3^2-a_3b_1b_3), & *, & \frac{1}{2}(a_2b_3^2-a_3b_2b_3), \\
\ldots, & \frac{1}{2}(a_2c_2c_3-a_3c_2^2), & 0, & \frac{1}{2}(a_1c_2^2-a_2c_1c_2), & *, & \frac{1}{2}(a_1c_3^2-a_3c_1c_3), & *, & \frac{1}{2}(a_2c_3^2-a_3c_2c_3), \\
\ldots, & \frac{1}{2}(a_2c_2c_3-a_3c_2^2), & 0, & \frac{1}{2}(a_1c_2^2-a_2c_1c_2), & *, & \frac{1}{2}(a_1c_3^2-a_3c_1c_3), & *, & \frac{1}{2}(a_2c_3^2-a_3c_2c_3), \\
\ldots, & \frac{1}{2}(b_2c_2c_3-b_3c_2^2), & 0, & \frac{1}{2}(b_1c_2^2-b_2c_1c_2), & *, & \frac{1}{2}(b_1c_3^2-b_3c_1c_3), & *, & \frac{1}{2}(b_2c_3^2-b_3c_2c_3), \\
\ldots, & \frac{1}{2}(b_2c_2c_3-b_3c_2^2), & 0, & \frac{1}{2}(b_1c_2^2-b_2c_1c_2), & *, & \frac{1}{2}(b_1c_3^2-b_3c_1c_3), & *, & \frac{1}{2}(b_2c_3^2-b_3c_2c_3),
\end{array}$$

The asterisks indicate columns each identical with the preceding column. It will be seen that the invariant matrix so formed is singular, having more than the necessary number of rows and columns. Some columns consist of zeros, others are repeated. However, if the columns of zeros and the corresponding rows are omitted, and if each repeated column is written once only, and the corresponding rows added, we obtain the following

invariant matrix, which is non-singular.

$$\begin{bmatrix}
a_1b_2c_3+a_2b_1c_3 & a_1b_3c_2+a_3b_1c_2 & 2a_1b_1c_2 & 2a_1b_1c_3 \\
\quad -a_3b_2c_1-a_2b_3c_1, & \quad -a_2b_3c_1-a_3b_2c_1, & \quad -a_1b_2c_1-a_2b_1c_1, & \quad -a_1b_3c_1-a_3b_1c_1, \\
a_1b_3c_2+a_2b_3c_1 & a_1b_2c_3+a_3b_2c_1 & 2a_1b_2c_1 & 2a_1b_3c_1 \\
\quad -a_3b_1c_2-a_2b_1c_3, & \quad -a_2b_1c_3-a_3b_1c_2, & \quad -a_1b_1c_2-a_2b_1c_1, & \quad -a_1b_1c_3-a_3b_1c_1, \\
a_1a_2b_3-a_2a_3b_1, & a_1a_3b_2-a_2a_3b_1, & a_1^2b_2-a_1a_2b_1, & a_1^2b_3-a_1a_3b_1, \\
a_1a_2c_3-a_2a_3c_1, & a_1a_3c_2-a_2a_3c_1, & a_1^2c_2-a_1a_2c_1, & a_1^2c_3-a_1a_3c_1, \\
b_1b_2c_3-b_2b_3c_1, & b_1b_3c_2-b_2b_3c_1, & b_1^2c_2-b_1b_2c_1, & b_1^2c_3-b_1b_3c_1, \\
a_1b_2b_3-a_3b_1b_2, & a_1b_2b_3-a_2b_1b_3, & a_1b_1b_2-a_2b_1^2, & a_1b_1b_3-a_3b_1^2, \\
a_1c_2c_3-a_3c_1c_2, & a_1c_2c_3-a_2c_1c_3, & a_1c_1c_2-a_2c_1^2, & a_1c_1c_3-a_3c_1^2, \\
b_1c_2c_3-b_3c_1c_2, & b_1c_2c_3-b_2c_1c_3, & b_1c_1c_2-b_2c_1^2, & b_1c_1c_3-b_3c_1^2,
\end{bmatrix}$$

$$\begin{bmatrix}
\ldots, & 2a_2b_2c_3 & a_1b_2c_2+a_2b_1c_2 & a_1b_3c_3+a_3b_1c_3 & a_2b_3c_3+a_3b_2c_3 \\
& \quad -a_2b_3c_2-a_3b_2c_2, & \quad -2a_2b_2c_1, & \quad -2a_3b_3c_1, & \quad -2a_3b_3c_2 \\
\ldots, & 2a_2b_3c_2 & a_1b_2c_2+a_2b_2c_1 & a_1b_3c_3+a_3b_3c_1 & a_2b_3c_3+a_3b_3c_2 \\
& \quad -a_2b_2c_3-a_3b_2c_2, & \quad -2a_2b_1c_2, & \quad -2a_3b_1c_3, & \quad -2a_3b_2c_3 \\
\ldots, & a_2^2b_3-a_2a_3b_2, & a_1a_2b_2-a_2^2b_1, & a_1a_3b_3-a_3^2b_1, & a_2a_3b_3-a_3^2b_2 \\
\ldots, & a_2^2c_3-a_2a_3c_2, & a_1a_2c_2-a_2^2c_1, & a_1a_3c_3-a_3^2c_1, & a_2a_3c_3-a_3^2c_2 \\
\ldots, & b_2^2c_3-b_2b_3c_2, & b_1b_2c_2-b_2^2c_1, & b_1b_3c_3-b_3^2c_1, & b_2b_3c_3-b_3^2c_2 \\
\ldots, & a_2b_2b_3-a_3b_2^2, & a_1b_2^2-a_2b_1b_2, & a_1b_3^2-a_3b_1b_3, & a_2b_3^2-a_3b_2b_3 \\
\ldots, & a_2c_2c_3-a_3c_2^2, & a_1c_2^2-a_2c_1c_2, & a_1c_3^2-a_3c_1c_3, & a_2c_3^2-a_3c_2c_3 \\
\ldots, & b_2c_2c_3-b_3c_2^2, & b_1c_2^2-b_2c_1c_2, & b_1c_3^2-b_3c_1c_3, & b_2c_3^2-b_3c_2c_3
\end{bmatrix} . \quad (9)$$

It may be verified that for either matrix $T(A).T(B) = T(AB)$.

THEOREM II. *If in the invariant matrix of Theorem I, columns of zeros and the corresponding rows are omitted, and repeated columns are written once only, and the corresponding rows added, we obtain a non-singular invariant matrix.*

It is obvious that the matrix is still invariant, for the omission of columns of zeros and corresponding rows cannot affect its invariancy, and an obvious transformation, if a column is repeated, will reduce all except

the first column to zeros, and add the corresponding rows. It is only necessary to show that the resulting matrix is non-singular. This we shall do by considering the invariant matrix of a matrix in diagonal form.

We shall later deduce Theorems III–V, which are of a very different nature, from Theorem II. Also, a direct proof of Theorem V will be given which, by a reversal of the reasoning, will lead to an alternative proof of Theorem II.

Let $(\tau) = (1^a 2^b 3^c \ldots)$. Use the symbols $\alpha_1, \alpha_2, \ldots, \alpha_a, \beta_1, \beta_2, \ldots, \beta_b, \gamma_1, \ldots, \gamma_c, \ldots$, etc., in this order, for the construction of the Young tableaux. Now for any tableau T_i in which there are two α's in the same column, say α_j and α_k, the product N_i will admit of a factor $1-(\alpha_j \alpha_k)$ on the right. Since the substitution $(\alpha_j \alpha_k)$ on the extreme right of a substitutional expression is equivalent to the interchange of the identical columns corresponding to α_j and α_k, any expression with a factor $1-(\alpha_j \alpha_k)$ on the right will give a zero term in the invariant matrix. In particular, to $M_s P_s \sigma_{si} N_i$ there corresponds a zero term for all s, and the complete column of the invariant matrix corresponding to the tableau T_i is zero. The same result holds if two β's or γ's, etc., are in the same column.

Next consider a tableau T_i and any other (standard) tableau T_j which can be obtained from it by rearrangement of the α's, β's, γ's, etc., among themselves. A factor σ_{ij} operating on the extreme right of any substitutional expression will have the effect of interchanging sets of identical columns. Hence $M_k P_k \sigma_{ki} N_i$ will give an identical expression in the invariant matrix to $M_k P_k \sigma_{ki} N_i \sigma_{ij} = M_k P_k \sigma_{kj} N_j$. Hence the two columns of the invariant matrix corresponding to T_i and T_j are identical.

It remains to show that if we choose one tableau only from each arrangement of the α's, β's, etc., irrespective of suffixes, omitting those with two α's or two β's, etc. in the same column, we obtain a non-singular invariant matrix.

Let the original matrix be the diagonal matrix

$$\Omega = \text{diag}\ (\omega_1, \omega_2, \ldots, \omega_m).$$

Clearly, to get a non-zero term of the invariant matrix we must have $(\sigma) = (\tau)$, or at least one term in each product P_S will have different suffixes and the expression will be zero.

For $(\sigma) = (\tau) = (1^a 2^b 3^c \ldots)$ there will be $a!b!c!\ldots$ products P_S which are non-zero, corresponding to the terms of the product of the symmetric groups on the α's, β's, γ's, ... respectively. Each product P_S will be equal to $\omega_1{}^a \omega_2{}^b \omega_3{}^c \ldots$.

Clearly $M_i P_i N_i$ will correspond to a multiple of $\omega_1{}^a \omega_2{}^b \omega_3{}^c \ldots$ with positive coefficient, for the only terms that will signify will correspond to partial products in which the identity is taken from M_i and from N_i, and only such expressions are taken from P_i as permute the α's, β's, γ's, etc., among themselves. For any other substitutional expression obtained from $M_i P_i N_i$ will permute symbols corresponding to different letters, and will give a zero result. Again it is clear that if σ_{ij} permutes the letters among the α's, β's, γ's, ..., then there is no expression in the product

$$M_i P_i \sigma_{ij} N_j$$

which will not give a zero result. The only case in which a non-zero result could possibly be obtained is when σ_{ik} belongs to $P_i N_k$ and σ_{jk} permutes the α's, β's, etc., among themselves only, and investigation shows that the definition of M_i is designed to make the result zero in this case also.

Thus it will be seen that the resulting invariant matrix is diagonal, every leading diagonal term differs from zero, and the matrix is non-singular.

That this invariant matrix is actually equivalent to the one that Schur* associates with the partition (λ) is seen from consideration of the diagonal terms in the invariant matrix of Ω. That term $\omega_1{}^a \omega_2{}^b \omega_3{}^c \ldots$ for which the value of a is greatest, and given this value of a, the value of b is greatest, etc., is obviously obtained from the Young tableau in which the first row consists of α's, the second row of β's, and so on, namely $\omega_1^{\lambda_1} \omega_2^{\lambda_2} \omega_3^{\lambda_3} \ldots$. Hence the partition associated with the invariant matrix by Schur is (λ), the same partition as we use.

The number of rows and columns is shown by Schur* to be

$$\left.\begin{array}{l} \phi_m^{(\lambda)} = f^{(\lambda)}/n! \text{ times the product of the first } \lambda_i \text{ terms from each } i\text{-th} \\ \quad \text{row of} \\ \\ \begin{array}{ccccccc} m, & & m+1, & m+2, & m+3, & \ldots \\ m-1, & m, & & m+1, & m+2, & \ldots \\ m-2, & m-1, & m, & & m+1, & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots, \end{array} \end{array}\right\} \quad (10)$$

where $f^{(\lambda)}$ is the degree of the corresponding character of the symmetric group.

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* Schur, *loc. cit.*

Hence we find an expression for $\phi_m^{(\lambda)}$ by counting the number of Young tableaux used in Theorem II.

THEOREM III. *The expression $\phi_m^{(\lambda)}$ gives the number of standard Young tableaux corresponding to a partition (λ) of n, that can be formed from any n of a set of m symbols, in which repetitions of the same symbol are allowed, but not in the same column.*

If $\alpha_1, \ldots, \alpha_m$ are the m symbols, take from each tableau, as defined in Theorem III, that part which consists entirely of α_1's; α_1's and α_2's; α_1's, α_2's, and α_3's; and so on. Each part corresponds to a definite partition. We obtain a sequence of m partitions such that

(1) zero is taken to be a partition,

(2) each partition is obtained from the previous partition by adding to any of the parts or leaving them unaltered,

(3) the first partition is either zero or contains one part,

(4) the $(i+1)$-th part of the $(r+1)$-th partition is less than or equal to the i-th part of the r-th partition.

(5) the m-th partition is (λ).

THEOREM IV. *The number of sequences of partitions satisfying the above five rules is $\phi_m^{(\lambda)}$.*

As an example we consider $\phi_3^{(21)}$. We have, from (10),

$$\phi_3^{(21)} = \tfrac{2}{6} \cdot 3 \cdot 4 \cdot 2 = 8,$$

as may be verified from the order of the invariant matrix already given as an example. The eight Young tableaux of Theorem III are

$$\begin{pmatrix} \alpha, & \alpha \\ \beta & \end{pmatrix}, \quad \begin{pmatrix} \alpha, & \alpha \\ \gamma & \end{pmatrix}, \quad \begin{pmatrix} \beta, & \beta \\ \gamma & \end{pmatrix},$$

$$\begin{pmatrix} \alpha, & \beta \\ \beta & \end{pmatrix}, \quad \begin{pmatrix} \alpha, & \gamma \\ \gamma & \end{pmatrix}, \quad \begin{pmatrix} \beta, & \gamma \\ \gamma & \end{pmatrix},$$

$$\begin{pmatrix} \alpha, & \beta \\ \gamma & \end{pmatrix}, \quad \begin{pmatrix} \alpha, & \gamma \\ \beta & \end{pmatrix}.$$

The eight sequences of partitions are

$$(2),\ (21),\ (21);\quad (2),\ (2),\ (21);\quad (0),\ (2),\ (21);$$

$$(1),\ (21),\ (21);\quad (1),\ (1),\ (21);\quad (0),\ (1),\ (21);$$

$$(1),\ (2),\ (21);\quad (1),\ (1^2),\ (21).$$

Now each partition in any sequence is one of the terms obtained with coefficient unity, on multiplying the previous partition by h_r for some r.

Consider the expansion of

$$[f(x)]^m = [1+h_1x+h_2x^2+h_3x^3+\ldots]^m,$$

and let the coefficient of x^n be expressed as a sum of S-functions. We pick out the coefficient of $\{\lambda\}x^n$. Each of the terms $\{\lambda\}x^n$ is obtained from a partial product

$$h_{a_1}x^{a_1}.h_{a_2}x^{a_2}.h_{a_3}x^{a_3}\ldots h_{a_m}x^{a^m}.$$

On multiplying, we get for each term $\{\lambda\}x^n$ a sequence of partitions as defined in Theorem IV, and conversely.

THEOREM V. *If $\{\lambda\}$ is an S-function associated with the series*

$$f(x) = 1+h_1x+h_2x^2+h_3x^3+\ldots,$$

then

$$[f(x)]^m = 1+\Sigma\phi_m^{(\lambda)}\{\lambda\}x^n$$

summed for all partitions (λ) of every integer n.

This is a somewhat remarkable result which, since it is true for every positive integral value of m, and since the coefficients can be shown, by the use of the binomial theorem, to be polynomials in the index, must clearly hold for fractional and negative indices.

A more direct proof seems indicated, and we give it here.

It is known that*

$$f(\rho x_1).f(\rho x_2)\ldots f(\rho x_m) = 1+\Sigma\{\lambda\}\{x;\ \lambda\}\rho^n,$$

where $\{x;\ \lambda\}$ denotes an S-function of the quantities $x_1, \ldots, x_m$.

Now, if

$$x_1 = x_2 = \ldots = x_m = 1,$$

$\{x;\ \lambda\}$ becomes an S-function of $(1-x)^{-m}$, which is known to be $\phi_m^{(\lambda)}$†.

* D. E. Littlewood, *Proc. London Math. Soc.* (2), 40 (1936), 54.

† D. E. Littlewood and A. R. Richardson, *Quart. J. of Math.* (Oxford), 5 (1934), 269.

Hence

$$[f(\rho)]^m = 1+\Sigma\phi_m^{(\lambda)}\{\lambda\}\rho^n,$$

which proves the theorem.

The same reasoning by which we deduced Theorem V from Theorem II, but reversed, will serve to deduce consecutively Theorems IV and III from Theorem V. Assuming Theorem V proved by the second method, and Theorems IV and III deduced, an alternative proof of Theorem II would be to show from Theorem III that the order of the invariant matrix is the correct order for invariant matrices corresponding to this partition, so that it must be non-singular.

Two other theorems may be deduced from Theorem II by using the fact that the spur of an invariant matrix is an S-function of the characteristic roots of the original matrix.

Take the original matrix as $X = \text{diag}\,(x_1, \ldots, x_m)$, and let $\{\lambda\}$ denote an S-function of $x_1, \ldots, x_m$. Let

$$\{\lambda\} = \Sigma k_{\lambda\mu} x_1^{\mu_1} x_2^{\mu_2} \ldots.$$

By counting the terms in the leading diagonal of $T(X)$ which correspond to the product $x_1^{\mu_1} x_2^{\mu_2} \ldots$, we have

THEOREM VI. *If*

$$\{\lambda\} = \Sigma k_{\lambda\mu} x_1^{\mu_1} x_2^{\mu_2} x_3^{\mu_3} \ldots,$$

then $K_{\lambda\mu}$ is the number of standard Young tableaux that can be formed corresponding to the partition (λ) from the symbol x_1 used μ_1 times, the symbol x_2 used μ_2 times, and so on, no symbol being repeated in the same column.

As an example,

$$\{31\} = \Sigma x_1^3 x_2 + \Sigma x_1^2 x_2^2 + 2\Sigma x_1^2 x_2 x_3 + 3\Sigma x_1 x_2 x_3 x_4,$$

and the corresponding tableaux are

$$\begin{pmatrix} x_1, & x_1, & x_1 \\ x_2 \end{pmatrix}, \quad \begin{pmatrix} x_1, & x_1, & x_2 \\ x_2 \end{pmatrix}, \quad \begin{pmatrix} x_1, & x_1, & x_2 \\ x_3 \end{pmatrix},$$

$$\begin{pmatrix} x_1, & x_1, & x_3 \\ x_2 \end{pmatrix}, \quad \begin{pmatrix} x_1, & x_2, & x_3 \\ x_4 \end{pmatrix}, \quad \begin{pmatrix} x_1, & x_2, & x_4 \\ x_3 \end{pmatrix},$$

$$\begin{pmatrix} x_1, & x_3, & x_4 \\ x_2 \end{pmatrix}.$$

Lastly, corresponding to a definite choice of (σ) and (τ) in Theorem II, with $(\sigma) = (\tau)$, the sum of the leading diagonal terms is the expression which corresponds to the sum of the diagonal terms, or the spur, of Young's matrix representation of the group. This is what is defined as the *immanant** of the minor. Hence

THEOREM VII. *The S-function $\{\lambda\}$ of the characteristic roots of a matrix A, is the sum of the corresponding immanants of the principal (coaxial) minors of A, repetitions of rows and columns being allowed, the conventional factor being introduced in such a case.*

Extension to integral equations.

It may be mentioned here that the whole of this work applies almost without modification to the theory of integral equations†. The application of Theorem VII is of especial interest.

The equation

$$[y_{sr}] = [a_{st}][x_{sr}],$$

where $[y_{sr}]$ and $[x_{sr}]$ are vectors and $[a_{st}]$ a square matrix, becomes, by a limiting process as the orders become infinite, the equation

$$g_r(u) = \int \psi(u, v)\, k_r(v)\, dv.$$

The characteristic equation of $[a_{st}]$ is replaced by the infinite series

$$f(x) = 1 + \Sigma(-1)^r a_r x^r,$$

where
$$a_r = \int\int \dots \int |\psi(u_s, u_t)|\, du_1 du_2 \dots du_r \quad (u_1 \leqslant u_2 \leqslant \dots \leqslant u_r),$$

the determinant being of order r^2.

We are able to assert that, given the necessary convergence conditions,

$$F(x) = 1/f(x) = 1 + \Sigma h_r x^r,$$

* D. E. Littlewood and A. R. Richardson, *Phil. Trans. Roy. Soc.* (A), 233 (1934), 99.

† See "Fredholm's equation" in any standard textbook on integral equations, *e.g.*, Lovitt, *Linear integral equations* (1924), ch. III.

where

$$h_r = \int\int \dots \int \overset{+}{|}\psi(u_s, u_t)\overset{+}{|} du_1 du_2 \dots du_r \quad (u_1 \leqslant u_2 \leqslant \dots \leqslant u_r),$$

and the S-function $\{\lambda\}$ associated with the series $F(x)$ is given by

$$\{\lambda\} = \int\int \dots \int |\psi(u_s, u_t)|^{(\lambda)} du_1 \dots du_r \quad (u_1 \leqslant u_2 \leqslant \dots \leqslant u_r),$$

(λ) being a partition of r, the determinant in the expression for a_r being replaced by the corresponding immanant. The range of integration is the same in each case, subject to the given inequalities, but is otherwise quite arbitrary.

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ON THE PARTITION FUNCTION $p(n)$.

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1. *Introduction.* The number $p(n)$ of unrestricted partitions of n appears in the generating function

$$(1.1) \qquad f(x) = 1 + \sum_{n=1}^{\infty} p(n) x^n = \frac{1}{(1-x)(1-x^2)(1-x^3)\dots},$$

which was discussed by Euler in his investigations on "Partitio numerorum"*. Several interesting formulae of recurrence were derived from (1.1), but no independent representation of $p(n)$ was known. Not even the order of magnitude of $p(n)$ for large n was ever examined, until, in 1917, G. H. Hardy and S. Ramanujan† applied to (1.1) their fundamental analytic method, which has been since used so successfully, in many variants, for numerous problems of additive number theory. Their result was an asymptotic formula for $p(n)$, which is surprising in the analytic theory of numbers in so far as it contains an error term approaching zero with increasing n. It reads as follows:

$$(1.2) \qquad p(n) = \frac{1}{2\pi\sqrt{2}} \sum_{k \leqslant a\sqrt{n}} A_k(n) k^{\frac{1}{2}} \frac{d}{dn}\left(\frac{e^{C\lambda_n/k}}{\lambda_n}\right) + O(n^{-\frac{1}{4}}),$$

with a an arbitrary constant,

$$(1.3) \qquad C = \pi\sqrt{(\tfrac{2}{3})}, \quad \lambda_n = \sqrt{(n - \tfrac{1}{24})},$$

* *Introductio in analysin infinitorum* (1748), cap. 16, "De partitione numerorum", *Opera omnia* (1), 8, in particular p. 318; "De partitione numerorum (1753)", *Opera omnia* (1), 2, 280.

† "Asymptotic formulae in combinatory analysis", *Proc. London Math. Soc.* (2), 17 (1918), 75–115, also *Collected papers of S. Ramanujan*, 276–309; "Une formule asymptotique pour le nombre des partitions de n", *Comptes rendus*, 164 (1917), 35–38.

and

$$A_k(n) = \sum_{\substack{h \bmod k \\ (h,k)=1}} \omega_{h,k}\, e^{-2\pi i h n/k}. \tag{1.4}$$

The symbol $\omega_{h,k}$ means here and in the following a $24k$-th root of unity given for odd h by

$$\omega_{h,k} = \left(\frac{-k}{h}\right) \exp\left(-\left\{\tfrac{1}{4}(2-hk-h)+\tfrac{1}{12}\left(k-\frac{1}{k}\right)(2h-h'+h^2h')\right\}\pi i\right), \tag{1.51}$$

and for odd k by

$$\omega_{h,k} = \left(\frac{-h}{k}\right) \exp\left(-\left\{\tfrac{1}{4}(k-1)+\tfrac{1}{12}\left(k-\frac{1}{k}\right)(2h-h'+h^2h')\right\}\pi i\right), \tag{1.52}$$

where $\left(\frac{a}{b}\right)$ denotes the Legendre-Jacobi symbol and h' is any solution of the congruence

$$hh' \equiv -1 \pmod{k}. \tag{1.6}$$

It may be mentioned, by the way, that $\omega_{h,k}$ admits also quite a different representation, viz.

$$\omega_{h,k} = \exp\left(\pi i \sum_{\mu=1}^{k-1} \frac{\mu}{k}\left(\frac{h\mu}{k} - \left[\frac{h\mu}{k}\right] - \tfrac{1}{2}\right)\right), \tag{1.7}$$

whose equivalence with (1.51) and (1.52) I proved some years ago*; $[u]$ means here, as usual, the greatest integer not exceeding u.

The object of the present paper is now to replace the asymptotic formula (1.2) by the *equality*

$$p(n) = \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{\infty} A_k(n)\, k^{\frac{1}{2}} \frac{d}{dn}\left(\frac{\sinh C\lambda_n/k}{\lambda_n}\right). \tag{1.8}$$

The asymptotic formula (1.2) will then appear as an immediate corollary of the equation (1.8).

2. *The Farey dissection.* By means of Cauchy's integral formula we obtain from (1.1)

$$p(n) = \frac{1}{2\pi i}\int_C \frac{f(x)}{x^{n+1}}\, dx. \tag{2.1}$$

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* "Bestimmung einer gewissen Einheitswurzel in der Theorie der Modulfunktionen", *Journal London Math. Soc.*, 7 (1932), 14–19.

The path of integration may be the circle C defined as

$$|x| = e^{-2\pi N^{-2}}, \tag{2.2}$$

where N is a certain positive integer at our disposal. We take the Farey series of order N and divide the circle C in the usual manner into Farey arcs $\xi_{h,k}$. We get thus

$$p(n) = \sum_{\substack{(h,k)=1 \\ 0 \leqslant h < k \leqslant N}} \frac{1}{2\pi i} \int_{\xi_{h,k}} \frac{f(x)}{x^{n+1}}\, dx. \tag{2.3}$$

The theory of the transformation of modular functions provides the formula*

$$f(e^{\frac{2\pi i h}{k} - \frac{2\pi z}{k}}) = \omega_{h,k}\, z^{\frac{1}{2}} \exp\left(\frac{\pi}{12kz} - \frac{\pi z}{12k}\right) f(e^{\frac{2\pi i h'}{k} - \frac{2\pi}{kz}}), \tag{2.4}$$

valid for $\Re(z) > 0$ and with the principal branch as the determination of $z^{\frac{1}{2}}$; $\omega_{h,k}$ and h' are explained by (1.51), (1.52), (1.6), (1.7).

We put
$$z = k(N^{-2} - i\phi)$$

and on the Farey arc $\xi_{h,k}$ therefore

$$x = e^{2\pi i h/k - 2\pi N^{-2} + 2\pi i \phi}, \tag{2.5}$$

which is in accordance with (2.2). The angle ϕ on $\xi_{h,k}$ is included between bounds

$$-\vartheta'_{h,k} \leqslant \phi \leqslant \vartheta''_{h,k}, \tag{2.6}$$

where

$$\frac{1}{2kN} \leqslant \vartheta_{h,k} < \frac{1}{kN}, \qquad \frac{1}{2kN} \leqslant \vartheta''_{h,k} < \frac{1}{kN}. \tag{2.7}$$

From (2.3), (2.4), (2.5), (2.6) we deduce

$$p(n) = \sum_{\substack{(h,k=1) \\ 0 \leqslant h < k \leqslant N}} e^{-2\pi n N^{-2}} e^{-2\pi i h n/k} \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} f(e^{2\pi i h/k - 2\pi(N^{-2} - i\phi)})\, e^{-2\pi i n \phi}\, d\phi,$$

$$p(n) = e^{-2\pi n N^{-2}} \sum_{\substack{(h,k)=1 \\ 0 \leqslant h < k \leqslant N}} \omega_{h,k}\, e^{-2\pi i h n/k} \times \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} \Psi\left(k(N^{-2} - i\phi)\right) f(e^{\frac{2\pi i h'}{k} - \frac{2\pi}{k^2(N^{-2} - i\phi)}})\, e^{-2\pi i n \phi}\, d\phi, \tag{2.81}$$

* Hardy-Ramanujan, *loc. cit.*, 93, Lemma 4.31.

where we use the abbreviation

$$\Psi(z) = z^{\frac{1}{2}} \exp\left(\frac{\pi}{12kz} - \frac{\pi z}{12k}\right). \tag{2.82}$$

3. *Approximation of the integrand.* The transformation formula was applied in order to get under the sign of integration in (2.81) a value of the function $f(x)$ which tends very rapidly to 1. Hence we replace (2.81) by

$$\begin{aligned} p(n) = {} & e^{2\pi n N^{-2}} \sum_{\substack{(h,k)=1 \\ 0 \leqq h < k \leqq N}} \omega_{h,k}\, e^{-2\pi i h n/k} \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} \Psi\Big(k(N^{-2} - i\phi)\Big)\, e^{-2\pi i n \phi}\, d\phi \\ & + e^{2\pi n N^{-2}} \sum_{\substack{(h,k)=1 \\ 0 \leqq h < k \leqq N}} \omega_{h,k}\, e^{-2\pi i h n/k} \\ & \times \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} \Psi\Big(k(N^{-2} - i\phi)\Big) \Big\{ f\big(e^{\frac{2\pi i h'}{k} - \frac{2\pi}{k(N^{-2} - i\phi)}}\big) - 1 \Big\}\, e^{-2\pi i n \phi}\, d\phi \\ = {} & e^{2\pi n N^{-2}} \sum_{h,k} \omega_{h,k}\, e^{-2\pi i h n/k}\, J_{h,k} + e^{2\pi n N^{-2}} \sum_{h,k} \omega_{h,k}\, e^{-2\pi i h n/k}\, J'_{h,k}, \end{aligned} \tag{3.1}$$

say. We shall estimate $J'_{h,k}$.

Putting

$$z = k(N^{-2} - i\phi), \tag{3.2}$$

we have to investigate

$$\Psi(z)\Big\{f\big(e^{\frac{2\pi i h'}{k} - \frac{2\pi}{kz}}\big) - 1\Big\} = z^{\frac{1}{2}} e^{\frac{\pi}{12k}\left(\frac{1}{z} - z\right)} \sum_{m=1}^{\infty} p(m)\, e^{\left(\frac{2\pi i h'}{k} - \frac{2\pi}{kz}\right)m},$$

from which we obtain

$$\Big|\Psi(z)\Big\{f\big(e^{\frac{2\pi i h'}{k} - \frac{2\pi}{kz}}\big) - 1\Big\}\Big| \leqq |z^{\frac{1}{2}}|\, e^{-\frac{\pi}{12N^2}} \sum_{m=1}^{\infty} p(m)\, e^{-\frac{2\pi}{k}(m - \frac{1}{24})\Re\left(\frac{1}{z}\right)}. \tag{3.3}$$

Now we have

$$\frac{1}{z} = \frac{1}{k(N^{-2} - i\phi)} = \frac{N^{-2} + i\phi}{k(N^{-4} + \phi^2)},$$

and for $-\vartheta'_{h,k} \leqq \phi \leqq \vartheta''_{h,k}$, by reason of (2.7),

$$\frac{1}{k}\,\Re\left(\frac{1}{z}\right) = \frac{N^{-2}}{k^2 N^{-4} + k^2 \phi^2} > \frac{N^{-2}}{k^2 N^{-4} + N^{-2}} = \frac{1}{k^2 N^{-2} + 1} \geqq \tfrac{1}{2}. \tag{3.4}$$

Furthermore,

$$|z^{\frac{1}{2}}| = |\sqrt{\{k(N^{-2} - i\phi)\}}| = \{k^2(N^{-4} + \phi^2)\}^{\frac{1}{4}} < (k^2 N^{-4} + N^{-2})^{\frac{1}{4}} \leqq 2^{\frac{1}{4}} N^{-\frac{1}{2}}. \tag{3.5}$$

The formulae (3.3), (3.4), (3.5) together yield the estimate

$$(3.6) \quad |\Psi(z)\{f(e^{\frac{2\pi i h'}{k}-\frac{2\pi}{kz}})-1\}| < 2^{\frac{1}{2}} N^{-\frac{1}{2}} \sum_{m=1}^{\infty} p(m)\, e^{-\pi(m-\frac{1}{24})} = B_1 N^{-\frac{1}{2}}.$$

The intervals of integration in (3.1) have together the length 1. Therefore (3.6) leads to

$$\left| e^{2\pi n N^{-2}} \sum_{\substack{(h,k)=1 \\ 0\leqslant h<k\leqslant N}} \omega_{h,k}\, e^{-2\pi i h n/k}\, J'_{h,k} \right| < B_1\, e^{2\pi n N^{-2}} N^{-\frac{1}{2}},$$

and from (3.1) we get the result

$$(3.71) \quad p(n) = e^{2\pi n N^{-2}} \sum_{\substack{(h,k)=1 \\ 0\leqslant h<k\leqslant N}} \omega_{h,k}\, e^{-2\pi i h n/k}\, J_{h,k} + \theta_1,$$

$$(3.72) \quad |\theta_1| < B_1\, e^{2\pi n N^{-2}}\, N^{-\frac{1}{2}}.$$

4. *A loop integral.* We have, by (3.1) and (2.82),

$$e^{2\pi n N^{-2}} J_{h,k} = e^{2\pi n N^{-2}} \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} \sqrt{\{k(N^{-2}-i\phi)\}}\, e^{\frac{\pi}{12k}\left(\frac{1}{k(N^{-2}-i\phi)}-k(N^{-2}-i\phi)\right)} e^{-2\pi i n\phi}\, d\phi.$$

This is transformed by the substitution

$$w = N^{-2}-i\phi$$

into

$$e^{2\pi n N^{-2}} J_{h,k} = e^{2\pi n N^{-2}} \int_{N^{-2}+i\vartheta'_{h,k}}^{N^{-2}-i\vartheta''_{h,k}} \sqrt{(kw)}\, e^{\frac{\pi}{12k}\left(\frac{1}{kw}-kw\right)} e^{2\pi n(w-N^{-2})}\, i dw,$$

$$(4.1) \quad e^{2\pi n N^{-2}} J_{h,k} = \frac{\sqrt{k}}{i} \int_{N^{-2}-i\vartheta''_{h,k}}^{N^{-2}+i\vartheta'_{h,k}} \sqrt{w}\, e^{\pi/12k^2 w}\, e^{2\pi(n-\frac{1}{24})w}\, dw.$$

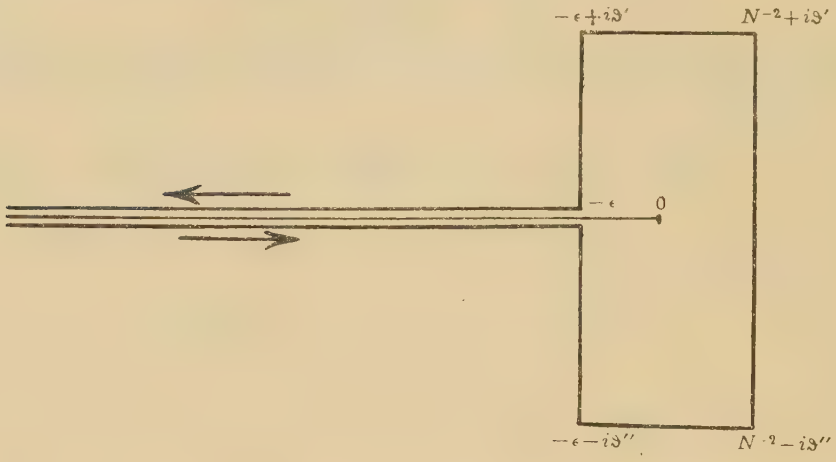
The integrand

$$(4.2) \quad \sqrt{w}\, e^{\pi/12k^2 w}\, e^{2\pi(n-\frac{1}{24})w}$$

is one-valued in the complex w-plane cut along the negative real axis from 0 to $-\infty$. In order to calculate the integral (4.1) we introduce a loop integral along the path which joins the following points by straight segments:

$$-\infty, \quad -\epsilon, \quad -\epsilon-i\vartheta''_{h,k}, \quad N^{-2}-i\vartheta''_{h,k}, \quad N^{-2}+i\vartheta'_{h,k}, \quad -\epsilon+i\vartheta'_{h,k}, \quad -\epsilon, \quad -\infty.$$

Because of the cut we have to follow first the lower border of the cut from $-\infty$ to $-\epsilon$ and end on the upper border from $-\epsilon$ to $-\infty$. The figure



indicates the path of integration in the w-plane. The arbitrary positive number ϵ may be chosen to satisfy

$$0 < \epsilon < N^{-2}. \tag{4.3}$$

If we denote (after Watson) by $\int_{-\infty}^{(0+)}$ a loop integral beginning at $-\infty$, surrounding the point 0 in the positive sense and returning to $-\infty$, we get, from (4.1),

$$e^{2\pi n N^{-2}} J_{h,k} \tag{4.4}$$

$$= \frac{k^{\frac{1}{2}}}{i} \int_{-\infty}^{(0+)} - \frac{k^{\frac{1}{2}}}{i} \left\{ \int_{-\infty}^{-\epsilon} + \int_{-\epsilon}^{-\epsilon-i\vartheta''_{h,k}} + \int_{-\epsilon-i\vartheta''_{h,k}}^{N^{-2}-i\vartheta''_{h,k}} + \int_{N^{-2}+i\vartheta'_{h,k}}^{-\epsilon+i\vartheta'_{h,k}} + \int_{-\epsilon+i\vartheta'_{h,k}}^{-\epsilon} + \int_{-\epsilon}^{-\infty} \right\}$$

$$= \frac{k^{\frac{1}{2}}}{i} L_k - \frac{k^{\frac{1}{2}}}{i} \{I_1+I_2+I_3+I_4+I_5+I_6\},$$

say. The integrals are understood all to contain the same integrand (4.2).

5. *Convergence of a series for $p(n)$.* Our next steps are to estimate the integrals I_2, I_3, I_4, I_5 and to make ϵ approach 0. The integrals I_1 and I_6 have later to be evaluated in exact form and will furnish an essential contribution to our final formula.

We have

$$(5.1)\quad I_1+I_6=\int_{-\infty}^{-\epsilon}\sqrt{|u|}\,e^{-\frac{1}{4}\pi i}\,e^{\pi/12k^2u}\,e^{2\pi(n-\frac{1}{24})u}\,du + \int_{-\epsilon}^{-\infty}\sqrt{|u|}\,e^{\frac{1}{4}\pi i}\,e^{\pi/12k^2u}\,e^{2\pi(n-\frac{1}{24})u}\,du$$
$$= -2i\int_{\epsilon}^{\infty} e^{-2\pi(n-\frac{1}{24})t}\,e^{-\pi/12k^2t}\,t^{\frac{1}{2}}\,dt.$$

The limit for $\epsilon\to 0$ will not involve any difficulty, since $e^{-\pi/12k^2t}$ tends to 0 for t approaching 0 through positive values. We have further

$$|I_2|\leqslant\int_0^{-\vartheta''_{h,k}}(\epsilon^2+v^2)^{\frac{1}{2}}\,e^{\frac{\pi}{12k^2}\,\mathfrak{R}\left(\frac{1}{-\epsilon+iv}\right)}\,e^{-2\pi(n-\frac{1}{24})\epsilon}\,|dv|.$$

But
$$\mathfrak{R}\left(\frac{1}{-\epsilon+iv}\right)=\mathfrak{R}\left(\frac{-\epsilon-iv}{\epsilon^2+v^2}\right)<0;$$

hence

$$(5.2)\quad |I_2|<(\epsilon^2+\vartheta''^2_{h,k})^{\frac{1}{2}}\,\vartheta''_{h,k}<\left(\epsilon^2+\frac{1}{k^2N^2}\right)^{\frac{1}{2}}\frac{1}{kN},$$

and similarly

$$(5.3)\quad |I_5|<(\epsilon^2+\vartheta'^2_{h,k})^{\frac{1}{2}}\,\vartheta'_{h,k}<\left(\epsilon^2+\frac{1}{k^2N^2}\right)^{\frac{1}{2}}\frac{1}{kN}.$$

For the two remaining integrals we find

$$|I_3|\leqslant\int_{-\epsilon}^{N-2}(u^2+\vartheta''^2_{h,k})^{\frac{1}{2}}\,e^{\frac{\pi}{12k^2}\,\mathfrak{R}\left(\frac{1}{u-i\vartheta''_{h,k}}\right)}\,e^{2\pi(n-\frac{1}{24})u}\,du.$$

Here we have

$$\frac{1}{k^2}\,\mathfrak{R}\left(\frac{1}{u-i\vartheta''_{h,k}}\right)=\frac{u}{k^2(u^2+\vartheta''^2_{h,k})}\leqslant\frac{N-2}{k^2\,\vartheta''^2_{h,k}}\leqslant 4.$$

This together with (4.3) leads to

$$|I_3|<(\epsilon+N^{-2})(N^{-4}+\vartheta''^2_{h,k})^{\frac{1}{2}}\,e^{\frac{1}{3}\pi}\,e^{2\pi nN^{-2}}$$
$$<(\epsilon+N^{-2})\left(N^{-4}+\frac{1}{k^2N^2}\right)^{\frac{1}{2}}e^{\frac{1}{3}\pi}\,e^{2\pi nN^{-2}},$$

i.e.

$$(5.4)\quad |I_3|<(\epsilon+N^{-2})\,2^{\frac{1}{2}}k^{-\frac{1}{2}}N^{-\frac{1}{2}}\,e^{\frac{1}{3}\pi}\,e^{2\pi nN^{-2}}.$$

The analogue is

$$(5.5)\quad |I_4|<(\epsilon+N^{-2})\,2^{\frac{1}{2}}k^{-\frac{1}{2}}N^{-\frac{1}{2}}\,e^{\frac{1}{3}\pi}\,e^{2\pi nN^{-2}}.$$

We now make $\epsilon \to 0$ in (5.1), (5.2), (5.3), (5.4), (5.5), and insert the results in (4.4). We get

$$e^{2\pi n N^{-2}} J_{h,k} = \frac{k^{\frac{1}{2}}}{i} L_k + 2k^{\frac{1}{2}} H_k + U_{h,k} + V_{h,k} \tag{5.61}$$

with

$$L_k = \int_{-\infty}^{(0+)} w^{\frac{1}{2}} e^{\pi/12k^2 w} e^{2\pi(n-\frac{1}{24})w}\, dw, \tag{5.62}$$

$$H_k = \int_0^{\infty} t^{\frac{1}{2}} e^{-\pi/12k^2 t} e^{-2\pi(n-\frac{1}{24})t}\, dt, \tag{5.63}$$

$$|U_{h,k}| \leqq 2N^{-\frac{3}{2}} k^{-1}, \tag{5.64}$$

$$|V_{h,k}| \leqq 2^{\frac{5}{4}} e^{\frac{1}{3}\pi} e^{2\pi n N^{-2}} N^{-\frac{5}{2}}. \tag{5.65}$$

Putting

$$\psi_k(n) = \frac{k^{\frac{1}{2}}}{i} L_k + 2k^{\frac{1}{2}} H_k \tag{5.7}$$

and introducing (5.61) into (3.71), we obtain

$$p(n) = \sum_{\substack{(h,k)=1 \\ 0 \leqq h < k \leqq N}} \omega_{h,k} e^{-2\pi i h n/k} \psi_k(n) + \theta_1 + \sum_{\substack{(h,k)=1 \\ 0 \leqq h < k \leqq N}} U_{h,k} + \sum_{\substack{(h,k)=1 \\ 0 \leqq h < k \leqq N}} V_{h,k}.$$

But we have, from (5.64),

$$\left| \sum_{\substack{(h,k)=1 \\ 0 \leqq h < k \leqq N}} U_{h,k} \right| \leqq 2N^{-\frac{3}{2}} \sum_{0 < k \leqq N} 1 = 2N^{-\frac{1}{2}},$$

and, from (5.65),

$$\left| \sum_{\substack{(h,k)=1 \\ 0 \leqq h < k \leqq N}} V_{h,k} \right| \leqq 2^{\frac{5}{4}} e^{\frac{1}{3}\pi} e^{2\pi n N^{-2}} N^{-\frac{5}{2}} \sum_{0 < k \leqq N} k \leqq 2^{\frac{5}{4}} e^{\frac{1}{3}\pi} e^{2\pi n N^{-2}} N^{-\frac{1}{2}},$$

and hence, observing (3.72), and with the notation (1.4), we get

$$p(n) = \sum_{k=1}^{N} A_k(n) \psi_k(n) + R(n, N), \tag{5.81}$$

$$|R(n, N)| < B_2 e^{2\pi n N^{-2}} N^{-\frac{1}{2}}. \tag{5.82}$$

If we keep n fixed, we can infer from (5.82) that when $N \to \infty$ the remainder tends to zero. This means that the series extended over all k is convergent and has $p(n)$ as its value: *i.e.*

$$p(n) = \sum_{k=1}^{\infty} A_k(n) \psi_k(n). \tag{5.9}$$

6. *Evaluation of an integral.* The final steps are directed towards the calculation of $\psi_k(n)$, *i.e.* of the integrals L_k and H_k. We have, from (5.62)*,

$$\begin{aligned}\frac{1}{i}L_k &= \frac{1}{i}\int_{-\infty}^{(0+)} w^{\frac{1}{2}}\, e^{\pi/12k^2 w}\, e^{2\pi(n-\frac{1}{24})w}\, dw \\ &= \frac{1}{i}\int_{-\infty}^{(0+)} w^{\frac{1}{2}}\, e^{2\pi(n-\frac{1}{24})w} \sum_{l=0}^{\infty} \frac{(\pi/12k^2 w)^l}{l!}\, dw \\ &= \frac{1}{i}\sum_{l=0}^{\infty}\frac{(\pi/12k^2)^l}{l!}\int_{-\infty}^{(0+)} w^{\frac{1}{2}}\, e^{2\pi(n-\frac{1}{24})w}\, w^{-l}\, dw \\ &= 2\pi\sum_{l=0}^{\infty}\frac{(\pi/12k^2)^l}{l!}\{2\pi(n-\tfrac{1}{24})\}^{l-\frac{3}{2}}\,\frac{1}{2\pi i}\int_{-\infty}^{(0+)} e^z z^{-l+\frac{1}{2}}\, dz.\end{aligned}$$

We apply Hankel's formula

$$\frac{1}{\Gamma(s)} = \frac{1}{2\pi i}\int_{-\infty}^{(0+)} e^z z^{-s}\, dz$$

and obtain

$$\frac{1}{i}L_k = \frac{1}{\sqrt{(2\pi)}}\,(n-\tfrac{1}{24})^{-\frac{3}{2}}\sum_{l=0}^{\infty}\frac{\{\pi^2(n-\frac{1}{24})/6k^2\}^l}{l!\,\Gamma(l-\frac{1}{2})}. \tag{6.1}$$

This infinite series may be studied separately: we have

$$\begin{aligned}\sum_{l=0}^{\infty}\frac{X^l}{l!\,\Gamma(l-\frac{1}{2})} &= \frac{1}{-2\pi^{\frac{1}{2}}} + \frac{X}{1!\,\pi^{\frac{1}{2}}} + \frac{X^2}{2!\,\frac{1}{2}\pi^{\frac{1}{2}}} + \frac{X^3}{3!\,\frac{1}{2}\cdot\frac{3}{2}\pi^{\frac{1}{2}}} + \dots \\ &= \frac{1}{\pi^{\frac{1}{2}}}\left\{-\tfrac{1}{2} + \frac{2X}{2!} + \frac{3\cdot 2^3 X^2}{4!} + \frac{5\cdot 2^5 X^3}{6!} + \dots\right\} \\ &= \frac{1}{2\pi^{\frac{1}{2}}}\left\{-1 + \frac{4X}{2!} + \frac{3(4X)^2}{4!} + \frac{5(4X)^3}{6!} + \dots\right\}.\end{aligned}$$

This yields, for $4X = Y^2$,

$$\begin{aligned}\sum_{l=0}^{\infty}\frac{(\frac{1}{4}Y^2)^l}{l!\,\Gamma(l-\frac{1}{2})} &= \frac{1}{2\pi^{\frac{1}{2}}}\left\{-1 + Y^2\left(\frac{1}{2!} + \frac{3Y^2}{4!} + \frac{5Y^4}{6!} + \dots\right)\right\} \\ &= \frac{1}{2\pi^{\frac{1}{2}}}\left\{-1 + Y^2\frac{d}{dY}\left(\frac{Y}{2!} + \frac{Y^3}{4!} + \frac{Y^5}{6!} + \dots\right)\right\} \\ &= \frac{1}{2\pi^{\frac{1}{2}}}\left\{-1 + Y^2\frac{d}{dY}\left(\frac{\cosh Y - 1}{Y}\right)\right\},\end{aligned}$$

$$\sum_{l=1}^{\infty}\frac{(\frac{1}{4}Y^2)^l}{l!\,\Gamma(l-\frac{1}{2})} = \frac{1}{2\pi^{\frac{1}{2}}}\,Y^2\frac{d}{dY}\left(\frac{\cosh Y}{Y}\right). \tag{6.2}$$

* The integral L_k is well known and is connected with Bessel functions of purely imaginary argument: *cf.* Watson, *Bessel functions*, 181. We prefer the direct calculation.

In order to use (6.2) for (6.1) we have to put

$$\tfrac{1}{4}Y^2 = \frac{\pi^2}{6k^2}\,(n-\tfrac{1}{24}),$$

$$Y = \frac{\pi}{k}\sqrt{(\tfrac{2}{3})}\,\sqrt{(n-\tfrac{1}{24})} = \frac{C\lambda_n}{k} \tag{6.3}$$

in the notation (1.3). Moreover, we have

$$\frac{d}{dn}\left(\frac{\cosh Y}{Y}\right) = \frac{d}{dY}\left(\frac{\cosh Y}{Y}\right)\frac{dY}{dn} = \frac{d}{dY}\left(\frac{\cosh Y}{Y}\right)\frac{C}{k\,.\,2\lambda_n}. \tag{6.4}$$

Combining (6.1), (6.2), (6.3), (6.4), we get

$$\frac{1}{i}\,L_k = \frac{1}{\sqrt{(2\pi)}}\,(n-\tfrac{1}{24})^{-\frac{3}{2}}\,\frac{1}{2\sqrt{\pi}}\,\frac{C^2\lambda_n{}^2}{k^2}\,\frac{2k\lambda_n}{C}\,\frac{d}{dn}\left(\frac{\cosh Y}{Y}\right)$$

$$= \frac{1}{\pi\sqrt{2}}\,\frac{C}{k}\,\frac{d}{dn}\left(\frac{\cosh Y}{Y}\right),$$

and finally

$$\frac{1}{i}\,L_k = \frac{1}{\pi\sqrt{2}}\,\frac{d}{dn}\left(\frac{\cosh C\lambda_n/k}{\lambda_n}\right). \tag{6.5}$$

7. *Evaluation of another integral and proof of the theorem.* There remains only the integral (5.63), namely

$$H_k = \int_0^\infty t^{\frac{1}{2}}\,e^{-\pi/12k^2 t}\,e^{-2\pi(n-\frac{1}{24})t}\,dt.$$

Now

$$\int_0^\infty e^{-c^2t-a^2/t}\,t^{-\frac{1}{2}}\,dt = 2\int_0^\infty e^{-c^2u^2-a^2/u^2}\,du = \frac{\sqrt{\pi}}{c}\,e^{-2ac},$$

and so

$$\int_0^\infty e^{-c^2t-a^2/t}\,t^{\frac{1}{2}}\,dt = -\frac{\sqrt{\pi}}{2c}\,\frac{d}{dc}\left(\frac{e^{-2ac}}{c}\right).$$

Putting $c = \sqrt{(2\pi)}\lambda_n$, $a = \frac{1}{k}\sqrt{\left(\frac{\pi}{12}\right)}$, we find that

$$H_k = -\frac{1}{2\sqrt{2}\,\lambda_n}\,\frac{d}{d\lambda_n}\left(\frac{e^{-C\lambda_n/k}}{\sqrt{(2\pi)}\,\lambda_n}\right)\frac{1}{\sqrt{(2\pi)}},$$

and hence

$$H_k = -\frac{1}{2\pi\sqrt{2}}\,\frac{d}{dn}\left(\frac{e^{-C\lambda_n/k}}{\lambda_n}\right). \tag{7.1}$$

From (5.7), (6.5), and (7.1) we deduce

$$\psi_k(n) = \frac{\sqrt{k}}{\pi\sqrt{2}} \frac{d}{dn}\left(\frac{\cosh(C\lambda_n/k)}{\lambda_n} - \frac{e^{-C\lambda_n/k}}{\lambda_n}\right),$$

so that, finally,

$$\psi_k(n) = \frac{\sqrt{k}}{\pi\sqrt{2}} \frac{d}{dn}\left(\frac{\sinh(C\lambda_n/k)}{\lambda_n}\right). \tag{7.2}$$

This together with (5.9) proves the

THEOREM. *The number $p(n)$ of unrestricted partitions of n is given by the convergent series*

$$p(n) = \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{\infty} A_k(n)\, k^{\frac{1}{2}} \frac{d}{dn}\left(\frac{\sinh(C\lambda_n/k)}{\lambda_n}\right), \tag{7.3}$$

where C, λ_n, $A_k(n)$ are defined by (1.3) *and* (1.4).

8. *Asymptotic formulae.* We proceed to discuss asymptotic formulae as consequences of our theorem. If we put, in (5.81), (5.82),

$$N = [\alpha\sqrt{n}]$$

we get

$$p(n) = \sum_{k=1}^{[\alpha\sqrt{n}]} A_k(n)\,\psi_k(n) + O(n^{-\frac{1}{4}}). \tag{8.1}$$

This is not yet the Hardy-Ramanujan formula (1.2). In order to obtain the latter we have to estimate the error which arises from replacing $\sinh(C\lambda_n/k)$ in (8.1) by $\frac{1}{2}\exp(C\lambda_n/k)$. We get

$$\begin{aligned}\sum_{1\leqq k\leqq \alpha\sqrt{n}} A_k n^{\frac{1}{2}} \frac{d}{du}\left(\frac{e^{-C\lambda_n/k}}{\lambda_n}\right) &= O\left\{\sum_{1\leqq k\leqq \alpha\sqrt{n}} k^{\frac{3}{2}}\left(\frac{1}{kn} + \frac{1}{n^{\frac{3}{2}}}\right)\right\} \\ &= O\left(n^{-1}\sum_{1\leqq k\leqq \alpha\sqrt{n}} k^{\frac{1}{2}}\right) + O\left(n^{-\frac{3}{2}}\sum_{1\leqq k\leqq \alpha\sqrt{n}} k^{\frac{3}{2}}\right) \\ &= O(n^{-1}n^{\frac{3}{4}}) + O(n^{-\frac{3}{2}}n^{\frac{5}{4}}) = O(n^{-\frac{1}{4}}).\end{aligned}$$

This together with (8.1) proves (1.2).

For the purpose of numerical computation, however, it is desirable to know suitable bounds for the constants involved in the O-terms. Such an estimate is better obtained from the convergent series (7.3) directly. We

have

$$\left|\frac{1}{\pi\sqrt{2}}\sum_{k=N+1}^{\infty}A_k(x)\,k^{\frac{1}{2}}\frac{d}{dn}\left(\frac{\sinh(C\lambda_n/k)}{\lambda_n}\right)\right|$$

$$\leqq\frac{1}{\pi\sqrt{2}}\sum_{k=N+1}^{\infty}k^{\frac{3}{2}}\frac{d}{dn}\sum_{\nu=0}^{\infty}\frac{(C/k)^{2\nu+1}(n-\frac{1}{24})^{\nu}}{(2\nu+1)!}$$

$$=\frac{1}{\pi\sqrt{2}}\sum_{k=N+1}^{\infty}k^{\frac{3}{2}}\sum_{\nu=1}^{\infty}\frac{\nu}{(2\nu+1)!}\left(\frac{C}{k}\right)^{2\nu+1}\lambda_{n\,i}^{2\nu-2}$$

$$<\frac{C^2}{\pi\sqrt{2}\,\lambda_n}\sum_{\nu=1}^{\infty}\frac{\nu}{(2\nu+1)!}(C\lambda_n)^{2\nu-1}\int_N^{\infty}\frac{dk}{k^{2\nu-\frac{1}{2}}}$$

$$=\frac{C^2}{\pi\sqrt{2}\,\lambda_n}\sum_{\nu=1}^{\infty}\frac{\nu}{(2\nu-\frac{3}{2})(2\nu+1)!}\frac{(C\lambda_n)^{2\nu-1}}{N^{2\nu-\frac{3}{2}}}$$

$$=\frac{C^2N^{\frac{1}{2}}}{\pi\sqrt{2}\,\lambda_n}\left\{\frac{1}{3}\frac{C\lambda_n}{N}+\sum_{\nu=2}^{\infty}\frac{(C\lambda_n/N)^{2\nu-1}}{(2\nu-1)!\,(2\nu+1)\,(4\nu-3)}\right\}$$

$$<\frac{C^2N^{\frac{1}{2}}}{\pi\sqrt{2}\,\lambda_n}\left\{\frac{1}{3}\frac{C\lambda_n}{N}+\frac{1}{5^2}\sum_{\nu=2}^{\infty}\frac{(C\lambda_n/N)^{2\nu-1}}{(2\nu-1)!}\right\}$$

$$=\frac{C^2N^{\frac{1}{2}}}{\pi\sqrt{2}\,\lambda_n}\left\{(\tfrac{1}{3}-\tfrac{1}{2\,5})\frac{C\lambda_n}{N}+\tfrac{1}{2\,5}\sinh\frac{C\lambda_n}{N}\right\},$$

which with (7.3) leads to

$$(8.2)\quad p(n)=\frac{1}{\pi\sqrt{2}}\sum_{k=1}^{N}A_k(n)\,k^{\frac{1}{2}}\frac{d}{dn}\left(\frac{\sinh(C\lambda_n/k)}{\lambda_n}\right)$$

$$+\vartheta'\left\{\frac{44\pi^2}{225\sqrt{3}}N^{-\frac{1}{2}}+\frac{\pi\sqrt{2}}{75}\left(\frac{N}{n-1}\right)^{\frac{1}{2}}\sinh\frac{C\sqrt{n}}{N}\right\}\quad(|\vartheta'|<1).$$

Finally we replace $\sinh C\lambda_n/k$ in (8.2) by $\frac{1}{2}\exp(C\lambda_n/k)$, committing thereby an error which lends itself to the following estimate:

$$\left|\frac{1}{2\pi\sqrt{2}}\sum_{k=1}^{N}A_k(n)\,k^{\frac{1}{2}}\frac{d}{dn}\left(\frac{e^{-C\lambda_n/k}}{\lambda_n}\right)\right|$$

$$\leqq\frac{1}{2\pi\sqrt{2}}\sum_{k=1}^{N}k^{\frac{3}{2}}\left(\frac{e^{-C\lambda_n/k}}{2\lambda_n{}^2}\frac{C}{k}+\frac{e^{-C\lambda_n/k}}{2\lambda_n{}^3}\right)$$

$$=\frac{C}{4\pi\sqrt{2}}\frac{1}{n-\frac{1}{24}}\sum_{k=1}^{N}k^{\frac{1}{2}}e^{-C\lambda_n/k}+\frac{1}{4\pi\sqrt{2}}\frac{1}{(n-\frac{1}{24})^{\frac{3}{2}}}\sum_{k-1}^{N}k^{\frac{3}{2}}e^{-C\lambda_n/k}$$

$$<\frac{C}{4\pi\sqrt{2}}\frac{1}{n-1}e^{-C\lambda_n/N}\int_0^{N+1}k^{\frac{1}{2}}dk+\frac{1}{4\pi\sqrt{2}}\frac{1}{(n-1)^{\frac{3}{2}}}e^{-C\lambda_n/N}\int_0^{N+1}k^{\frac{3}{2}}dk$$

$$<\frac{1}{2\sqrt{3}}\frac{(N+1)^{\frac{3}{2}}}{n-1}e^{-C\lambda_n/N}\left(\frac{1}{3}+\frac{\sqrt{3}}{5\pi\sqrt{3}}\frac{N+1}{(n-1)^{\frac{1}{2}}}\right).$$

This together with (8.2) yields the

COROLLARY.

$$(8.3)\qquad p(n)=\frac{1}{2\pi\sqrt{2}}\sum_{k=1}^{N}A_k(n)\,k^{\frac{1}{2}}\,\frac{d}{dn}\left(\frac{e^{C\lambda_n/k}}{\lambda_n}\right)$$
$$+\vartheta\left\{\frac{44\pi^2}{225\sqrt{3}}N^{-\frac{1}{2}}+\frac{\pi\sqrt{2}}{75}\left(\frac{N}{n-1}\right)^{\frac{1}{2}}\sinh\frac{C\sqrt{n}}{N}\right.$$
$$\left.+\frac{1}{2\sqrt{3}}\,\frac{(N+1)^{\frac{3}{2}}}{n-1}\,e^{-C(n-1)^{\frac{1}{2}}/N}\left(\tfrac{1}{3}+\frac{\sqrt{3}}{5\pi\sqrt{2}}\,\frac{N+1}{(n-1)^{\frac{1}{2}}}\right)\right\}\quad(|\vartheta|<1).$$

9. *A numerical application.* Recently D. H. Lehmer* has used (1.2) for the computation of $p(599)$ and $p(721)$. By some special cases of Ramanujan's conjecture which are already proved we know that

$$(9.1)\qquad p(599)\equiv 0 \pmod{5^3},$$

$$(9.2)\qquad p(721)\equiv 0 \pmod{11^2}.$$

Now (1.2) asserts only that the error after $[a\sqrt{n}]$ terms tends to zero like $n^{-\frac{1}{4}}$. But since the constants in the O-term were not known, a conclusive control of the numerical result was not possible. Lehmer had to admit the possibility that his result for $p(599)$ differed from the true value by a multiple of 5^3, in accordance with (9.1), and $p(721)$ differed by a multiple of 11^2, in accordance with (9.2).

Our corollary (8.3) will now easily justify Lehmer's use of the Hardy-Ramanujan formula for the computation of $p(599)$ and $p(721)$, since the possible errors turn out to be not only less than $\frac{1}{2}5^3$ and $\frac{1}{2}11^2$ respectively, which obviously would suffice, but even less than $\frac{1}{2}$ each.

If we modify the error term in (8.3) slightly by substituting $\frac{1}{2}\exp(C\sqrt{n}/N)$ for $\sinh(C\sqrt{n}/N)$, we introduce a certain increase of the error, which in the present cases is not very noticeable. We get then:

(1) For $n=599$, $N=18$, an error of at most

$$\frac{44\pi^2}{225\sqrt{3}\sqrt{18}}+\frac{\pi\sqrt{2}}{150}\left(\frac{18}{598}\right)\;e^{\pi\sqrt{(\frac{2}{3})}\frac{\sqrt{(599)}}{18}}$$
$$+\frac{19^{\frac{3}{2}}}{2\sqrt{3}.598}\,e^{-\pi\sqrt{(\frac{2}{3})}\frac{\sqrt{(598)}}{18}}\left(\tfrac{1}{3}+\frac{\sqrt{3}.19}{5\pi\sqrt{2}\sqrt{598}}\right)$$
$$=0{\cdot}26265+0{\cdot}16810+0{\cdot}00048=0{\cdot}43123$$

* "On a conjecture of Ramanujan", *Journal London Math. Soc.*, 11 (1936), 114–118.

in absolute value. This renders Lehmer's hypothetical figure

$$p(599) = 4353\ 50207\ 84031\ 73482\ 70000$$

definitive.

(2) For $n = 721$, $N = 21$, at most an absolute value

$$\frac{44\pi^2}{225\sqrt{3}\sqrt{21}} + \frac{\pi\sqrt{2}}{150}\left(\frac{21}{720}\right)^{\frac{1}{2}} e^{\pi\sqrt{(\frac{2}{3})}\frac{\sqrt{(721)}}{21}} + \frac{22^{\frac{3}{2}}}{2\sqrt{3}.720}\, e^{-\pi\sqrt{(\frac{2}{3})}\frac{\sqrt{(720)}}{21}}\left(\frac{1}{3} + \frac{\sqrt{3}.22}{5\pi\sqrt{2}\sqrt{720}}\right)$$

$$= 0{\cdot}24316 + 0{\cdot}13441 + 0{\cdot}00062 = 0{\cdot}37819$$

for the error. This vindicates Lehmer's result of

$$p(721) = 16\ 10617\ 55750\ 27947\ 76355\ 34762.$$

The smallness of the error estimates in both cases, which made it indeed unnecessary to refer to the divisibility property (9.1), (9.2), seems to promise an applicability of (8.3) for numerical computation of $p(n)$ in a wider range.

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THE EIGEN-VALUES AND SINGULAR VALUES OF INTEGRAL EQUATIONS†

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Introduction.

1.1. In this paper I propose to investigate generalizations of some well-known results in the theory of linear integral equations, including (i) the classical theory of E. Schmidt, (ii) the work of Weyl and Hille and Tamarkin on the distribution of the eigen-values, and (iii) Hammerstein's theorem on the expansion of a kernel in terms of its eigen-functions.

1.2. Mercer's theorem‡ states that, if $K(s, t)$ is real, continuous, and symmetric, and is further positive definite, *i.e.*

$$\int_a^b \int_a^b K(s, t) f(s) f(t)\, ds\, dt \geqslant 0$$

for every real continuous function $f(s)$, then the series

$$\sum_{\nu=1}^{\infty} \frac{\phi_\nu(s)\, \phi_\nu(t)}{\lambda_\nu} \qquad (1.2.1)$$

is absolutely and uniformly convergent to the sum $K(s, t)$, where $\{\phi_n(s)\}$ is a complete normal and orthogonal set of eigen-functions of $K(s, t)$ and $\{\lambda_n\}$ is the set of the corresponding eigen-values.

1.3. Several other conditions have been given which are sufficient for the convergence of (1.2.1). Hobson§ obtained a generalization of

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‡ Mercer (5).

§ Hobson (4).

Mercer's theorem by retaining definiteness and abandoning continuity. However, he requires to assume conditions concerning the behaviour of the kernel $K(s, t)$ on the line $s = t$.

Hammerstein† supposed $K(s, t)$ continuous, and introduced a kind of integrated Lipschitz condition, viz.

$$\int_a^b [K(s_1, t) - K(s_2, t)]^2 dt \leqslant A(s_1 - s_2)^2. \qquad (1.3.1)$$

These hypotheses he showed to be sufficient for the series (1.2.1) to converge uniformly to $K(s, t)$.

1.4. In this paper results will be obtained concerning the convergence of (1.2.1) almost everywhere. The hypotheses made on the kernel are of the same general type as Hammerstein's. In some of the cases considered the restriction of continuity is abandoned.

1.5. This problem is approached through a discussion of the distribution of the eigen-values $\{\lambda_n\}$ as $n \to \infty$, when the sequence $\{\lambda_n\}$ is arranged in order of increasing absolute magnitude. Appeal is then made to a general theorem on systems of orthogonal functions.

The order of magnitude of the eigen-values of a symmetric kernel is discussed in a classical paper of Weyl‡. In this he proves that, if

$$\frac{\partial^r K(s, t)}{\partial s^r}$$

exists and is continuous, then

$$\frac{1}{|\lambda_n|} = o(n^{-r-\frac{1}{2}}).$$

More recently the eigen-values of a general kernel have been treated in an important paper by Hille and Tamarkin§.

One of the conditions used in this is as follows:

Write $$\frac{\partial^r K(s, t)}{\partial s^r} = K^{(r)}(s, t) \quad (r = 0, 1, 2, \ldots).$$

† Hammerstein (1).
‡ Weyl (8).
§ Hille and Tamarkin (2).

Let $1 < p \leqslant 2$. Then, for some r,

$$\left\{\int_a^b |K^{(r)}(s+\theta, t)-K^{(r)}(s, t)|^p\, ds\right\}^{1/p} \leqslant |\theta|^\alpha g(t), \qquad (1.5.1)$$

where $\int_a^b |g(t)|^{p'} dt$ exists, and $\frac{1}{p}+\frac{1}{p'}=1$.

Under this hypothesis, they show that

$$\frac{1}{|\lambda_n|} = O(n^{-r-\alpha-1+(1/p)}(\log n)^{r+\alpha}).$$

It is conjectured that the logarithmic factor can be removed†.

The method used by Hille and Tamarkin applies equally well to the eigen-values of symmetric and non-symmetric kernels. In the present paper the problem is attacked in a different way, and in the case of the non-symmetric kernel it turns out that the results apply to the singular values, *i.e.* E. Schmidt's eigen-values of the non-symmetric kernel, and not to the eigen-values in the ordinary sense.

So far as the symmetric kernel is concerned, the results of the present paper are an improvement on those of Hille and Tamarkin. A more general hypothesis is introduced, and their conjecture is shown to hold even under less restrictive conditions.

I have so far been unable to establish any direct connection between the orders of magnitude of the eigen-values and the singular values when the kernel is not symmetric.

1.6. In order to prove the above results in their full generality, I have found it necessary to rewrite a considerable part of the classical theory of integral equations in a slightly more general form. E. Schmidt, in his fundamental paper‡, supposes that $\int_a^b K^2(s, t)\, dt$ is a continuous function of s. As is well known from the abstract theory of linear transformations§, the classical theory remains valid under the hypothesis that $\int_a^b \int_a^b K^2(s, t)\, ds\, dt$ exists.

I have felt it worth while, however, to show that this theory can be derived without using either abstract Hilbert space or infinite matrices.

† Hille and Tamarkin (2), 60.

‡ Schmidt (6).

§ Stone (7), 193–197.

This task is carried out in Part I of this paper. In every case where the proof of a theorem can be obtained by a straightforward translation of Schmidt's paper into slightly more general language, only the result will be stated, with a reference to the appropriate point in Schmidt's memoir. In the case of a few theorems, however, the generalisation is not quite trivial, and the proof is given more fully.

For reasons of space I have confined myself to those parts of the theory which will actually be needed in Part II of this paper. In particular, I have omitted the solution of the non-homogeneous equation, and have in certain cases proved only mean-square convergence of a series which converges in the ordinary sense.

1.7. The Lebesgue integral is used throughout, and equalities and inequalities between functions will generally be understood as holding "almost everywhere", *i.e.* at all points of the range considered except those of a set of measure zero. All functions are supposed measurable whenever measurability is not implied by the context.

A function $f(s)$ or $K(s, t)$ is said to be a *nul-function* if it vanishes almost everywhere in the appropriate range.

By "$f(s) \in L^p$ in (a, b)" we shall understand that $\int_a^b |f(s)|^p\, ds$ exists. Similarly, "$K(s, t) \in L^p(s, t)$" means that $\int_a^b \int_a^b |K(s, t)|^p\, ds\, dt$ exists. Again, "$K(s, t) \in L^p(s)$ for almost all t" will mean that $\int_a^b |K(s, t)|^p\, ds$ exists for almost all values of t. We may also use such a phrase as "the function $f(s) \in L^p$" to mean "the function $f(s)$, belonging to the class L^p". Here "$\in$" may be read "of". Other similar phrases may also be used.

We shall assume as much as we require of the theories of mean-square convergence and normal and orthogonal systems of functions†. The word "orthonormal" will be used as a convenient abbreviation for "normal and orthogonal".

We also employ the notation "l.i.m." (limit in mean). *E.g.*, the formulae

$$\text{l.i.m.}_{n\to\infty} f_n(s) = f(s),$$

$$\text{l.i.m.}_{n\to\infty} K_n(s, t) = K(s, t),$$

† For these, see Hobson (3).

mean that, when $n \to \infty$,

$$\int_a^b [f(s) - f_n(s)]^2 \, ds \to 0,$$

$$\int_a^b \int_a^b [K(s, t) - K_n(s, t)]^2 \, ds \, dt \to 0,$$

respectively.

PART I.

The theory of symmetric kernels and its extensions.

Eigen-functions and eigen-values.

2.1. We suppose $K(s, t)$ real, non-nul, and symmetric, *i.e.*

$$K(s, t) = K(t, s),$$

in the fundamental square $a \leqslant s \leqslant b$, $a \leqslant t \leqslant b$. Further, we suppose that

$$k^2 = \int_a^b \int_a^b K^2(s, t) \, ds \, dt < \infty.$$

When no confusion is possible, we shall frequently omit the limits of integration a, b.

Definition. $\phi(s)$ is said to be an *eigen-function* of the kernel $K(s, t)$ with *eigen-value* λ if (i) $\phi(s)$ is not nul, (ii) $\phi(s) \in L^2$, and (iii)

$$\phi(s) = \lambda \int K(s, t) \, \phi(t) \, dt. \qquad (2.1.1)$$

If $\phi(s)$ is an eigen-function, so also is $A\phi(s)$, for arbitrary constant $A \neq 0$. If $\phi(s)$, $\psi(s)$ are eigen-functions with the same eigen-value λ, then any non-nul linear combination $A\phi(s) + B\psi(s)$ is an eigen-function with the same eigen-value.

2.2. It is easily proved by standard methods† that eigen-functions belonging to different eigen-values are orthogonal to one another, and that all the eigen-values of a symmetric kernel $K(s, t)$ are real.

We can in consequence suppose that every eigen-function $\phi(s)$ is real. For, if $\phi(s)$ is an eigen-function, so also are $R\phi(s)$ and $I\phi(s)$, its real and imaginary parts, whenever they are non-nul.

† Schmidt (6), 444–445.

2.3. THEOREM 1. *The eigen-values of $K(s, t)$ form at most an enumerable set with infinity as its sole limiting-point. The number of linearly independent eigen-functions corresponding to a given eigen-value λ is finite. Denoting the set of eigen-values by $\{\lambda_n\}$, and including each eigen-value once for each of its set of linearly independent eigen-functions, we have*

$$\sum_{\nu=1}^{\infty} \frac{1}{\lambda_\nu^2} \leqslant k^2 < \infty.$$

This again is proved by standard methods†.

Without loss of generality, we can suppose the set of linearly independent eigen-functions belonging to each eigen-value to be orthonormal.

Definition. A *complete orthonormal system* of eigen-functions of $K(s, t)$ is an orthonormal set of eigen-functions such that every eigen-function of $K(s, t)$ is a finite linear combination of functions of the set.

Such a system clearly exists, and whenever we mention the system of eigen-functions of $K(s, t)$, we shall understand a complete orthonormal system.

We shall from now on suppose the eigen-values enumerated in order of increasing absolute magnitude, so that

$$|\lambda_1| \leqslant |\lambda_2| \leqslant \ldots \leqslant |\lambda_n| \leqslant \ldots.$$

Each eigen-value is included in the sequence once for each of its set of linearly independent eigen-functions.

Iterated kernels.

3.1. We define, whenever the integrals exist in the sense of Lebesgue, the *iterated kernels*

$$K_2(s, t) = \int K(s, r) K(r, t)\, dr,$$

$$K_3(s, t) = \int K_2(s, r) K(r, t)\, dr,$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

We may also write $K_1(s, t) = K(s, t)$.

† Schmidt (6), 445–446.

It follows immediately from the Schwarz inequality that $K_n(s, t)$ exists for almost all s, t, and $K_n(s, t) \in L^2(s, t)$. Further, if we define $k_n \geqslant 0$ by the equation

$$k_n^2 = \iint K_n^2(s, t)\, ds\, dt,$$

then

$$k_n \leqslant k^n.$$

It may be noted that

$$K_{m+n}(s, t) = \int K_m(s, r)\, K_n(r, t)\, dr.$$

3.2. THEOREM 2. *If $K(s, t)$ is not nul, then $K_n(s, t)$ is not nul for any n.*

We prove this first for $n = 2$. Suppose that

$$K_2(s, t) = 0$$

almost everywhere.

Then

$$\iint K_2(s, t) f(s) f(t)\, ds\, dt = 0$$

for all $f(s) \in L^2$, *i.e.*

$$\int dr \int K(s, r) f(s)\, ds \int K(r, t) f(t)\, dt = 0. \tag{3.2.1}$$

Since $K(s, t)$ is symmetric, (3.2.1) becomes

$$\int dr \left\{ \int K(r, s) f(s)\, ds \right\}^2 = 0.$$

Hence

$$\int K(r, s) f(s)\, ds = 0, \tag{3.2.2}$$

for almost all r. Now take $f(s) = K(r, s)$ for any r for which (3.2.2) holds and $K(r, s) \in L^2(s)$. These statements are true for almost all r.

Then

$$\int K^2(r, s)\, ds = 0,$$

so that $K(s, t)$ must be nul.

The proof is now complete for $n = 2$. The case $n > 2$ can now be treated in the usual way†.

† Schmidt (6), 447.

THEOREM 3. *If $\phi(s)$ is an eigen-function of $K_2(s, t)$ with eigen-value λ^2, then either*

$$\phi(s) = \phi_1(s) + \phi_2(s),$$

where $\phi_1(s)$, $\phi_2(s)$ are eigen-functions of $K(s, t)$ with eigen-values $\pm\lambda$ respectively, or $\phi(s)$ is an eigen-function of $K(s, t)$ with eigen-value $\pm\lambda$.

This is a particular case of a classical result†.

The fundamental theorem.

4.1. THEOREM 4. *If $K(s, t)$ is not nul, it possesses at least one eigen-function.*

We define $$\kappa_n = \int K_n(s, s)\, ds \quad (n = 2, 3, \ldots).$$

$\int K(s, s)\, ds$ does not necessarily exist, but, for $n > 1$,

$$\kappa_n = \iint K_{n-1}(s, t)\, K_1(s, t)\, ds\, dt,$$

and therefore exists, by the Schwarz inequality.

We note that

$$\kappa_{m+n} = \iint K_m(s, t)\, K_n(s, t)\, ds\, dt,$$

and that $$\kappa_{2n} = \int\int K_n^2(s, t)\, ds\, dt.$$

We first state a lemma.

4.2. LEMMA. *There exist finite numbers c, C, such that*

$$\frac{\kappa_{2n}}{c^{2n}} \to C$$

as $n \to \infty$. Further, $C \geqslant 1$.

This is proved as in Schmidt's original paper‡.

4.3. From this point our argument differs somewhat from Schmidt's, which is not applicable to the more general case considered here.

Let $$\chi_n(s, t) = c^{-2n} K_{2n}(s, t).$$

† Schmidt (6), 448–449.

‡ Schmidt (6), 455–456.

Then

$$\begin{aligned}\iint (\chi_m - \chi_n)^2 \, ds\, dt &= \iint (c^{-4m} K_{2m}^2 - 2c^{-2m-2n} K_{2m} K_{2n} + c^{-4n} K_{2n}^2)\, ds\, dt \\ &= c^{-4m} \kappa_{4m} - 2c^{-2m-2n} \kappa_{2m+2n} + c^{-4n} \kappa_{4n} \\ &\to C + C - 2C = 0\end{aligned}$$

as $m,\ n \to \infty$.

Hence the sequence $\{\chi_n(s, t)\}$ is convergent in mean square in the two variables s, t to a limit function $\chi(s, t) \in L^2(s, t)$. Further,

$$\begin{aligned}\iint \chi^2(s, t)\, ds\, dt &= \lim_{n\to\infty} c^{-4n} \iint K_{2n}^2(s, t)\, ds\, dt \\ &= \lim_{n\to\infty} c^{-4n} \kappa_{4n} = C \geqslant 1.\end{aligned}$$

Thus $\chi(s, t)$ is not nul.

Now

$$c^{-2n} K_{2n}(s, t) = \frac{1}{c^2} \int K_2(s, r)\, c^{-2n+2} K_{2n-2}(r, t)\, dr,$$

i.e.

$$\chi_n(s, t) = \frac{1}{c^2} \int K_2(s, r)\, \chi_{n-1}(r, t)\, dr. \tag{4.3.1}$$

By familiar mean-square convergence arguments we can easily show that

$$\text{l.i.m.}_{n\to\infty} \int K_2(s, r)\, \chi_{n-1}(r, t)\, dr = \int K_2(s, r)\, \chi(r, t)\, dr.$$

Hence, by (4.3.1),

$$c^2 \chi(s, t) = \int K_2(s, r)\, \chi(r, t)\, dr \tag{4.3.2}$$

for almost all s, t.

We can now choose a value T of t such that (i) (4.3.2) holds for almost all s, (ii) $\chi(s, T)$ is not a nul-function of s, and (iii) $\chi(s, T) \in L^2(s)$. Write $\chi(s, T) = \phi(s)$. Then

$$c^2 \phi(s) = \int K_2(s, r)\, \phi(r)\, dr. \tag{4.3.3}$$

The proof can now be completed by reference to Theorem 3.

Expansion theorems.

5.1. THEOREM 5.

$$K(s, t) = \text{l.i.m.}_{n\to\infty} \sum_{\nu=1}^{n} \frac{\phi_\nu(s)\, \phi_\nu(t)}{\lambda_\nu}. \tag{5.1.1}$$

We carry out the proof in a number of stages.

(i) $$\iint \left\{ \sum_{\nu=m}^{n} \frac{\phi_\nu(s)\,\phi_\nu(t)}{\lambda_\nu} \right\}^2 ds\,dt = \sum_{\nu=m}^{n} \frac{1}{\lambda_\nu^2} \to 0$$

as $m, n \to \infty$. Hence the series (5.1.1) converges in mean square to a limit function $Q(s, t)$.

Let $$K(s, t) = Q(s, t) + R(s, t).$$

Then $Q(s, t) \in L^2(s, t)$, $R(s, t) \in L^2(s, t)$, and $Q(s, t)$, $R(s, t)$ are both symmetric.

(ii) Let $\chi(s)$ be any function of L^2. Then it follows easily by mean-square convergence arguments that

$$\int Q(s, t)\,\chi(t)\,dt = \underset{n\to\infty}{\text{l.i.m.}} \sum_{\nu=1}^{n} \frac{\phi_\nu(s)}{\lambda_\nu} \int \chi(t)\,\phi_\nu(t)\,dt. \tag{5.1.2}$$

(iii) In (5.1.2) take $\chi(s) = \phi_m(s)$. Then

$$\begin{aligned} \int Q(s, t)\,\phi_m(t)\,dt &= \underset{n\to\infty}{\text{l.i.m.}} \sum_{\nu=1}^{n} \frac{\phi_\nu(s)}{\lambda_\nu} \int \phi_m(t)\,\phi_\nu(t)\,dt \\ &= \frac{1}{\lambda_m}\,\phi_m(s) = \int K(s, t)\,\phi_m(t)\,dt. \end{aligned}$$

Hence $$\int R(s, t)\,\phi_m(t)\,dt = 0 \tag{5.1.3}$$

for every eigen-function ϕ_m.

(iv) By Theorem 4, if $R(s, t)$ is not nul, it possesses an eigen-function $\psi(s)$. Then

$$\psi(s) = c \int R(s, t)\,\psi(t)\,dt,$$

where $c \neq 0$.

Consequently

$$\int \phi_n(s)\,\psi(s)\,ds = c \int \psi(t)\,dt \int R(s, t)\,\phi_n(s)\,ds = 0, \tag{5.1.4}$$

since $R(s, t)$ is symmetric.

(v) Now take $\chi(s) = \psi(s)$ in (5.1.2). Then

$$\int Q(s, t)\,\psi(t)\,dt = \underset{n\to\infty}{\text{l.i.m.}} \sum_{\nu=1}^{n} \frac{\phi_\nu(s)}{\lambda_\nu} \int \psi(t)\,\phi_\nu(t)\,dt = 0,$$

by (5.1.4).

Hence

$$\int K(s,t)\psi(t)\,dt = \int R(s,t)\psi(t)\,dt = \frac{1}{c}\psi(s),$$

i.e. $\psi(s)$ is an eigen-function of K, which contradicts (5.1.4).

Thus $R(s,t)$ must be nul, and $K(s,t) = Q(s,t)$.

5.2. THEOREM 6. *Let* $q(t) \epsilon L^2$, $g(s) = \int K(s,t)q(t)\,dt$.

Then

$$\left.\begin{aligned} g(s) &= \text{l.i.m.}_{n\to\infty} \sum_{\nu=1}^{n} \phi_\nu(s) \int g(t)\phi_\nu(t)\,dt \\ &= \text{l.i.m.}_{n\to\infty} \sum_{\nu=1}^{n} \frac{\phi_\nu(s)}{\lambda_\nu} \int q(t)\phi_\nu(t)\,dt. \end{aligned}\right\} \qquad (5.2.1)$$

This follows from (5.1.2) and the fact that

$$K(s,t) = Q(s,t).$$

It can easily be shown that (5.2.1) is convergent in the ordinary sense almost everywhere; we shall not require this result.

5.3. THEOREM 7. *Let* $p(s)$, $q(s) \epsilon L^2$. *Then*

$$\iint K(s,t)p(s)q(t)\,ds\,dt = \sum_{\nu=1}^{\infty} \frac{1}{\lambda_\nu} \int \phi_\nu(s)p(s)\,ds \int \phi_\nu(t)q(t)\,dt. \qquad (5.3.1)$$

Define $g(s)$ as in Theorem 6. Then

$$\iint K(s,t)p(s)q(t)\,ds\,dt = \int p(s)g(s)\,ds.$$

The theorem now follows from a standard result on mean-square convergence, combined with the second form of (5.2.1).

Equation (5.3.1) is known as the Hilbert formula.

The non-symmetric kernel.

6.1. Let $K(s,t)$ be real and $K(s,t) \epsilon L^2(s,t)$. We no longer suppose K to be symmetric.

If $\phi(s)$, $\psi(s) \epsilon L^2$ satisfy the equations

$$\left.\begin{aligned} \phi(s) &= \lambda \int_a^b K(s,t)\psi(t)\,dt, \\ \psi(s) &= \lambda \int_a^b K(t,s)\phi(t)\,dt, \end{aligned}\right\} \qquad (6.1.1)$$

and are not nul, they are said to be a pair of *adjoint singular functions* of $K(s, t)$ belonging to the *singular value* λ. $\phi(s)$, $\psi(s)$, and λ may at present be real or complex.

We now define

$$K^*(s, t) = \int_a^b K(s, r)\, K(t, r)\, dr,$$

$$K_*(s, t) = \int_a^b K(r, s)\, K(r, t)\, dr.$$

K^* and K_* are clearly symmetric in s, t.

6.2. Following Schmidt's paper†, we can now show that λ is real, so that we can confine our attention to real pairs of adjoint singular functions.

We do not distinguish between the singular values λ and $-\lambda$, since they correspond respectively to the pairs (ϕ, ψ) and $(\phi, -\psi)$.

Next, if (ϕ, ψ) is a pair of adjoint singular functions with singular value λ, then ϕ, ψ are eigen-functions of K^*, K_* respectively, with eigen-value λ^2. Consequently, if (ϕ_1, ψ_1) and (ϕ_2, ψ_2) correspond to different singular values, then ϕ_1 is orthogonal to ϕ_2, and ψ_1 to ψ_2.

We can now introduce the concept of a complete orthonormal system of pairs of adjoint singular functions. It is defined in the same way as a complete orthonormal system of eigen-functions of a symmetric kernel.

6.3. THEOREM 8. *If $\{\phi_n(s), \psi_n(s)\}$ is a complete system of pairs of adjoint singular functions of $K(s, t)$ with singular values $\{\lambda_n\}$, then $\{\phi_n(s)\}$ is a complete orthonormal system of eigen-functions of $K^*(s, t)$ with eigen-values $\{\lambda_n^2\}$, and $\{\psi_n(s)\}$ is a complete orthonormal system of eigen-functions of $K_*(s, t)$ with the same eigen-values.*

We need add nothing to Schmidt's discussion.

6.4. THEOREM 9. *Let $q(s) \in L^2$, $g(s) = \int K(s, t)\, q(t)\, dt$.*

Then

$$g(s) = \underset{n\to\infty}{\text{l.i.m.}} \sum_{\nu=1}^{n} \phi_\nu(s) \int g(t)\, \phi_\nu(t)\, dt$$

$$= \underset{n\to\infty}{\text{l.i.m.}} \sum_{\nu=1}^{n} \frac{\phi_\nu(s)}{\lambda_\nu} \int q(t)\, \psi_\nu(t)\, dt.$$

† Schmidt (6), 461–463.

Again, if $p(s) \in L^2$ *and* $h(s) = \int K(t, s)\, p(t)\, dt$, *then*

$$h(s) = \underset{n\to\infty}{\text{l.i.m.}} \sum_{\nu=1}^{n} \psi_\nu(s) \int h(t)\, \psi_\nu(t)\, dt$$

$$= \underset{n\to\infty}{\text{l.i.m.}} \sum_{\nu-1}^{n} \frac{\psi_\nu(s)}{\lambda_\nu} \int p(t)\, \phi_\nu(t)\, dt.$$

Further,

$$\iint K(s, t)\, p(s)\, q(t)\, ds\, dt = \sum_{\nu=1}^{\infty} \frac{1}{\lambda_\nu} \int p(s)\, \phi_\nu(s)\, ds \int q(t)\, \psi_\nu(t)\, dt. \quad (6.4.1)$$

It may be noted that the above series are, in fact, convergent almost everywhere.

It is easily seen that

$$\int ds \left\{ \int K(s, t)\, q(t)\, dt - \sum_{\nu=1}^{n} \frac{\phi_\nu(s)}{\lambda_\nu} \int q(t)\, \psi_\nu(t)\, dt \right\}^2$$

$$= \iint K_*(s, t)\, q(s)\, q(t)\, ds\, dt - \sum_{\nu-1}^{n} \frac{1}{\lambda_\nu^2} \left\{ \int q(t)\, \psi_\nu(t)\, dt \right\}^2$$

$$\to 0$$

as $n \to \infty$, by (5.3.1).

Similarly for the second part. The third part follows as in Theorem 7.

6.5. THEOREM 10.

$$K(s, t) = \underset{n\to\infty}{\text{l.i.m.}} \sum_{\nu=1}^{n} \frac{\phi_\nu(s)\, \psi_\nu(t)}{\lambda_\nu}. \quad (6.5.1)$$

(i) We begin by showing that

$$\sum_{\nu=1}^{\infty} \frac{1}{\lambda_\nu^2} \leqslant \iint K^2(s, t)\, ds\, dt. \quad (6.5.2)$$

For $$\int K(s, t)\, \psi_\nu(t)\, dt = \frac{\phi_\nu(s)}{\lambda_\nu}.$$

By Bessel's inequality,

$$\sum_{\nu=1}^{n} \left\{ \frac{\phi_\nu(s)}{\lambda_\nu} \right\}^2 \leqslant \int K^2(s, t)\, dt$$

for all n. Hence, integrating over s, we have

$$\sum_{\nu=1}^{n} \frac{1}{\lambda_\nu^2} \leqslant \iint K^2(s, t)\, ds\, dt.$$

The result follows.

(ii) Next $$\iint \left\{ \sum_{\nu=m}^{n} \frac{\phi_\nu(s)\,\psi_\nu(t)}{\lambda_\nu} \right\}^2 ds\, dt = \sum_{\nu=m}^{n} \frac{1}{\lambda_\nu^2} \to 0$$

as $m,\ n \to \infty$, by (6.5.2).

Hence $\sum_{\nu=1}^{\infty} \frac{\phi_\nu(s)\,\psi_\nu(t)}{\lambda_\nu}$ converges in mean square to some function $Q(s, t) \in L^2(s, t)$.

Let $$K(s, t) = Q(s, t) + R(s, t).$$

(iii) By arguments belonging to the theory of mean-square convergence, we obtain

$$\iint Q(s, t)\, p(s)\, q(t)\, ds\, dt = \sum_{\nu=1}^{\infty} \frac{1}{\lambda_\nu} \int p(s)\, \phi_\nu(s)\, ds \int q(t)\, \psi_\nu(t)\, dt$$

for all $p(s),\ q(s) \in L^2$.

(iv) By (ii) and (6.4.1),

$$\iint K(s, t)\, p(s)\, q(t)\, ds\, dt = \iint Q(s, t)\, p(s)\, q(t)\, ds\, dt,$$

i.e. $$\iint R(s, t)\, p(s)\, q(t)\, ds\, dt = 0$$

for all $p(s),\ q(s) \in L^2$.

Hence $R(s, t)$ is a nul-function, and the theorem is proved.

COROLLARY 1.

$$\sum_{\nu=1}^{\infty} \frac{1}{\lambda_\nu^2} = \iint K^2(s, t)\, ds\, dt. \tag{6.5.3}$$

For $$\iint \left\{ K(s, t) - \sum_{\nu=1}^{n} \frac{\phi_\nu(s)\,\psi_\nu(t)}{\lambda_\nu} \right\}^2 ds\, dt \to 0,$$

i.e. $$\iint K^2(s, t)\, ds\, dt - \sum_{\nu=1}^{n} \frac{1}{\lambda_\nu^2} \to 0.$$

COROLLARY 2. *If the series* $\sum_{\nu=1}^{\infty} \frac{\phi_\nu(s)\,\psi_\nu(t)}{\lambda_\nu}$ *converges in the ordinary sense almost everywhere, then its sum is equal to* $K(s, t)$ *for almost all* s, t.

This is an immediate consequence of Theorem 10.

6.6. It is important to notice that the theory of the symmetric kernel, as developed in this Part, may be regarded as a particular case of the theory of the non-symmetric kernel. In consequence, Theorem 11 and the results of Part II hold both for the eigen-functions of a symmetric kernel and for the singular functions of a non-symmetric kernel.

For, if $K(s, t)$ is symmetric, then

$$K_*(s, t) = K^*(s, t) = K_2(s, t).$$

Let $\{\phi_\nu(s)\}$ be a complete orthonormal system of eigen-functions of $K(s, t)$. Then it is also a complete system for the kernel $K_2(s, t) = K^*(s, t)$, and it is identical with its adjoint system $\{\psi_\nu(s)\}$, since

$$\psi_\nu(s) = \lambda_\nu \int K(t, s)\, \phi_\nu(t)\, dt = \phi_\nu(s).$$

Thus $\{\phi_\nu(s), \phi_\nu(s)\}$ forms a complete system of pairs of adjoint singular functions for the kernel $K(s, t)$.

The approximation theorem.

7.1. THEOREM 11. *Let $K(s, t) \in L^2(s, t)$, K not being necessarily symmetric. Let $\alpha_\nu(s) \in L^2(s)$, $\beta_\nu(t) \in L^2(t)$ $(\nu = 1, 2, \ldots, n)$. When n is fixed, and α_ν, β_ν vary, the integral*

$$\int\int \left| K(s, t) - \sum_{\nu=1}^{n} \alpha_\nu(s)\, \beta_\nu(t) \right|^2 ds\, dt$$

is least if we take

$$\sum_{\nu=1}^{n} \alpha_\nu(s)\, \beta_\nu(t) = \sum_{\nu=1}^{n} \frac{\phi_\nu(s)\, \psi_\nu(t)}{\lambda_\nu},$$

where the $\{\lambda_\nu\}$ are arranged in order of increasing absolute magnitude,

i.e. $$|\lambda_\nu| \leqslant |\lambda_{\nu+1}|.$$

The value of the integral is then

$$\int\int K^2(s, t)\, ds\, dt - \sum_{\nu=1}^{n} \frac{1}{\lambda_\nu^2}.$$

The proof is practically the same as that of the corresponding theorem in Schmidt's paper†. Equation (39) on p. 469 must be replaced by

$$\int K(s, t)\, \beta_\nu\, dt = \text{l.i.m.}_{r\to\infty} \sum_{\rho=1}^{r} \frac{\phi_\rho(s)}{\lambda_\rho} \int \beta_\nu \psi_\rho(t)\, dt.$$

† Schmidt (6), 467–470.

This formula follows from Theorem 9, and equation (40) results from it immediately by the Parseval formula.

PART II.

Distribution of the singular values and expansion of the kernel.

Distribution of the singular values.

8.1. Let $K(s, t)$ be defined in $a \leqslant s \leqslant b$, $a \leqslant t \leqslant b$. Without loss of generality, we may take $a = 0$, $b = \pi$.

Let r be a non-negative integer. Write

$$K(s, t) = K^{(0)}(s, t),$$

$$\frac{\partial^\rho K(s, t)}{\partial s^\rho} = K^{(\rho)}(s, t). \quad (\rho = 1, 2, \ldots, r).$$

Then, if $r > 0$, we suppose that $K^{(1)}, K^{(2)}, \ldots, K^{(r-1)}$ exist and are absolutely continuous functions of s for almost all t, and that $K^{(r)}(s, t) \in L^\nu(s)$ for almost all t and for some p such that $1 < p \leqslant 2$. The latter hypothesis is also supposed to hold when $r = 0$.

It is convenient to suppose that $K(s, t)$ is extended, as an even function of s if r is even, and as an odd function of s if r is odd, into the range

$$-\pi \leqslant s \leqslant 0,$$

and thence as a periodic function of period 2π.

8.2. THEOREM 12. *If $K(s, t)$ satisfies the above conditions for some p such that $1 < p \leqslant 2$, and*

$$\int_0^\pi \left\{ \int_0^\pi |K^{(r)}(s+\theta, t) - K^{(r)}(s-\theta, t)|^p \, ds \right\}^{2/p} dt \leqslant A |\theta|^{2\alpha} \qquad (8.2.1)$$

for some A and for all sufficiently small θ, where either $r > 0$, $\alpha > 0$, or $r = 0$, $\alpha > 1/p - \frac{1}{2}$, then $K(s, t) \in L^2(s, t)$, and, when $n \to \infty$,

$$\frac{1}{|\lambda_n|} = O(n^{-\beta-r}),$$

where

$$\beta = \alpha + 1 - \frac{1}{p}.$$

It may be of interest to indicate the relation of this theorem to known results.

Hammerstein's condition [(1.3.1) above] gives

$$\int_0^\pi [K(s+\theta, t)-K(s-\theta, t)]^2 dt \leqslant A\theta^2.$$

This implies (8.2.1) with $r=0, p=2, \alpha=1$, and we obtain the conclusion

$$\frac{1}{|\lambda_n|} = O(n^{-\frac{3}{2}}).$$

We shall not, however, attempt to draw any conclusions about the uniform convergence of the series

$$\sum_{\nu=1}^{\infty} \frac{\phi_\nu(s)\,\phi_\nu(t)}{\lambda_\nu}.$$

Theorem 14 will show that, when the above conditions are satisfied, this series is absolutely convergent for almost all s, t.

The condition given by Hille and Tamarkin [(1.5.1) above] is

$$\left\{\int_a^b |K^{(r)}(s+\theta, t)-K^{(r)}(s-\theta, t)|^p\, ds\right\}^{1/p} < |\theta|^\alpha g(t),$$

where $g(t) \epsilon L^{p'}$. *A fortiori* $g(t) \epsilon L^2$, and (8.2.1) follows. We obtain the conclusion $\frac{1}{|\lambda_n|} = O(n^{-\beta-r})$, which is slightly stronger than their result:

$$\frac{1}{|\lambda_n|} = O\{n^{-\beta-r}(\log n)^{r+\alpha}\}.$$

As we indicated in the introduction, the two results are, so far as is known at present, strictly comparable only when $K(s, t)$ is symmetric.

Weyl (8) supposes that $\partial K(s, t)/\partial t$ is continuous, and shows that

$$\frac{1}{|\lambda_n|} = o(n^{-\frac{3}{2}}).$$

This hypothesis implies (8.2.1) with $r=0, p=2, \alpha=1$, and the conclusion is

$$\frac{1}{|\lambda_n|} = O(n^{-\frac{3}{2}}).$$

However, it can be easily seen that the proof of Theorem 12 can be altered slightly in such a way as to give Weyl's full result.

8.3. We shall at first impose the following restrictions on $K(s, t)$: (i) if r is even,

$$\left.\begin{aligned} \int_0^\pi K(s, t)\,ds &= 0, \\ K^{(1)}(0, t) = K^{(1)}(\pi, t) &= 0, \\ K^{(3)}(0, t) = K^{(3)}(\pi, t) &= 0, \\ \cdots \quad \cdots \quad \cdots \quad \cdots & \\ K^{(r-1)}(0, t) = K^{(r-1)}(\pi, t) &= 0; \end{aligned}\right\} \tag{8.3.1}$$

and (ii) if r is odd,

$$\left.\begin{aligned} K(0, t) = K(\pi, t) &= 0, \\ K^{(2)}(0, t) = K^{(2)}(\pi, t) &= 0, \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots & \\ K^{(r-1)}(0, t) = K^{(r-1)}(\pi, t) &= 0. \end{aligned}\right\} \tag{8.3.2}$$

8.4. We now require two lemmas. The first of these belongs to the theory of Fourier series.

LEMMA 1. *If $f(s) \in L^p$ in $(0, \pi)$, where $1 < p \leqslant 2$, and $f(s)$ has the Fourier sine series $\Sigma c_\nu \sin \nu s$, then*

$$\sum_{\nu=1}^{\infty} \nu^{(p-2)/p} c_\nu{}^2 \leqslant A \left\{ \int_0^\pi |f(s)|^p\,ds \right\}^{2/p}, \tag{8.4.1}$$

where A is an absolute constant.

This is an immediate consequence of the corresponding result for complex Fourier series, which is a particular case of a theorem due to Hardy and Littlewood†.

8.5. LEMMA 2. *Let $\{a_\nu(t)\}$ $(\nu = 0, 1, 2, \ldots)$ be the Fourier coefficients of $K(s, t)$ with respect to the orthogonal system $\{\chi_\nu(s)\}$, where if r is even, $\chi_\nu(s) = \cos \nu s$, and if r is odd, $\chi_\nu(s) = \sin \nu s$, so that*

$$a_\nu(t) = \frac{2}{\pi} \int_0^\pi K(s, t)\,\chi_\nu(s)\,ds.$$

† Hardy and Littlewood (9), Theorem 10, 199.

Then, under the hypotheses of Theorem 12,

$$\sum_{\nu=n}^{\infty} \int_0^{\pi} [a_\nu(t)]^2 \, dt \leqslant A n^{1-2r-2\beta}, \tag{8.5.1}$$

where A is independent of n.

For almost all t, $K^{(r)}(s, t) \in L^p(s)$, and so possesses a Fourier cosine series in $0 \leqslant s \leqslant \pi$, say

$$K^{(r)}(s, t) \sim \sum_{\nu=0}^{\infty} g_\nu(t) \cos \nu s.$$

Hence

$$K^{(r)}(s+\theta, t) - K^{(r)}(s-\theta, t) \sim -2 \sum_{\nu=1}^{\infty} g_\nu(t) \sin \nu s \sin \nu\vartheta,$$

a Fourier sine series in $0 \leqslant s \leqslant \pi$. We may take $\theta > 0$.

In (8.4.1) take

$$f(s) = K^{(r)}(s+\theta, t) - K^{(r)}(s-\theta, t).$$

For almost all t the hypothesis of Lemma 1 holds. Hence

$$\sum_{\nu=1}^{\infty} \nu^{(p-2)/p} [g_\nu(t)]^2 \sin^2 \nu\theta \leqslant A \left\{ \int | K^{(r)}(s+\theta, t) - K^{(r)}(s-\theta, t) |^p \, ds \right\}^{2/p}.$$

Integrate with respect to t, and apply (8.2.1). Then

$$\int \left\{ \sum_{\nu=1}^{\infty} \nu^{(p-2)/p} [g_\nu(t)]^2 \sin^2 \nu\theta \right\} dt \leqslant A\theta^{2\alpha}.$$

A fortiori

$$\sum_{\nu \leqslant 1/\theta} \nu^{(p-2)/p} \sin^2 \nu\theta \int [g_\nu(t)]^2 \, dt \leqslant A\theta^{2\alpha}. \tag{8.5.2}$$

Now, when $\nu \leqslant 1/\theta$, we have

$$\frac{\sin \nu\theta}{\nu\theta} > \frac{2}{\pi},$$

i.e.

$$\sin \nu\theta > \frac{2\nu\theta}{\pi}.$$

Hence

$$\sum_{\nu \leqslant 1/\theta} \nu^{(3p-2)/p} \int g_\nu^2(t) \, dt \leqslant A\theta^{2\alpha-2}.$$

In this inequality take $\theta = 1/n$. Then

$$\sum_{\nu=1}^{n} \nu^{(3p-2)/p} \int g_\nu^2(t)\,dt \leqslant An^{2-2\alpha}.$$

We therefore have, with a new A,

$$\sum_{\nu=n}^{2n-1} \nu^{(3p-2)/p} \int g_\nu^2(t)\,dt \leqslant An^{2-2\alpha}.$$

Consequently

$$\begin{aligned}\sum_{\nu=n}^{2n-1} \int g_\nu^2(t)\,dt &= \sum_{\nu=n}^{2n-1} \nu^{-(3p-2)/p}\, \nu^{(3p-2)/p} \int g_\nu^2(t)\,dt \\ &\leqslant n^{-(3p-2)/p} \sum_{\nu=n}^{2n-1} \nu^{(3p-2)/p} \int g_\nu^2(t)\,dt \\ &\leqslant An^{-(3p-2)/p}\, n^{2-2\alpha} \\ &= An^{1-2\beta}, \qquad (8.5.3)\end{aligned}$$

since

$$\beta = \alpha + 1 - \frac{1}{p}.$$

8.6. Taking account of the restrictions (8.3.1), (8.3.2), we see that

$$|a_\nu(t)| = \nu^{-r}|g_\nu(t)| \quad (\nu > 0),$$

and that

$$a_0(t) = 0.$$

Hence, by (8.5.3),

$$\begin{aligned}\sum_{\nu=n}^{2n-1} \int a_\nu^2(t)\,dt &= \sum_{\nu=n}^{2n-1} \nu^{-2r} \int g_\nu^2(t)\,dt \\ &\leqslant n^{-2r} \sum_{\nu=n}^{2n-1} \int g_\nu^2(t)\,dt \\ &\leqslant An^{1-2\beta-2r}.\end{aligned}$$

Consequently

$$\begin{aligned}\sum_{\nu=n}^{\infty} \int a_\nu^2(t)\,dt &= \sum_{\nu=n}^{2n-1} \int a_\nu^2(t)\,dt + \sum_{\nu=2n}^{4n-1} \int a_\nu^2(t)\,dt + \sum_{\nu=4n}^{8n-1} \int a_\nu^2(t)\,dt + \ldots \\ &\leqslant An^{1-2r-2\beta}(1 + 2^{1-2r-2\beta} + 4^{1-2r-2\beta} + \ldots).\end{aligned}$$

By the conditions of Theorem 12, either $r > 0$ or $r = 0$ and $\beta > \frac{1}{2}$. Hence the series

$$1 + 2^{1-2r-2\beta} + 4^{1-2r-2\beta} + \ldots$$

is convergent. We therefore have, with a new A,

$$\sum_{\nu=n}^{\infty} \int a_\nu^2(t)\,dt \leqslant An^{1-2r-2\beta}. \qquad (8.6.1)$$

The proof of the lemma is now complete.

8.7. We next come to the proof of the main theorem.
Taking $n=1$ in (8.6.1), we see that

$$\sum_{\nu=1}^{\infty} \int a_\nu^2(t)\,dt < \infty. \qquad (8.7.1)$$

By the Parseval formula,

$$\sum_{\nu=1}^{\infty} a_\nu^2(t) = A \int K^2(s, t)\,ds.$$

Hence
$$\iint K^2(s, t)\,ds\,dt < \infty,$$

i.e. $K(s, t) \epsilon L^2(s, t)$. We have therefore proved one part of Theorem 12.

8.8. We now consider the $(n-1)$-th partial sum of the Fourier series of $K(s, t)$ with respect to $\{\chi_\nu(s)\}$, viz.

$$a_1(t)\chi_1(s)+\ldots+a_{n-1}(t)\chi_{n-1}(s).$$

Since the functions $\chi_\nu(s)$ are bounded, they belong to the class $L^2(s)$. From (8.7.1) it follows that $a_\nu(t) \epsilon L^2(t)$. The sum $\sum_{\nu=1}^{n-1} a_\nu(t)\chi_\nu(s)$ consequently satisfies the conditions of Theorem 11. Hence

$$\iint K^2(s, t)\,ds\,dt - \sum_{\nu=1}^{n-1} \frac{1}{\lambda_\nu^2} \leqslant \iint \left\{K(s, t) - \sum_{\nu=1}^{n-1} a_\nu(t)\chi_\nu(s)\right\}^2 ds\,dt$$

$$= A \int \sum_{\nu=n}^{\infty} a_\nu^2(t)\,dt,$$

by the Parseval formula. Applying Theorem 10, Corollary 1, and the inequality (8.6.1), we have

$$\sum_{\nu=n}^{\infty} \frac{1}{\lambda_\nu^2} \leqslant An^{1-2r-2\beta} \quad (n=1, 2, \ldots). \qquad (8.8.1)$$

A fortiori

$$\sum_{\nu=n}^{2n-1} \frac{1}{\lambda_\nu^2} \leqslant An^{1-2r-2\beta},$$

so that

$$\frac{n}{\lambda_{2n-1}^2} \leqslant \sum_{\nu=n}^{2n-1} \frac{1}{\lambda_\nu^2} \leqslant An^{1-2r-2\beta},$$

and

$$\frac{1}{|\lambda_{2n-1}|} \leqslant An^{-r-\beta}.$$

A similar argument leads to the corresponding result for λ_{2n}. Consequently, when $n \to \infty$,

$$\frac{1}{|\lambda_n|} = O(n^{-r-\beta}).$$

This completes the proof of the Theorem under the special restrictions (8.3.1), (8.3.2).

8.9. We now suppose that $K(s, t)$ satisfies the hypotheses of Theorem 12, but not the special end conditions (8.3.1), (8.3.2).

Define

$$H(s, t) = K(s, t) - \sum_{\rho=0}^{r} s^\rho \omega_\rho(t), \tag{8.9.1}$$

choosing the $r+1$ functions $\omega_\rho(t)$ so that $H(s, t)$ satisfies (8.3.1) or (8.3.2), as the case may be.

The functions $\omega_\rho(t)$ are linear combinations of

$$K^{(r-1)}(0, t), \quad K^{(r-1)}(\pi, t), \quad K^{(r-3)}(0, t), \ldots,$$

and

$$\int_0^\pi K(s, t)\, ds.$$

Since

$$H^{(r)}(s, t) = K^{(r)}(s, t) - r!\, \omega_r(t),$$

H satisfies all the hypotheses of the theorem.

As in the main proof above, we approximate to $H(s, t)$ by a sum of $n-1$ terms

$$a_1(t)\chi_1(s) + a_2(t)\chi_2(s) + \ldots + a_{n-1}(t)\chi_{n-1}(s).$$

Then

$$a_1(t)\chi_1(s) + \ldots + a_{n-1}(t)\chi_{n-1}(s) + \omega_0(t) + s\omega_1(t) + \ldots + s^r \omega_r(t)$$

is a sum of $n+r$ terms approximating to $K(s, t)$, satisfying the conditions of Theorem 11. That $\omega_\rho(t) \in L^2$ follows from the fact that

$$H(s, t) \in L^2(s, t), \quad K(s, t) \in L^2(s, t).$$

Consequently

$$\sum_{\nu=n+r+1}^{\infty} \frac{1}{\lambda_\nu^2} \leqslant A n^{1-2r-2\beta}.$$

From this we deduce, as before, that

$$\frac{1}{|\lambda_n|} = O(n^{-r-\beta}).$$

This result completes the proof.

Expansion of the kernel in terms of its singular functions.

9.1. We have already shown in Theorem 10 that

$$K(s, t) = \text{l.i.m.}_{n\to\infty} \sum_{\nu=1}^{n} \frac{\phi_\nu(s)\psi_\nu(t)}{\lambda_\nu}. \tag{9.1.1}$$

We now proceed to determine conditions under which the series is almost everywhere convergent in the ordinary sense.

THEOREM 13. *If $K(s, t)$ satisfies the hypotheses of Theorem 12, then*

$$\sum_{\nu=1}^{\infty} \frac{\phi_\nu(s)\psi_\nu(t)}{\lambda_\nu} = K(s, t) \tag{9.1.2}$$

almost everywhere.

It may be noted that the set $\{\lambda_\nu\}$ is supposed to be arranged in order of increasing absolute magnitude, and that absolute convergence is not asserted.

By Theorem 12, $$\frac{1}{|\lambda_n|} \leqslant A n^{-r-\beta} \quad (n = 1, 2, \ldots)$$

for some constant A.

Hence $$\sum_{\nu=1}^{\infty} \frac{(\log \nu)^2}{\lambda_\nu^2} < \infty,$$

provided that $2r+2\beta > 1$. By hypothesis, this inequality is satisfied. It follows that

$$\sum_{\nu=1}^{\infty} \int \frac{(\log \nu)^2}{\lambda_\nu^2} \phi_\nu^2(s)\, ds < \infty.$$

The series

$$\sum_{\nu=1}^{\infty} \frac{(\log \nu)^2}{\lambda_\nu^2} \phi_\nu^2(s) \qquad (9.1.3)$$

must therefore be convergent for almost all s.

Now it has been proved by Rademacher† that, if $\{\psi_\nu(t)\}$ is any orthonormal system of functions and $\{a_\nu\}$ is any system of constants such that

$$\sum_{\nu=1}^{\infty} a_\nu^2 (\log \nu)^2 < \infty,$$

then the series $\sum_{\nu=1}^{\infty} a_\nu \psi_\nu(t)$ converges for almost all t.

Take $$a_\nu = \frac{\phi_\nu(s)}{\lambda_\nu}$$

for any s for which the series (9.1.3) converges. By Rademacher's theorem,

$$\sum_{\nu=1}^{\infty} \frac{\phi_\nu(s)\psi_\nu(t)}{\lambda_\nu}$$

is, for almost all s, convergent for almost all t, *i.e.* it is convergent for almost all s, t.

Hence, by Theorem 10, Corollary 2,

$$\sum_{\nu=1}^{\infty} \frac{\phi_\nu(s)\psi_\nu(t)}{\lambda_\nu} = K(s, t)$$

almost everywhere.

9.2. THEOREM 14. *If $K(s, t)$ satisfies the hypotheses of Theorem 12 and, in addition, either $r > 0$ or $r = 0$ and $\alpha > 1/p$, then*

$$\sum_{\nu=1}^{\infty} \frac{\phi_\nu(s)\psi_\nu(t)}{\lambda_\nu}$$

is almost everywhere absolutely convergent.

For $$\frac{1}{|\lambda_n|} \leqslant A n^{-r-\beta}.$$

Under the above conditions $\sum_{\nu=1}^{\infty} \frac{1}{|\lambda_\nu|}$ is convergent, *i.e.*

$$\sum_{\nu=1}^{\infty} \int \frac{\phi_\nu^2(s)}{|\lambda_\nu|}\, ds < \infty.$$

† Hobson (3), 763.

Hence $\sum_{\nu=1}^{\infty} \frac{\phi_\nu^2(s)}{|\lambda_\nu|}$ converges for almost all s. Similarly $\sum_{\nu=1}^{\infty} \frac{\psi_\nu^2(t)}{|\lambda_\nu|}$ converges for almost all t.

Further,

$$\sum_{\nu=n+1}^{n+p} \frac{|\phi_\nu(s)|\,|\psi_\nu(t)|}{|\lambda_\nu|} \leqslant \left\{\sum_{\nu=n+1}^{n+p} \frac{\phi_\nu^2(s)}{|\lambda_\nu|}\right\}^{\frac{1}{2}} \left\{\sum_{\nu=n+1}^{n+p} \frac{\psi_\nu^2(t)}{|\lambda_\nu|}\right\}^{\frac{1}{2}}$$

for almost all s and t. The required result follows.

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PARTIAL DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS (I)

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1. The solution of differential equations with constant coefficients occupies so elementary and fundamental a place in the theory of ordinary differential equations that it seems desirable to attempt something similar for partial differential equations with constant coefficients. In this and subsequent papers I hope to give, as contributions to a general theory, solutions of certain types of partial differential equations with arbitrary constant coefficients.

I shall take the field of two independent variables (x, y) as the standard field, extending the results, where possible, to fields of more variables. Now, in a field of two variables (x, y), the terms of highest order in a partial differential equation (with constant coefficients) can be written in a form

$$\Pi \left(\frac{\partial}{\partial x} - a_r \frac{\partial}{\partial y}\right) V.$$

With this expression we associate the families of parallel lines

$$a_r x + y = \text{constant},$$

which are termed the *characteristics* of the differential equation. If, by writing everywhere x, y in place of $\partial/\partial x$, $\partial/\partial y$, we convert the differential equation into an algebraic equation, the asymptotic directions of the plane curve defined by the algebraic equation evidently correspond to "characteristic" directions of the differential equation. These characteristics are fundamental in the theory of the partial differential equation.

2. In the first place it is important to classify partial differential equations by the number of their *distinct* characteristics according as

this number (i) exceeds, (ii) is equal to, (iii) is less than the number of independent variables of the field. I shall say (tentatively) that the characteristics are respectively (i) "excessive", (ii) "sufficient", (iii) "deficient". In the present paper I consider only equations in which the characteristics are *sufficient*, *i.e.* are equal in number to the independent variables, and so, by a suitable change of variables, can be taken to be the axi-parallels $x =$ constant, $y =$ constant. The most general equation of this type that I consider is the equation

$$\sum_{r=0}^{p} \sum_{s=0}^{q} A_{rs} \frac{\partial^{p+q-r-s} V}{\partial x^{p-r} \partial y^{q-s}} = 0, \tag{1}$$

where, of course, the coefficients A_{rs} are constants; and, in particular, I take $A_{00} = -1$, thus fixing the coefficients absolutely. It should be noted that in (1) the term of highest order $\partial^{p+q} V / \partial x^p \partial y^q$ prescribes also the highest orders of differentiations in x and in y separately. Thus I exclude such an equation as

$$\frac{\partial^{p+q} V}{\partial x^p \partial y^q} = \frac{\partial^{p+m} V}{\partial x^{p+m}} + \frac{\partial^{q+n} V}{\partial y^{q+n}},$$

where the corresponding algebraic equation

$$x^p y^q = x^{p+m} + y^{q+n}$$

has "parabolic" branches indicated by $y^q = x^m$ and $x^p = y^n$.

If the characteristics are "parabolic" the equation is not, at present, amenable to my methods.

3. The simplest equation of the type (1) is Riemann's equation*

$$\Delta V \equiv \frac{\partial^2 V}{\partial x \partial y} + a \frac{\partial V}{\partial x} + b \frac{\partial V}{\partial y} + cV = 0, \tag{2}$$

and it is the method given by Riemann for solving (2) that I shall extend to the more general equation (1). Riemann associates with this equation its adjoint equation

$$\Delta' V \equiv \frac{\partial^2 U}{\partial x \partial y} - a \frac{\partial U}{\partial x} - b \frac{\partial U}{\partial y} + cU = 0, \tag{3}$$

and obtains a "fundamental solution" $U(X, Y; x, y)$ with these properties: if (x, y) are the variables and (X, Y) parametric constants,

* For a discussion of this equation see **3**, 71–111, chap. 4, or **4**, 147–58.

U is a solution of the adjoint equation (3); if now (X, Y) are the variables and (x, y) parametric constants, U is a solution of the primitive equation (2); moreover, U or certain expressions derived from it, have prescribed values along the characteristics $x = X$, $y = Y$, and at their intersection.

In a differential equation with constant coefficients the adjoint is obtained merely by changing the sign of every $\partial/\partial x$, $\partial/\partial y$, *i.e.* by changing the signs of x, y themselves. Thus, if $V(x, y)$ is a solution of the primitive equation (1), then the adjoint equation has a solution $V(-x, -y)$ and, more generally, $V(X-x, Y-y)$, if X, Y are constants, since x, y enter the differential equation only through their differentials. The values of the latter solution along the characteristics $x = X$, $y = Y$ are, of course, the values of $V(x, y)$ itself along the axes $x = 0$, $y = 0$.

4. In constructing the fundamental solution of (1) we therefore first look for a particular solution of (1) having the appropriate behaviour along the axes $x = 0$, $y = 0$. This particular solution turns out to be

$$\Phi(x, y) \equiv \sum_{\tau} \frac{T!\, x^{R+p-1} y^{S+q-1}}{(R+p-1)!\,(S+q-1)!} \prod_{r,s}{}' \frac{A_{rs}^{\tau(r,s)}}{\tau(r, s)!}; \tag{4}$$

where Σ is a multiple summation over all non-negative integer values of every $\tau(r, s)$ taken independently; Π' omits the single term $(0, 0)$; and

$$R \equiv \sum_{r,s} r\tau(r, s), \quad S \equiv \sum_{r,s} s\tau(r, s), \quad T \equiv \sum_{r,s} \tau(r, s), \tag{5}$$

the term $(0, 0)$ being again omitted in these last three summations. Then

$$\frac{\partial^{p+q}\Phi}{\partial x^p\, \partial y^q} = \sum_{\tau} \frac{T!\, x^{R-1} y^{S-1}}{(R-1)!\,(S-1)!} \prod_{r,s}{}' \frac{A_{rs}^{\tau(r,s)}}{\tau(r, s)!},$$

and, if (ρ, σ) is any (r, s) other than $(0, 0)$,

$$A_{\rho\sigma} \frac{\partial^{p+q-\rho-\sigma}\Phi}{\partial x^{p-\rho}\, \partial y^{q-\sigma}} = \sum_{\tau} \frac{T!\, x^{R+\rho-1} y^{S+\sigma-1}}{(R+\rho-1)!\,(S+\sigma-1)!} \frac{A_{\rho\sigma}^{1+\tau(\rho,\sigma)}}{\tau(\rho, \sigma)!} \prod_{r,s}{}'' \frac{A_{rs}^{\tau(r,s)}}{\tau(r, s)!},$$

where Π'' omits the term (ρ, σ) as well as the term $(0, 0)$. On the right put $\tau(\rho, \sigma)$ in place of $1+\tau(\rho, \sigma)$, and we get, in virtue of (5),

$$\tau(\rho, \sigma) \sum_{\tau}{}' \frac{(T-1)!\, x^{R-1} y^{S-1}}{(R-1)!\,(S-1)!} \prod_{r,s}{}' \frac{A_{rs}^{\tau(r,s)}}{\tau(r, s)!},$$

where the summation Σ' is now restricted to *positive* values of $\tau(\rho, \sigma)$, though zero values remain admissible in any other $\tau(r, s)$. But, in view of the factor $\tau(\rho, \sigma)$ outside the summation, we can arbitrarily add in terms

in which $\tau(\rho, \sigma) = 0$, and so the summation Σ' can be replaced by the original summation Σ. Thus

$$\sum_{\rho=0}^{p} \sum_{\sigma=0}^{q} A_{\rho\sigma} \frac{\partial^{p+q-\rho-\sigma}\Phi}{\partial x^{p-\rho}\partial y^{q-\sigma}} = \left\{-T + \sum_{\rho,\sigma} \tau(\rho, \sigma)\right\} \sum_{\tau} \frac{(T-1)!\, x^{R-1} y^{S-1}}{(R-1)!\,(S-1)!} \prod_{r,s}{}' \frac{A_{rs}^{\tau(r,s)}}{\tau(r, s)!}$$

$$= 0, \quad \text{in virtue of (5) defining } T.$$

Thus $\Phi(x, y)$, as defined in (4), is certainly a solution of (1).

5. Consider now its behaviour along the axes $x = 0$, $y = 0$. Since, in (4), R, S are, by definition, never negative, we see that $\Phi(0, y) = 0$, $\Phi(x, 0) = 0$; and, more generally, that

$$\left.\begin{aligned} \Phi_{m0}(0, y) &= 0 \quad (m = 0, 1, \ldots, p-2), \\ \Phi_{0n}(x, 0) &= 0 \quad (n = 0, 1, \ldots, q-2), \end{aligned}\right\} \tag{6}$$

where $$\Phi_{rs} \equiv \frac{\partial^{r+s}\Phi}{\partial x^r \partial y^s}.$$

Again, in $\Phi_{p-1,0}(0, y)$ only those terms remain in which $R = 0$, *i.e.* in which $\tau(r, s) = 0$ or $r = 0$. Thus

$$\Phi_{p-1,0}(0, y) = \sum_{\tau} \frac{T!\, y^{S+q-1}}{(S+q-1)!} \prod_{s}{}' \frac{A_{0s}^{\tau(0,s)}}{\tau(0, s)!}, \tag{7}$$

and, by an argument similar to that used above*, we show that

$$\sum_{s=0}^{q} A_{0s} \frac{d^{q-s}}{dy^{q-s}} \Phi_{p-1,0}(0, y) = 0. \tag{8}$$

Interchanging x, y we find an analogous differential equation satisfied by $\Phi_{0,q-1}(x, 0)$. Finally, at the origin we have, if $m \leqslant p-1$, $n \leqslant q-1$,

$$\Phi_{mn}(0, 0) = 0, \quad \text{unless} \quad m = p-1 \quad \text{and} \quad n = q-1; \tag{9}$$

$$\Phi_{p-1,q-1}(0, 0) = 1. \tag{10}$$

This completes the description of the function Φ. In terms of it we define the *fundamental solution*

$$U = U(X, Y; x, y) \equiv \Phi(X-x, Y-y), \tag{11}$$

and thence obtain the *general solution* of (1) as follows.

* Actually the proof is covered by the case $q = 0$ of the preceding discussion.

6. If $V(x, y)$ is any solution of (1), *i.e.* if $\Delta V = 0$, then

$$U\Delta V - V\Delta' U = 0, \tag{12}$$

since U, in variables (x, y), is a solution of the adjoint equation $\Delta' U = 0$*.

By the defining property of adjoint operators we can rewrite (12) in the form

$$\frac{\partial \Theta_1}{\partial x} - \frac{\partial \Theta_2}{\partial y} = 0,$$

or, as I prefer†,

$$\Theta_1 = \frac{\partial \Omega}{\partial x}, \quad \Theta_2 = \frac{\partial \Omega}{\partial y},$$

for some Θ_1, Θ_2, Ω.

These functions Θ_1, Θ_2, Ω can be chosen with a certain freedom. On noting as a lemma that

$$U\frac{\partial^{P+Q} V}{\partial x^P \partial y^Q} - V\left(-\frac{\partial}{\partial x}\right)^P \left(-\frac{\partial}{\partial y}\right)^Q U \quad (P, Q \geqslant 1)$$

can be written indifferently as

$$\frac{\partial}{\partial x}\left\{\sum_{m=0}^{P-1} \frac{\partial^m V}{\partial x^m}\left(-\frac{\partial}{\partial x}\right)^{P-m-1}\left(-\frac{\partial}{\partial y}\right)^Q U\right\} + \frac{\partial}{\partial y}\left\{\sum_{n=0}^{Q-1} \frac{\partial^{P+n} V}{\partial x^P \partial y^n}\left(-\frac{\partial}{\partial y}\right)^{Q-n-1} U\right\}$$

or as

$$\frac{\partial}{\partial x}\left\{\sum_{m=0}^{P-1} \frac{\partial^{m+Q} V}{\partial x^m \partial y^Q}\left(-\frac{\partial}{\partial x}\right)^{P-m-1} U\right\} + \frac{\partial}{\partial y}\left\{\sum_{n=0}^{Q-1} \frac{\partial^n V}{\partial y^n}\left(-\frac{\partial}{\partial x}\right)^P \left(-\frac{\partial}{\partial y}\right)^{Q-n-1} U\right\},$$

we can deduce, from (12), the alternative pairs

$$-\frac{\partial F}{\partial x} = \sum_{r=0}^{p}\sum_{s=0}^{q-1} A_{rs} \sum_{n=0}^{q-s-1} \frac{\partial^{p-r+n} V}{\partial x^{p-r} \partial y^n}\left(-\frac{\partial}{\partial y}\right)^{q-s-n-1} U, \tag{13}$$

$$\frac{\partial F}{\partial y} = \sum_{r=0}^{p-1}\sum_{s=0}^{q} A_{rs} \sum_{m=0}^{p-r-1} \frac{\partial^m V}{\partial x^m}\left(-\frac{\partial}{\partial x}\right)^{p-r-m-1}\left(-\frac{\partial}{\partial y}\right)^{q-s} U; \tag{14}$$

$$\frac{\partial G}{\partial x} = \sum_{r=0}^{p}\sum_{s=0}^{q-1} A_{rs} \sum_{n=0}^{s-1} \frac{\partial^n V}{\partial y^n}\left(-\frac{\partial}{\partial x}\right)^{p-r}\left(-\frac{\partial}{\partial y}\right)^{q-s-n-1} U, \tag{15}$$

$$-\frac{\partial G}{\partial y} = \sum_{r=0}^{p-1}\sum_{s=0}^{q} A_{rs} \sum_{m=0}^{p-r-1} \frac{\partial^{q-s+m} V}{\partial x^m \partial y^{q-s}}\left(-\frac{\partial}{\partial x}\right)^{p-r-m-1} U, \tag{16}$$

* I have now extended Δ, Δ', introduced for the special equation (2), to the general equation (1).

† At this point I depart from Riemann's method in detail but not in principle.

where F, G are functions whose sum is given by

$$F+G=-\sum_{r=0}^{p-1}\sum_{s=0}^{q-1}A_{rs}\sum_{m=0}^{p-r-1}\sum_{n=0}^{q-s-1}\frac{\partial^{m+n}V}{\partial x^m\partial y^n}\left(-\frac{\partial}{\partial x}\right)^{p-r-m-1}\left(-\frac{\partial}{\partial y}\right)^{q-s-n-1}U. \tag{17}$$

7. Now, by definition in (11),

$$U=\Phi(X-x,\ Y-y),$$

and so, from (6), every

$$\left(-\frac{\partial}{\partial x}\right)^{p-r-m-1}U \quad (r+m>0)$$

vanishes along the characteristic $x=X$. Thus every term vanishes on the right of (14) except those in which $r=0=m$. Hence, along the characteristic $x=X$,

$$\frac{\partial F}{\partial y}=V\left\{\sum_{s=0}^{q}A_{0s}\left(-\frac{\partial}{\partial y}\right)^{q-s}\right\}\left(-\frac{\partial}{\partial x}\right)^{p-1}U$$

$$=0, \quad \text{by (8)}.$$

Similarly $\partial G/\partial x=0$ along the characteristic $y=Y$. Thus, taking an arbitrary origin $(\xi,\ \eta)$, we have

$$F(X,\ Y)=F(X,\ \eta), \quad G(X,\ Y)=G(\xi,\ Y). \tag{18}$$

Finally, at $x=X$, $y=Y$, every

$$\left(-\frac{\partial}{\partial x}\right)^{p-r-m-1}\left(-\frac{\partial}{\partial y}\right)^{q-s-n-1}U$$

on the right of (17) vanishes, by (9), unless $r+m=0$ and $s+n=0$. This reduces (17) to

$$F(X,\ Y)+G(X,\ Y)=-A_{00}\left[V\left(-\frac{\partial}{\partial x}\right)^{p-1}\left(-\frac{\partial}{\partial y}\right)^{q-1}U\right]_{(X,\ Y)}$$

$$=V(X,\ Y)\Phi_{p-1,\,q-1}(0,\ 0)$$

$$=V(X,\ Y), \quad \text{by (10)}.$$

Hence, from (18),

$$V(X, Y) = F(X, \eta)+G(\xi, Y)$$

$$= \int_{\xi}^{X} F_x(x, \eta)\, dx+\int_{\eta}^{Y} G_y(\xi, y)\, dy+F(\xi, \eta)+G(\xi, \eta).$$

Substituting for F_x, G_y, $F+G$ from (13), (15), (17), and modifying the notation a little, we can write

$$V(x, y) = \int_{\xi}^{x} \sum_{r=0}^{p} \sum_{s=0}^{q-1} (-A_{rs}) \sum_{n=0}^{q-s-1} \frac{\partial^{p-r+n} V(t, \eta)}{\partial t^{p-r} \partial \eta^n}\, \Phi_{0,\, q-s-n-1}(x-t, y-\eta)\, dt$$

$$+\int_{\eta}^{y} \sum_{r=0}^{p-1} \sum_{s=0}^{q} (-A_{rs}) \sum_{m=0}^{p-r-1} \frac{\partial^{q-s+m} V(\xi, t)}{\partial \xi^m \partial t^{q-s}}\, \Phi_{p-r-m-1,\, 0}(x-\xi, y-t)\, dt$$

$$+\sum_{r=0}^{p-1} \sum_{s=0}^{q-1} (-A_{rs}) \sum_{m=0}^{p-r-1} \sum_{n=0}^{q-s-1} \frac{\partial^{m+n} V(\xi, \eta)}{\partial \xi^m \partial \eta^n}\, \Phi_{p-r-m-1,\, q-s-n-1}(x-\xi, y-\eta). \quad (19)$$

8. This is the formula giving the general solution $V(x, y)$ when the *data*, *i.e.* the prescribed values of V and its derivatives, lie on the characteristics $x = \xi$, $y = \eta$ through the arbitrary origin (ξ, η). If, more generally, the data are borne by an arbitrary curve C, and the characteristics $x = X$, $y = Y$ through (X, Y) cut this curve (suppose uniquely) in (X, η), (ξ, Y), then writing, as before,

$$V(X, Y) = F(X, \eta)+G(\xi, Y),$$

we now have

$$V(X, Y) = \tfrac{1}{2} \int_{(\xi, Y)}^{(X, \eta)} \{(F_x-G_x)\, dx+(F_y-G_y)\, dy\}$$

$$+\tfrac{1}{2}\{F(X, \eta)+G(X, \eta)+F(\xi, Y)+G(\xi, Y)\}, \quad (20)$$

where the integration is taken along the curve C. Substitution for F_x, G_y, $F+G$ from (13), (15), (17) then gives the value of V at the arbitrary point (X, Y) in terms of its value and those of certain of its derivatives at points of the curve C, and in particular at (X, η), (ξ, Y), the limits of the integration on the curve.

If, finally, we desire the general solution in terms of arbitrary functions, we have it at once from (19), in terms of the $p+q$ arbitrary functions

$$\phi_m(t) \equiv \frac{\partial^m V(\xi, t)}{\partial \xi^m} \quad (m = 0, 1, \ldots, p-1),$$

$$\psi_n(t) \equiv \frac{\partial^n V(t, \eta)}{\partial \eta^n} \quad (n = 0, 1, \ldots, q-1),$$

and their derivatives.

9. As a simple example of the equation (1) we can take the equation

$$\frac{\partial^{p+q} V}{\partial x^p \partial y^q} = V. \tag{21}$$

This we can rewrite as

$$\delta(\delta-1)\ldots(\delta-p+1)\,\delta'(\delta'-1)\ldots(\delta'-q+1)\,V = x^p y^q V, \tag{22}$$

where, as usual, δ, δ' denote $x\partial/\partial x$, $y\partial/\partial y$. The equation (22) is a simple example of the equation

$$f(\delta)\,F(\delta')\,V = x^p y^q g(\delta)\,G(\delta')\,V,$$

which I have discussed elsewhere (**1**, **2**) under the title of "hypergeometric" partial differential equations. The two forms of the fundamental solution, and therefore also of the general solution, do not, of course, differ*.

10. The extension of these methods to the equation, analogous to (1), in any number of independent variables is sufficiently indicated by considering briefly the equation in the field of three variables (x, y, z).

$$\sum_{l=0}^{p} \sum_{m=0}^{q} \sum_{n=0}^{r} A_{lmn} \frac{\partial^{p+q+r-l-m-n} V}{\partial x^{p-l} \partial y^{q-m} \partial z^{r-n}} = 0. \tag{23}$$

We find that the fundamental solution is

$$U(X, Y, Z;\ x, y, z) \equiv \Phi(X-x,\ Y-y,\ Z-z),$$

where now

$$\Phi(x, y, z) \equiv \sum_{\tau} \frac{T!\, x^{L+p-1} y^{M+q-1} z^{N+r-1}}{(L+p-1)!\,(M+q-1)!\,(N+r-1)!} \prod'_{l, m, n} \frac{A_{lmn}^{\tau(l, m, n)}}{\tau(l, m, n)!}, \tag{24}$$

* Except that I go a little further in 2 and express the fundamental solution as a contour integral.

in which $L \equiv \Sigma l\tau$, $M \equiv \Sigma m\tau$, $N \equiv \Sigma n\tau$, $T \equiv \Sigma\tau$, and Π' omits the term (0, 0, 0). It is not difficult to show that Φ has properties analogous to (6), (8), (9), (10) on the characteristics $x = X$, $y = Y$, $z = Z$. In terms of Φ we obtain the general solution

$$
\begin{aligned}
V(x, y, z) &= \sum_{x,y,z} \int_{\eta}^{y} \int_{\zeta}^{z} \sum_{l=0}^{p-1} \sum_{m=0}^{q} \sum_{n=0}^{r} (-A_{lmn}) \sum_{i=0}^{p-l-1} \frac{\partial^{i+q+r-m-n} V(\xi, t, u)}{\partial \xi^i \partial t^{q-m} \partial u^{r-n}} \\
&\qquad \times \Phi_{p-l-i-1,\,0,\,0}(x-\xi, y-t, z-u)\, dt\, du \\
&\quad + \sum_{x,y,z} \int_{\xi}^{x} \sum_{l=0}^{p} \sum_{m=0}^{q-1} \sum_{n=0}^{r-1} (-A_{lmn}) \sum_{j=0}^{q-m-1} \sum_{k=0}^{r-n-1} \frac{\partial^{j+k+p-l} V(t, \eta, \zeta)}{\partial t^{p-l} \partial \eta^j \partial \zeta^k} \\
&\qquad \times \Phi_{0,\,q-m-j-1,\,r-n-k-1}(x-t, y-\eta, z-\zeta)\, dt \\
&\quad + \sum_{l=0}^{p-1} \sum_{m=0}^{q-1} \sum_{n=0}^{r-1} (-A_{lmn}) \sum_{i=0}^{p-l-1} \sum_{j=0}^{q-m-1} \sum_{k=0}^{r-n-1} \frac{\partial^{i+j+k} V(\xi, \eta, \zeta)}{\partial \xi^i \partial \eta^j \partial \zeta^k} \\
&\qquad \times \Phi_{p-l-i-1,\,q-m-j-1,\,r-n-k-1}(x-\xi, y-\eta, z-\zeta). \qquad (25)
\end{aligned}
$$

This result can be obtained by substituting Green's theorem for Stokes theorem in Riemann's method or by a process analogous to that which I have used* in treating the hypergeometric equation in the field of three variables.

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* 1, 301–3, § 6.

ON THE FUNDAMENTAL REGION OF AN ORTHOGONAL REPRESENTATION OF A FINITE GROUP

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Introduction.

The fundamental region of an orthogonal group of finite order has been the subject of much investigation in recent years by H. S. M. Coxeter and others. The results seem to have been put in a definitive form, and throw much light on the abstract definitions. It is proved that, if the fundamental region is definite, it must be a simplex and the group may be generated by reflections*. But this is a very drastic restriction; for consider the symmetric group on four symbols. One representation of degree 3 is the group of symmetries of the tetrahedron, with a triangle as fundamental region which is perfectly definite. As will appear in the sequel, the fundamental region of the other representation of degree 3 is a pair of triangles whose vertices only are definite.

In this paper we show that those elements of the fundamental region of an irreducible orthogonal representation (α) of G which are definite may be determined from a study of the table of functions S^{α}_{H}, where this is the number of times that (α) appears as an irreducible component of the representation g_H of G with respect to the sub-group H. If the degree of (α) is f_α, it follows that a necessary condition for the fundamental region to be completely defined is that S^{α}_{H} takes all values from 1 to f_α.

In the geometrical theory as originated by Klein the characteristic has never played a part, although it is fundamental from the algebraic point of view. Its geometrical interpretation is given in Theorem 3.

At the end of the paper are tables of the sub-groups of S_4 and of the functions χ^{α}_{R}, S^{α}_{H}, and m^{H}_{R} derived therefrom.

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* Coxeter, 595–597.

I should like to express my thanks to the Referees for their valuable criticisms of the manuscript.

1. It has long been known that with each sub-group H of a group G may be associated a representation of G as a transitive permutation group. If h is the order of H and g that of G, the degree of this representation g_H is g/h.

If an operation R of G is represented by a permutation $R(H)$ in g_H, $R(H)$ will leave a certain number of the g/h symbols unaltered. Call this number the *mark* (the name is due to Burnside) of R in g_H and denote it by m_R^H. Frobenius* developed the theory of marks in 1898; the principal results are as follows:

$$\sum_{R \subset G} m_R^H = g, \tag{1.1}$$

$$\sum_{R \subset G} m_R^{H_1} m_R^{H_2} = lg, \tag{1.2}$$

where l is the number of sets of the form $H_1 S H_2$ whose sum is G.

Evidently such a permutation group is immediately expressible as an orthogonal group of permutation matrices, which is reducible. The irreducible components of this group, which we shall also denote by g_H, may easily be determined. If S_H^α denotes the multiplicity with which such a component appears in g_H, then

$$m_R^H = \sum_\alpha S_H^\alpha \chi_R^\alpha, \tag{1.3}$$

where χ_R^α is the characteristic of R in the representation (α) of G. Let $\chi_I^\alpha = f_\alpha$. Multiplying by $\chi_R^{\alpha'}$ and summing over R, we have

$$S_H^\alpha = \frac{1}{g} \sum_{R \subset G} m_R^H \chi_R^{\alpha'} = \frac{1}{g} \sum_{R \subset G} m_R^H \chi_{R^{-1}}^\alpha = S_H^{\alpha'}. \tag{1.4}$$

This expression for S_H^α may be put in a more convenient form. It is known that

$$m_R^H = \frac{g\, n_\gamma^{(H)}}{h\, n_\gamma},$$

where n_γ is the number of members in the conjugate set which contains R,

* Frobenius, § 1; *cf.* Speiser, 120 and 198.

and $n_\gamma^{(H)}$ is the number of these which appear in H. Hence*

$$S_H^\alpha = \frac{1}{h} \sum_\gamma \chi_\gamma^\alpha n_\gamma^{(H)} = \frac{1}{h} \sum_{R \subset H} \chi_R^\alpha, \tag{1.5}$$

which is more suitable for calculation than (1.4).

In what follows we are concerned only with the real irreducible components of g_H. If imaginary ones appear, then the conjugate representations will also be present and $S_H^\alpha = S_H^{\alpha'}$. Thus in a projection on to a real component, the linear relations among the vertices (2.6) are all real. In the last section we consider the imaginary components.

2. Let† us suppose that K is a sub-group of G of order k. We have

$$G = R_1 K + R_2 K + \ldots + R_r K, \tag{2.1}$$

where $r = g/k$. It is well known that g_K is primitive if, and only if, K is a maximum sub-group of G. If H is a sub-group of K,

$$K = \rho_1 H + \rho_2 H + \ldots + \rho_s H, \tag{2.2}$$

where $s = k/h$, and also

$$G = R_1 (\Sigma \rho_i H) + R_2 (\Sigma \rho_i H) + \ldots + R_r (\Sigma \rho_i H), \tag{2.3}$$

yielding the representation g_H. Any permutation of K permutes the sets $R_1 \rho_i H$ according to (2.2), and permutations of the form $R_j K R_j^{-1}$ permute the sets $R_j \rho_i H$.

From‡ such a representation of G as a permutation group g_K, we may construct a set of permutation matrices. The simplex

$$(0^{j-1},\ 1,\ 0^{r-j}) \quad (j = 1, 2, \ldots, r) \tag{2.4}$$

constitutes an invariant configuration for the linear group which we may still denote by g_K. Supposing (α) to be orthogonal, we can find an orthogonal matrix $A = (a_{ij})$, such that $A g_K A^{-1}$ takes the reduced form

$$\begin{bmatrix} [f_\alpha] & 0 \\ 0 & [r - f_\alpha] \end{bmatrix} \tag{2.5}$$

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* Weyl, 287.

† Burnside, 234. For questions of transitivity *cf.* Miller.

‡ Robinson, 407.

if, and only if, $S_K^a > 0$. Under such conditions the configuration (2.4) becomes

$$\Pi_j : (a_{1j}, a_{2j}, \ldots, a_{f_a j}, \ldots, a_{rj}),$$

whose projection on $[f_a]$,

$$(2.6) \qquad {}_aP_j : (a_{1j}, a_{2j}, \ldots, a_{f_a j}),$$

is invariant under the transformations of (a). For brevity we shall speak of "the projection of g_K on to $[f_a]$".

If H is a sub-group of K, then $S_K^a \leqslant S_H^a$.* This results immediately if we make the change of variables in g_H given by the matrix

$$T: \begin{array}{c} \begin{matrix} R_1\rho_1 H & . & R_1\rho_s H, & R_2\rho_1 H & . & R_2\rho_s H & . \ . \ . & R_r\rho_1 H & . & R_r\rho_s H \end{matrix} \\ \begin{bmatrix} 1/\sqrt{s} & . & 1/\sqrt{s} & 0 & . & 0 & . \ . \ . & 0 & . & 0 \\ 0 & . & 0 & 1/\sqrt{s} & . & 1/\sqrt{s} & . \ . \ . & 0 & . & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & . & 0 & 0 & . & 0 & . \ . \ . & 1/\sqrt{s} & . & 1/\sqrt{s} \\ a_{11} & . & a_{1s} & a_{1s+1} & . & & & & & \\ a_{21} & . & & & & & & & & \\ . & & & & & & & & & \end{bmatrix} \end{array}.$$

The a's may be chosen so that $Tg_H T^{-1}$ takes the form

$$\begin{bmatrix} g_K & 0 \\ 0 & A \end{bmatrix},$$

and the statement follows.

Let us suppose that $S_H^a = 1$. If $[f_a]$ appears as an irreducible component of g_K, then $S_K^a = 1$. Projecting g_H on to the space of g_K as above described, we see from the form of T that the vertices coincide in sets of s, and finally projecting on to $[f_a]$, we see that the configuration is the same as that obtained by projecting g_K on to $[f_a]$. If K is the largest sub-group of G containing H, such that $S_K^a = 1$, the number of points in the configuration is g/k and we may call K the *group of stability* of the corresponding point. The groups of stability of the other points of the configuration are the conjugates of K in G. As a simple illustration put $K = G$ and we obtain

* Frobenius, 505 (17).

the identical representation of G; $S^1_G = 1$ while all other $S^a_G = 0$. This representation appears just once* in each g_H, *i.e.* $S^1_H = 1$. The transformation T has $1/\sqrt{(g/h)}$ in each place of its first row. $Tg_H T^{-1}$ may be written

$$\begin{bmatrix} g_G & 0 \\ 0 & A \end{bmatrix}$$

and if we project on to the space of g_G, *i.e.* a 1-dimensional sub-space, all the vertices coincide in a single point.

If $S^a_H > 1$, we may take the irreducible components X_i of g_H equivalent to (a) to be the same, and if†

$$(2.7)\quad X = \begin{bmatrix} X_1 & 0 & . & . & 0 \\ 0 & X_2 & . & . & 0 \\ \cdots & \cdots & \cdots & & \cdots \\ 0 & 0 & . & . & X_{S^a_H} \end{bmatrix}, \qquad P = \begin{bmatrix} p_{11} I_a & . & . & p_{1 S^a_H} I_a \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ p_{S^a_H 1} I_a & . & . & p_{S^a_H S^a_H} I_a \end{bmatrix},$$

then $PXP^{-1} = X$, where (p_{ij}) is an arbitrary orthogonal matrix and I_a is the unit matrix of degree f_a. Thus, making the projection on to $[f_a]$, instead of obtaining as before when $S^a_H = 1$ a unique point configuration, we here obtain a configuration any vertex of which is a linear function of S^a_H independent points‡. With such a projection we associate a sub-space of the surface Σ of the invariant hyper-sphere in $[f_a]$ of $S^a_H - 1$ dimensions. As before, it follows that if K is the largest sub-group of G containing H, such that $S^a_H = S^a_K = \rho$, then the projections of g_H and g_K on to $[f_a]$ are the same. The configuration which we thus obtain is generated by applying the transformations of (a) to one of the sub-spaces $[\rho - 1]$ on Σ. Every point of $[\rho - 1]$ is left invariant by the transformations of K. On the other hand, it is clear that K is the largest sub-group of G which leaves every point of the sub-space invariant. We shall call K the *group of stability* of $[\rho - 1]$.

Conversely, we may show that, if K in (a) is the group of stability of each of s linearly independent points and no more, then the irreducible orthogonal representation (a) appears just s times in g_K.

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* *Cf.* Speiser, 183, Theorem 163.

† Schur, 418.

‡ That the point configuration so obtained might vary was pointed out to me by Dr. Ferns; this suggested the writing of the present paper. For the misinterpretation of the theorem in Ferns's foot-note (12), I take the responsibility.

Thus we have proved*

THEOREM 1. *The maximum number of linearly independent points on the hyper-sphere in* $[f_a]$ *left invariant by the sub-group* K *is* S^a_K.

3. This theorem has an interesting application to the concept of the fundamental region. The existence of such a region follows from the Fricke-Klein† construction, but this construction yields little information about its nature. We take the familiar definition: a connected region‡ R of the surface Σ of the hyper-sphere in $[f_a]$, such that no two points of R are equivalent under (α), and such that every point of Σ is equivalent to just one point of R, is called a *fundamental region* of (α). Since R and its equivalents under (α) must cover the whole of Σ, it follows that R must be of f_a-1 dimensions.

We have seen that with every point P of Σ is associated a group of stability K, and that K leaves invariant a sub-space $[S^a_K-1]$ containing P. If $S^a_K<f_a$, this must be equivalent to at least one sub-space *belonging to the boundary of* R. This is a consequence of the fact that a point distant ϵ from $[S^a_K-1]$ is transformed by K into one or more equivalent points also distant ϵ from $[S^a_K-1]$, and no two equivalent points belong to R. If the boundary of R is thus completely defined, we say that R is *definite*. Its interior points are equivalent to those of Σ for which K is the group of stability of Σ, and $S^a_K=f_a$. Thus R is composed of all its interior points and a representative set of non-equivalent sub-spaces of its boundary. Hence we have§

THEOREM 2. *The fundamental region of an irreducible orthogonal representation* (α) *of* G *cannot be definite unless* S^a_K *takes all values from* 1 *to* f_a.

It would be interesting if this condition were sufficient as well as necessary for the group to be generated by reflections.

4. We turn to a more detailed investigation of R. So far we have required that $S^a_H=S^a_K=\rho$, but it may happen that K is contained in a sub-group K' for which $0<S^a_{K'}<\rho$. If this is so, the groups of stability of certain points of $[\rho-1]$ are larger than those of other points, and the

* This may also be considered as a corollary of Frobenius's reciprocity theorem, as was pointed out by a Referee. *Cf.* Speiser, 198, Theorem 175, putting $\Delta=1$.

† Fricke-Klein, 108, 216; also Got, 7.

‡ This does not limit the generality for our purposes.

§ *Cf.* also (A) and (B) in the following section.

truth of the following statements is clear.

(A) *If K_i is the group of stability of a sub-space π_i of $S^n_{K_i}-1$ dimensions, and π_3 is the meet of π_1 and π_2, then $K_1 \subset K_3$ and $K_2 \subset K_3$.*

(B) *If K_i is the group of stability of a sub-space π_i, and π_3 is the join of π_1 and π_2, then K_3 is the cross cut of K_1 and K_2.*

It may happen that the dimensionality of π_3 in (B) is less than $S^a_{K_3}-1$, as will appear below. If we replace K_1 by RK_1R^{-1} in general K_3 will vary, and we may trace this variation, handling all possibilities at once.

If* g_{H_i} is represented on the permutation symbols x_r and g_{H_j} on the symbols y_s, then $g_{H_i}g_{H_j}$ may be represented on the symbols $x_r y_s$. Further,

$$g_{H_i}g_{H_j} = \sum_k a^{ij}_k g_{H_k}, \tag{4.1}$$

where the a's are numerical. Evidently the projection of $g_{H_i}g_{H_j}$ on to a sub-space $[f_a]$ of one of the g_{H_k} appearing on the right is the same as that of the g_{H_k} on $[f_a]$, as in the proof of Theorem 1. But we could choose symbols on which to represent $g_H\, g_{H_j}$ which are suggested by the geometry. If we interpret x_r, y_s as vertices† of two configurations of points on Σ whose groups of stability are respectively H_i, H_j, then

$$\lambda x_r + \mu y_s$$

represents a point on the join of x_r and y_s, whose group of stability is the cross-cut of H_i and H_j by (B). Varying y_s, *i.e.* choosing different members of the conjugate set defined by H_j in G, we arrive at different cross-cuts H_k according to the right-hand side of (4.1), yielding one or more distinct configurations on Σ. Such a geometrical interpretation does not admit linear transformations, so the a's in (4.1) are unique‡.

Let us consider as an illustration the symmetric group on four symbols, whose sub-groups are given in Table I, with their orders. Table II gives the characteristics, and Tables III and IV give the functions S^a_H and m^H_R. For the projection of g_H on to $[f_{3.1}]$ then, all pertinent information is contained in the last column but one of Table III. $H_3 \subset H_7$, which is the group of stability of a vertex of an octahedron. $H_4 \subset H_8$, yielding a tetrahedron§. Since

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* Burnside, 240.

† *Cf.* § 2.

‡ *Cf.* Burnside, 288. The identities there referred to may easily be obtained from the table of m^H_K by treating it as a matrix and reducing it.

§ Ferns, § 3.2 and § 3.3.

the mark of R in $g_{H_i} g_{H_j}$ is $m_R^{H_i} m_R^{H_j}$, from Table IV, we may easily see that

$$g_7 g_8 = 2g_2. \tag{4.2}$$

Now, if we consider RH_8R^{-1}, it is evident that it has in common with H_7 either $H_2 = \{I, (12)\}$, or $H_2' = \{I, (34)\}$, each contributing one g_2 to (4.2). Since $S_{H_7}^{3,1} = 2$, we have a linear sub-space whose group of stability is H_2, and which, along with its equivalents, yields the edges of the fundamental region. On account of the self-duality of the tetrahedron, it is interesting to consider

$$g_8 g_8 = g_2 + g_8. \tag{4.3}$$

Again,

$$g_2 g_8 = g_1 + 2g_2, \tag{4.4}$$

and the component g_2 may be explained as before, while g_1 appears by the choice of a vertex x_i of the tetrahedron along with a point y_s in the opposite edge of the fundamental region.

Consider next the projection on to $[f_{2,1,1}]$. $H_3 \subset H_5$ and we obtain an octahedron from g_5, while g_4 yields a cube and g_2 a cuboctahedron*, neighbouring vertices of which we shall denote by O, C, and CO. This representation is, in fact, an isomorphism with the group of the octahedron. From the product

$$g_4 g_5 = 2g_1, \tag{4.5}$$

we see that the group of stability of the sub-space joining C and O is the identity. Similarly

$$g_4 g_2 = 4g_1, \text{ and } g_2 g_5 = 3g_1, \tag{4.6}$$

and none of the sub-spaces so defined is definite. This is borne out by the fact that there is no two appearing in the column (2, 1, 1) of Table III.

The projection on to $[f_{2.2}]$ is not so interesting, for here the fundamental region is merely an arc of the circle. $H_1 \subset H_3 \subset H_6$ and H_6 is the group of stability of the general point. H_2, H_5, H_7 are all sub-groups of H_9, yielding a triangle which with its dual divides the circumference into the six fundamental regions.

5. We may ask for what representations $(\alpha_1, \alpha_2, \ldots, \alpha_h)$ of the symmetric group S_n on n symbols is the fundamental region definite? The alternating

* Ferns, § 3.4.

group for $n > 4$ is simple. Hence the isomorphism between S_n and its representation must be simple except for the representations corresponding to the partitions (1^n), (n) of n. From Coxeter's enumeration* of the groups generated by reflections, it follows that only these and the representation corresponding to the partition $(n-1, 1)$ can have definite fundamental regions.

If we hold one letter fixed in a representation $(\alpha_1, \alpha_2, \ldots, \alpha_h)$ of S_n we obtain a reducible representation of S_{n-1}, and the irreducible components correspond† to those partitions of $n-1$ such that

$$(5.1) \qquad f_{\alpha_1, \alpha_2, \ldots, \alpha_h} = f_{\alpha_1 - 1, \alpha_2, \ldots, \alpha_h} + f_{\alpha_1, \alpha_2 - 1, \ldots, \alpha_h} + \cdots + f_{\alpha_1, \alpha_2, \ldots, \alpha_h - 1},$$

where those f's whose suffixes β_i do not satisfy $\beta_i \geqslant \beta_{i+1}$ are to be omitted. If K is a sub-group of S_{n-1}, then, summing the characteristics over K, we obtain by (1.5)

$$(5.2) \qquad S_K^{\alpha_1, \alpha_2, \ldots, \alpha_h} = S_K^{\alpha_1 - 1, \alpha_2, \ldots, \alpha_h} + S_K^{\alpha_1, \alpha_2 - 1, \ldots, \alpha_h} + \ldots + S_K^{\alpha_1, \alpha_2, \ldots, \alpha_h - 1}.$$

The maximum number of linearly independent points that have K as group of stability is given by (5.2). The fundamental region of S_{n-1} in $[f_{\alpha_1, \alpha_2, \ldots, \alpha_h}]$ is the product space π of the fundamental regions of S_{n-1} in $[f_{\alpha_1 - 1, \alpha_2, \ldots, \alpha_h}]$, $[f_{\alpha_1, \alpha_2 - 1, \ldots, \alpha_h}]$, etc. The boundary of π must be equivalent to part of the boundary of R, and, in fact, just n fundamental regions equivalent to R are contained in and completely fill a properly chosen π. Analogous statements apply to any other sub-group of S_n. Thus we may relate the fundamental region of an irreducible orthogonal representation of any group to that of an appropriate S_n.

6. Closely connected with the ideas which we have been developing is

THEOREM 3. *If the point P_1 on the unit hyper-sphere in $[f_a]$ gives rise under (α) to a configuration of points P_j, permuted according to g_K, then*

$$\chi_{R^a} = \frac{f_a k}{g} \left(\sum_{j=1}^{g/k} P_j P_{j'} \right),$$

where R^{-1} is represented in g_K by the permutation $\begin{pmatrix} P_j \\ P_{j'} \end{pmatrix}$.

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* Coxeter, 618.

† Young, 269.

With the notation of § 2 consider the orthogonal matrix $A = (a_{ij})$ which transforms g_K into the reduced form (2.5). If R is a permutation matrix of g_K, ARA^{-1} is orthogonal and the product AR is obtained by permuting the columns of A according to R^{-1}. Hence $AR.A^{-1}$ will be obtained by multiplying a "permuted" row of A into some "unpermuted" row of A. Thus the characteristic is given by

$$\chi_R = m_R^K = \sum_i \sum_j a_{ij} a_{ij'} = \sum_j \sum_i a_{ij} a_{ij'} = \sum_{j=1}^{g/k} \pi_j . \pi_{j'}, \tag{6.1}$$

where, as usual, we denote scalar multiplication by a dot. If $R = I$, $j = j'$, and $\chi_I = m_I^K = g/k$.

Projecting the configuration π_j on to $[f_\alpha]$ yields, as in § 2, a configuration ${}_\alpha P_j$, which is invariant under the transformations of (α). But clearly it does not lie on the unit hyper-sphere in $[f_\alpha]$, for

$$\sum_{j=1}^{g/k} {}_\alpha P_j . {}_\alpha P_j = \frac{g}{k} ({}_\alpha P_j . {}_\alpha P_j) = f_\alpha,$$

since we are summing the squares of the elements in the first f_α rows of A. We may radially project the configuration ${}_\alpha P_j$ on to the unit hyper-sphere by writing

$${}_\alpha P_j = \left(\frac{f_\alpha k}{g}\right)^{\frac{1}{2}} P_j,$$

where now $P_j . P_j = 1$. In (6.1) χ_R breaks up into the irreducible character χ_R^α and another according to (2.5). Hence

$$\chi_R^\alpha = \sum_{j=1}^{g/k} {}_\alpha P_j . {}_\alpha P_{j'} = \frac{f_\alpha k}{g} \sum_{j=1}^{g/k} P_j . P_{j'}. \tag{6.2}$$

We could have obtained this result by transforming the regular representation, since any invariant configuration in $[f_\alpha]$ can be obtained from its projection as in the proof of Theorem 1. We may illustrate the theorem by considering the representation (3, 1) of S_4. Choosing P_1 to be a vertex of the tetrahedron, we obtain P_2, P_3, P_4. It is easily seen that $P_i . P_j = -\frac{1}{3}$. Thus

$$\chi_{(12)}^{3,1} = \frac{3.6}{24} (P_1 . P_2 + P_2 . P_1 + P_3 . P_3 + P_4 . P_4)$$

$$= \tfrac{3}{4}(-\tfrac{1}{3} - \tfrac{1}{3} + 1 + 1) = 1.$$

Similarly,

$$\chi_{(12)(34)}^{3,1} = \tfrac{3}{4}(P_1 . P_2 + P_2 . P_1 + P_3 . P_4 + P_4 . P_3)$$

$$= \tfrac{3}{4}(-\tfrac{1}{3} - \tfrac{1}{3} - \tfrac{1}{3} - \tfrac{1}{3}) = -1.$$

If we choose the initial point to be a vertex of the octahedron, we may call its vertices $a_1, b_1;\ a_2, b_2;\ a_3, b_3$. Then we have

$$\chi^{3,1}_{(12)} = \frac{3\,.\,4}{24}(a_1\,.\,a_1+a_2\,.\,b_3+a_3\,.\,b_2+b_1\,.\,b_1+b_2\,.\,a_3+b_3\,.\,a_2)$$

$$= \tfrac{1}{2}(1+0+0+1+0+0) = 1.$$

Similarly,

$$\chi^{3,1}_{(12)(34)} = \tfrac{1}{2}(a_1\,.\,a_1+a_2\,.\,b_2+a_3\,.\,b_3+b_1\,.\,b_1+b_2\,.\,a_2+b_3\,.\,a_3)$$

$$= \tfrac{1}{2}(1-1-1+1-1-1) = -1.$$

Since (a) is orthogonal, we may replace R^{-1} by R in any application of Theorem 3.

7. We have previously remarked that, if (a) is comp ex, then the conjugate representation (a') also appears in g_H, and $S^a_H = S^{a'}_H$. From* two such irreducible representations X_i, $\bar{X}_i$ of G may be constructed an orthogonal representation $\underline{X}_i$ which is irreducible in the real field. This orthogonal representation of G is of degree $2f_a$, and its fundamental region is subject to the same degree of arbitrariness as discussed in § 2, but with one difference. $\underline{X}_i$ is not absolutely irreducible and there exist orthogonal matrices permutable with it which are different from identity, and leave no point of the surface of the hyper-sphere unaltered. For example, consider the representation g_1 of the cyclic group of order 3. Its irreducible components are

$$e^{2\pi i/3}, \quad e^{-2\pi i/3}, \quad 1.$$

$\underline{X}$ is here the rotation group generated by

$$\begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix},$$

and clearly the fundamental region of this representation is completely indefinite. We have then

THEOREM 4. *The fundamental region of an orthogonal representation of G which is irreducible in the real field, but not absolutely irreducible, is completely indefinite.*

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* Loewy.

TABLE I.

H	Sub-group	h
H_1	I	1
H_2	I, (12)	2
H_3	I, (12)(34)	2
H_4	I, (123), (132)	3
H_5	I, (1234), (13)(24), (1432)	4
H_6	I, (12)(34), (14)(23), (13)(24)	4
H_7	I, (12), (34), (12)(34)	4
H_8	I, (12), (13), (23), (123), (132)	6
H_9	I, (12), (34), (12)(34), (14)(23), (13)(24), (1324), (1423)	8
H_{10}	$\frac{1}{2}S_4$	12
H_{11}	S_4	24

TABLE II.

Class	1.1.1.1	2.1.1	2.2	3.1	4
Order	1	6	3	8	6
1.1.1.1	1	−1	1	1	−1
2.1.1	3	−1	−1	0	1
2.2	2	0	2	−1	0
3.1	3	1	−1	0	−1
4	1	1	1	1	1

$\chi_R{}^a$

TABLE III.

	1.1.1.1	2.1.1	2.2	3.1	4
g_1	1	3	2	3	1
g_2	.	1	1	2	1
g_3	1	1	2	1	1
g_4	1	1	.	1	1
g_5	.	1	1	.	1
g_6	1	.	2	.	1
g_7	.	.	1	1	1
g_8	.	.	.	1	1
g_9	.		1	.	1
g_{10}	1	.	.	.	1
g_{11}	.	.	.	.	1

$S_H{}^a$

TABLE IV.

	1.1.1.1	2.1.1	2.2	3.1	4
g_1	24	.	.	.	.
g_2	12	2	.	.	.
g_3	12	.	4	.	.
g_4	8	.	.	2	.
g_5	6	.	2	.	2
g_6	6	.	6	.	.
g_7	6	2	2	.	.
g_8	4	2	.	1	.
g_9	3	1	3	.	1
g_{10}	2	.	2	2	.
g_{11}	1	1	1	1	1

$m_R{}^H$

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NOTES ON A NET OF QUADRIC SURFACES: I. THE CREMONA TRANSFORMATION

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1. Three quadric linearly independent surfaces in three-dimensional space determine a net of quadrics. The polar planes of a point O with respect to the quadrics have in common, in general, a single point O'; moreover, the point common to the polar planes of O' is O. The net of quadrics thus sets up an involutory Cremona transformation T. We speak of O and O' as a pair of *conjugate points*; they are indeed conjugate, in the long accepted sense of the term, with respect to every quadric of the net.

This transformation has been known to mathematicians for a century and more; yet there still remain results to be found, and that quite simply, by means of it. The chief purpose of this note is to obtain, by geometrical reasoning, several combinantal covariants of the net of quadrics. The covariants that we obtain are, with one exception, scrolls generated by chords of the Jacobian curve of the net of quadrics; the other covariant is a surface, of order sixteen, that is self-conjugate (*i.e.* unchanged by the transformation T), and of which we give three geometrical definitions. Before discussing these covariants we give a short description (§§ 3–5) of the simplest self-conjugate loci. Those quartic surfaces that are self-conjugate are indeed already known; but among them are twenty-eight surfaces whose properties entitle them to be noticed separately (§ 4).

A Cremona transformation, unless it is a collineation, has certain *fundamental points* which are transformed not into points but into curves or surfaces. The definition of T admits, *a priori*, of this happening in two ways: a point may be such that its polar planes with respect to the quadrics all coincide, or it may be such that its polar planes have a line in common. The first possibility is not realized unless the net of quadrics has some special property, and we shall assume throughout our work that the net is of the most general kind. But the polar planes of a point P have

a line in common when, and only when, P is the vertex of a cone belonging to the net. The locus of fundamental points for T is therefore the Jacobian curve, ϑ say, of the net of quadrics; it is a twisted sextic of genus 3. The line t which is common to the polar planes of P is a trisecant of ϑ; we speak of it as *the trisecant conjugate to* P. The point conjugate to any point of t is P, and the three trisecants that are conjugate to the intersections of t and ϑ all pass through P. The trisecants generate a scroll R^8, of order eight, on which ϑ is a triple curve; the curve ϑ and the scroll R^8 are of great importance for the study of the transformation T.

2. When a point O lies on a curve or surface, its conjugate point lies on a conjugate curve or surface. If, to take the simplest examples first, O lies on a line λ which does not meet ϑ, O' lies on a twisted cubic* λ'; since λ meets eight trisecants of ϑ, λ' meets ϑ in eight points. Also, if O lies on a plane π, O' lies on a cubic surface* π'; since every trisecant of ϑ meets

* These statements are of known properties of T: see, for example, Bobillier, *Annales de mathématiques*, 18 (1827–8), 268–9; Reye, *Die Geometrie der Lage*, 3 (Leipzig, 1892), 136. They follow at once by elementary considerations of projective geometry. Take, for example, the line λ. The range of points on λ is related projectively to the pencil of polar planes of the points with respect to any quadric. Hence λ' may be obtained as the locus of intersections of corresponding planes of three projectively related pencils; it is, therefore, a twisted cubic. The chords of λ' are the polar lines of λ with respect to the quadrics of the net.

Suppose the net of quadrics is given by $\xi Q_0+\eta Q_1+\zeta Q_2 = 0$, where Q_0, Q_1, Q_2 are homogeneous quadratic functions of four coordinates x_0, x_1, x_2, x_3 and $\xi : \eta : \zeta$ are parameters. Then the equations of ϑ are

$$\begin{vmatrix} \frac{\partial Q_0}{\partial x_0} & \frac{\partial Q_0}{\partial x_1} & \frac{\partial Q_0}{\partial x_2} & \frac{\partial Q_0}{\partial x_3} \\ \frac{\partial Q_1}{\partial x_0} & \frac{\partial Q_1}{\partial x_1} & \frac{\partial Q_1}{\partial x_2} & \frac{\partial Q_1}{\partial x_3} \\ \frac{\partial Q_2}{\partial x_0} & \frac{\partial Q_2}{\partial x_1} & \frac{\partial Q_2}{\partial x_2} & \frac{\partial Q_2}{\partial x_3} \end{vmatrix} = 0.$$

The cubic surface conjugate to the plane $a_0x_0+a_1x_1+a_2x_2+a_3x_3 = 0$ is

$$a_0U_0+a_1U_1+a_2U_2+a_3U_3 \equiv \begin{vmatrix} a_0 & a_1 & a_2 & a_3 \\ \frac{\partial Q_0}{\partial x_0} & \frac{\partial Q_0}{\partial x_1} & \frac{\partial Q_0}{\partial x_2} & \frac{\partial Q_0}{\partial x_3} \\ \frac{\partial Q_1}{\partial x_0} & \frac{\partial Q_1}{\partial x_1} & \frac{\partial Q_1}{\partial x_2} & \frac{\partial Q_1}{\partial x_3} \\ \frac{\partial Q_2}{\partial x_0} & \frac{\partial Q_2}{\partial x_1} & \frac{\partial Q_2}{\partial x_2} & \frac{\partial Q_2}{\partial x_3} \end{vmatrix} = 0;$$

we thus obtain ∞^3 cubic surfaces passing through ϑ. The Cremona transformation itself is given by the equations $\frac{x_0'}{U_0} = \frac{x_1'}{U_1} = \frac{x_2'}{U_2} = \frac{x_3'}{U_3}$.

π, π' must contain ϑ. The six trisecants that are conjugate to the intersections of π and ϑ lie on π', forming one half of a double-six.

Having obtained the curve and surface conjugate to a line and plane, we may use the results to obtain the loci conjugate to any curve and surface.

Suppose that a curve, not lying on R^8, is of order ν and has ι intersections with ϑ. Then the order of the conjugate curve is $\nu' = 3\nu - \iota$, and the number of its intersections with ϑ is $\iota' = 8\nu - 3\iota$. For the order ν', being the number of intersections of the conjugate curve with a plane, is the number of intersections of the original curve with a cubic surface through ϑ, apart from intersections on ϑ itself; while ι', being the number of points common to the conjugate curve and ϑ, is the number of trisecants of ϑ that meet the original curve at points not lying on ϑ.

If a surface is of order n and contains ϑ as a curve of multiplicity i, the conjugate surface is of order $n' = 3n - 8i$ and contains ϑ as a curve of multiplicity $i' = n - 3i$. For the order n', being the number of intersections of the conjugate surface with a line, is the number of intersections, other than points on ϑ, of the original surface with a twisted cubic meeting ϑ at eight points; while i', the multiplicity of ϑ on the conjugate surface, is the number of intersections, other than points of ϑ, of a trisecant of ϑ with the original surface.

Self-conjugate loci.

3. There are certain loci that are unchanged by T, and the investigation of these affords some interesting results.

The first inquiry naturally concerns *points* that are self-conjugate. If O and O' coincide, the polar plane of O with respect to any quadric of the net must pass through O itself; hence O must be common to all the quadrics. Conversely, if O lies on every quadric of the net it coincides with its conjugate point. Hence *the eight base points of the net, and only these points, are unchanged by the transformation**.

Let us now pass to the consideration of self-conjugate *curves*. If two surfaces are conjugate to one another, then their curve of intersection is a self-conjugate curve; for example, the plane cubic in which a plane is met by its conjugate cubic surface is a self-conjugate curve. A curve may, however, be self-conjugate without being the complete intersection of two surfaces.

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* These points are those whose coordinates satisfy $\frac{x_0}{U_0} = \frac{x_1}{U_1} = \frac{x_2}{U_2} = \frac{x_3}{U_3}$.

For a self-conjugate curve, $\nu = \nu'$ and $\iota = \iota'$; hence $\iota = 2\nu$. Taking $\nu = 1$, we see that *if a line is self-conjugate it must be a chord of* ϑ. Suppose now that O lies on a line joining two of the eight base points; then the polar plane of O with respect to any quadric of the net passes through the harmonic conjugate of O with respect to the two base points, so that this is the point O'. Hence the point conjugate to any point of a line joining two base points also lies on this line; the line is therefore self-conjugate. This is in agreement with the known fact* that *the line joining two base points of a net of quadrics is a chord of the Jacobian curve.* We have thus found twenty-eight self-conjugate lines. Conversely: if a line is self-conjugate, the points of the line must be conjugate to one another in pairs. These pairs of conjugate points form an involution on the line, the two double points of which must be self-conjugate. Hence the line must join a pair of base points, and there are no self-conjugate lines other than the twenty-eight joins of pairs of base-points.

Since ϑ is a triple curve on R^8, a chord of ϑ meets two trisecants other than those which pass through its intersections with ϑ. This is true, in particular, when the chord joins a pair of base points, but then, since the chord is self-conjugate, the two trisecants must be those that are conjugate to the intersections of the chord with ϑ.

If a conic is self-conjugate, it must meet ϑ at four points. Moreover, it must lie on the cubic surface conjugate to the plane in which it lies, and the residual intersection of the plane and cubic surface is a line, which must also be self-conjugate. Thus, if a conic is self-conjugate, its plane must pass through one of the twenty-eight self-conjugate lines. Conversely: any plane passing through a self-conjugate line is met by its conjugate cubic surface in this line and a self-conjugate conic; hence there is one self-conjugate conic in any plane passing through a self-conjugate line.

The points of any self-conjugate conic are conjugate to one another in pairs; these conjugate pairs of points form an involution on the conic, and the two double points of the involution, being self-conjugate, are base points. Hence we have the following result. *Any plane through the line joining two base points has four intersections with* ϑ *that do not lie on this line; there is a conic passing through these four points and the two base points,* this conic being the self-conjugate conic that lies in the plane.

Take then a plane β passing through two of the eight base points B_i and B_j. The joins of pairs of conjugate points on the self-conjugate conic in this plane all pass through X, the pole of $B_i B_j$ with respect to this

* Hesse, *Journal für Math.*, 49 (1855), 298.

conic. Now lines which join pairs of conjugate points belong, as is well known*, to a cubic complex, and the cubic cone generated by the lines of this complex that pass through a point O is the cone that projects, from O, the quartic base curve of the pencil of quadrics passing through O. The base curve of the pencil of quadrics passing through X must therefore consist of $B_i B_j$ and the twisted cubic γ_{ij} through the other six base points, and so this twisted cubic passes through X; γ_{ij} passes through the two intersections of $B_i B_j$ with ϑ, and X is its remaining intersection with β.

4. As β varies in the pencil of planes through $B_i B_j$ the conic traces out a surface Γ_{ij} containing ϑ, and this surface, being generated by self-conjugate curves, must be a self-conjugate surface. For the plane joining $B_i B_j$ to a third base point B_k, the conic that passes through B_i, B_j and the four intersections of the plane with ϑ consists of the two lines $B_i B_k$ and $B_j B_k$; thus Γ_{ij} contains the twelve lines that join B_i and B_j to the other six base points. If N is the order of Γ_{ij}, it is the number of points in which Γ_{ij} meets a line λ and also the number of points, other than points of ϑ, in which Γ_{ij} meets the twisted cubic λ' conjugate to λ. Hence

$$N = 3N - 8,$$

so that Γ_{ij} *is a quartic surface.* Moreover, any line that meets $B_i B_j$ has only two other intersections with Γ_{ij}, namely the two points in which it meets the self-conjugate conic lying in the plane joining the line to $B_i B_j$; hence $B_i B_j$ *is a nodal line on* Γ_{ij}. Also a line passing through either B_i or B_j has only one further intersection with Γ_{ij}, so that B_i *and* B_j *are triple points of* Γ_{ij}.

The self-conjugate conic in the plane joining $B_i B_j$ to one of the two trisecants that meet it (other than at the two intersections of ϑ with $B_i B_j$) must consist of the trisecant and the line $B_i B_j$ itself; this plane therefore touches Γ_{ij} at every point of $B_i B_j$, and Γ_{ij} has the same two tangent planes at every point of its nodal line.

We can also identify the tangent cones of Γ_{ij} at its two triple points. The line through B_i or B_j which lies in the plane β and touches the self-conjugate conic at B_i or B_j is a generator of the tangent cone at the point, since it has four, and not merely three, intersections with the surface there. The tangent cones of Γ_{ij} at B_i and B_j are therefore the cubic cones that project the twisted cubic γ_{ij} from these two points.

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* Reye, *loc. cit.*, 137.

5. If we now pass to the consideration of self-conjugate surfaces, it is first to be noted that, for such a surface, $n = n'$ and $i = i'$, and so $n = 4i$. Thus a surface cannot be self-conjugate unless its order is a multiple of 4. The first possibility is therefore a quartic surface; this must have $i = 1$, and so pass through ϑ.

We have noticed already twenty-eight self-conjugate quartic surfaces containing ϑ, and it is easy to give a more general example of such a surface. Let λ be any line; conjugate to it is a twisted cubic λ'. Associated with any plane through λ there is a conjugate cubic surface through λ' and ϑ; these intersect in a self-conjugate curve. As the plane varies in the pencil of planes through λ this cubic curve generates a self-conjugate quartic surface, containing λ, λ', and ϑ. Moreover, this surface also contains the eight base points, for when a plane joins λ to a base point the conjugate cubic surface also passes through this base point. We thus have a system of ∞^4 quartic surfaces, all passing through ϑ and the base points; each surface is self-conjugate, and is associated with a line λ. When λ has a special position the quartic surface has special properties; for example, when λ joins two base points the quartic surface is one of the twenty-eight surfaces Γ_{ij}.

This system of ∞^4 quartic surfaces belongs to a system, of ∞^5 self-conjugate quartic surfaces, which is one of two distinct linear systems of self-conjugate quartic surfaces. These two systems, which together include all self-conjugate quartic surfaces, were found by Godeaux*; they arise as follows.

The locus of a point O whose conjugate point O' lies in the polar plane of O with respect to a fixed linear complex is a quartic surface; this surface, from its definition, is self-conjugate and passes through ϑ and the eight base points. We thus obtain a system of ∞^5 quartic surfaces, one surface being associated with each linear complex; if the linear complex is special, consisting of those lines which meet a given line λ, the quartic surface is the one previously found. The other linear system of self-conjugate quartic surfaces is of freedom six; the surfaces of this system are associated not with linear complexes but with quadrics. The locus of a point O whose conjugate point O' lies in the polar plane of O with respect to a fixed quadric (not belonging to the net) is a quartic surface which, from its

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* *Bulletins de la classe des sciences de l'académie royale de Belgique* (5), 13 (1927), 123.

definition, is self-conjugate; the surface contains ϑ but does not, in general, pass through any of the base points*.

The covariant F^{16}.

6. If a point is on R^8, its conjugate point is on ϑ. Consider now the surface which is the locus, apart from R^8, of points O such that the line OO' meets ϑ. The surface so defined is clearly self-conjugate; it is also, since it bears a relation to the net of quadrics that is unchanged by a linear transformation, a combinantal covariant of the net of quadrics.

The order of the surface is the number of its intersections with a line λ. Suppose then that O lies on λ, so that O' lies on the conjugate twisted cubic λ'; then the lines OO', since they join pairs of corresponding points in a (1, 1) correspondence between a line and a twisted cubic, generate a quartic scroll; this scroll has twenty-four intersections with ϑ. These include the eight points common to ϑ and λ', and the generators of the quartic scroll through these points arise from the eight intersections of λ with R^8. The remaining sixteen intersections of ϑ with the quartic scroll are on generators of this scroll that arise from points O of λ that do *not* lie on R^8, and which therefore lie on the surface with which we are concerned. Hence the surface is of order sixteen, and may be denoted by F^{16}. Since it is self-conjugate, the multiplicity of ϑ on the surface must be one quarter of the order of the surface; hence ϑ is a quadruple curve on F^{16}.

We can also find the multiplicity of the base points on F^{16}. Suppose that λ passes through a base point B, so that λ' also passes through B. The joins of pairs of conjugate points on λ and λ' now generate a *cubic* scroll, since the (1, 1) correspondence between λ and λ' has a united point at B. The cubic scroll meets ϑ at eighteen points, of which eight are the points

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* The equation of the quartic surface associated with the linear complex

$$a'l+b'm+c'n+al'+bm'+cn' = 0$$

is

$$a(x_2U_3-x_3U_2)+b(x_3U_1-x_1U_3)+c(x_1U_2-x_2U_1)+a'(x_0U_1-x_1U_0)$$
$$+b'(x_0U_2-x_2U_0)+c'(x_0U_3-x_3U_0) = 0,$$

the left-hand side being a *skew-symmetric* bilinear form in the two sets of variables x and U. The left-hand side of the equation of the quartic surface associated with a quadric is, on the other hand, a *symmetric* bilinear form. The equation of any quartic surface passing through ϑ may be written $S_1+S_2 = 0$, where S_1 is a symmetric and S_2 a skew-symmetric bilinear form in x and U; the equation of the conjugate quartic surface is then $S_1-S_2 = 0$.

If we take a quadric not belonging to the net and passing through six of the eight base points, the resulting self-conjugate surface is a Weddle surface; thus there are twenty-eight self-conjugate Weddle surfaces.

common to λ' and ϑ; the remaining ten arise from points where λ meets F^{16}. Thus λ and F^{16} intersect in ten points other than B, so that the base points are sextuple points on F^{16}. A similar argument shows that any point on the join of two base points is a double point on F^{16}; hence the twenty-eight self-conjugate lines are nodal lines on the surface.

7. It is, from our point of view, of especial interest to study those covariants of three quadrics which are *combinants*, bearing an invariant relation not merely to the three quadrics individually but to the whole of the net of quadrics determined by these three. Any combinantal covariant, when equated to zero, gives a surface; and any surface, such as R^8 or F^{16}, which bears a relation, to the net of quadrics as a whole, that is unchanged by linear transformation, must be a combinantal covariant. Now if any surface, other than R^8, is a combinant, its conjugate surface must be also. Thus, except for R^8, those combinants which are not individually self-conjugate occur in conjugate pairs. Of combinantal covariants that are self-conjugate, the one of lowest order that has so far been found is F^{16}; two others, both of order 48, occur below.

8. The line joining a pair of conjugate points O and O' is a generator of a quadric of the net, and any other quadric of the net meets the line in a pair of points harmonically conjugate with respect to O and O'; if then OO' meets ϑ in P, the cone whose vertex is P must be the quadric of the net which contains OO'. Thus F^{16} is the locus of points O such that OO' lies on a *cone* of the net.

There is another way of obtaining F^{16} which may be given here.

A pencil of quadrics has a quartic base curve, and on this curve there are sixteen *stalls**, or points where the osculating plane is stationary, having four-point contact instead of the usual three-point contact. The position of these stalls on the curve is known; they are the intersections of the curve with the four faces of the tetrahedron which is self-polar for all the quadrics of the pencil. Moreover, the four tangents of the curve at the stalls where it is met by a face of the tetrahedron all pass through the opposite vertex of the tetrahedron, this being the vertex of a cone containing the quartic curve. Now there are ∞^2 pencils of quadrics belonging to the net, and hence ∞^2 base curves, on each of which there is a finite number of stalls.

* This nomenclature is introduced by Baker, *Principles of geometry*, 5 (Cambridge, 1933), 186.

The locus of the stalls of the base curves is therefore a surface, and this surface is F^{16}, as we now prove.

Those quadrics of the net which pass through a point O belong to a pencil; the base curve of this pencil passes through O and, when O is not a base point, it is the only base curve passing through O. The conjugate point O' is common to the polar planes of O, and therefore lies on the tangent planes at O to all the quadrics passing through O; these tangent planes have in common the tangent of the base curve at O. Hence, if O is any point other than a base point, the line OO' is the tangent at O of the base curve which passes through O. If then O is a stall of a base curve, the line OO', being the tangent of the curve at its stall O, passes through the vertex of a cone containing the curve, and is a generator of this cone. Hence, since OO' lies on a cone of the net, O lies on F^{16}. The locus of the stalls of the base curves is therefore F^{16}, as we desired to prove. The intersections of F^{16} and a base curve consist of the stalls and of the eight base points, each base point counting for six intersections.

The definition of F^{16} as the locus of stalls of base curves shows at once that it is a combinantal covariant, although it does not show so immediately that it is self-conjugate. But, knowing the self-conjugacy of F^{16}, we have the result that *the point conjugate to a stall of a base curve is itself a stall of a base curve.*

The chords of the Jacobian curve.

9. We see from § 2 that the curve conjugate to a chord of ϑ is itself a chord of ϑ, so that *the chords of ϑ are conjugate in pairs.* The relation between a conjugate pair of chords can be stated precisely, as the following considerations quickly show.

Let PQ be any chord of ϑ. We denote the cone of the net whose vertex is at P by (P), and similarly for any other point of ϑ. The cones (P) and (Q) determine a pencil of quadrics, and to this pencil there belong two other cones, (R) and (S). Take any point O on PQ; its conjugate point is the intersection of its polar planes with respect to all the quadrics of the net. The polar plane of O with respect to (P), however, is PRS, and its polar plane with respect to (Q) is QRS; the line common to these two polar planes is RS, on which therefore O' must lie. Hence *the chord conjugate to PQ is RS.* This result may be stated as follows: *two chords of ϑ are conjugate when and only when the two pairs of points joined by the chords together form a canonical set of four points*; it is not necessary for the four points of the canonical set to be distinct.

In this connection Hesse's representation of the quadrics of the net by the points of a plane is helpful; the cones of the net are then represented by the points of a non-singular plane quartic δ in (1, 1) correspondence with ϑ. Corresponding to any two points P, Q of ϑ are two points p, q of δ; the line pq meets δ in two further points r, s, which correspond to two points R, S of ϑ; then RS is the chord conjugate to PQ. If, for example, a chord of ϑ is self-conjugate, its two intersections with ϑ must correspond to two points of δ that are points of contact of a bitangent and, conversely, such a chord is self-conjugate. We thus obtain the twenty-eight chords previously noticed.

10. The four cones belonging to the pencil of quadrics of the net that pass through a point P of ϑ consist of the cone (P) itself, counted twice, and of two other cones (A) and (B); the vertices A and B of these two cones correspond to the two points a and b in which the plane quartic δ is met by its tangent at p. We call the chord AB the *satellite chord* of P; *the chord conjugate to the tangent of ϑ at P is the satellite chord of P.* The scroll generated by the tangents of ϑ is therefore conjugate to the scroll generated by the satellite chords; we may speak of these scrolls as the *tangent scroll* and the *satellite scroll*; the former is developable, and both scrolls are combinantal covariants. Now the tangent scroll is of order $n = 16$ and has ϑ as a cuspidal curve, and so of multiplicity $i = 2$; hence the satellite scroll is of order $n' = 3n - 8i = 32$, and has ϑ as a curve of multiplicity $i' = n - 3i = 10$. We have then a tangent scroll S^{16} and a satellite scroll R^{32}, conjugate to one another. Among the generators of R^{32} are the tangents of ϑ at the fifty-six points corresponding to the points of contact of δ with its bitangents.

We can also identify those trisecants that are generators of these two scrolls. There are twenty-four trisecants which touch ϑ; these are generators of S^{16}. If a trisecant is a generator of R^{32}, then it must join two points of ϑ such that the canonical set determined by these two points includes some other point of ϑ twice over. But the canonical set determined by two points on a trisecant of ϑ is such that its remaining two points also lie on a trisecant*; it follows that there are twenty-four trisecants of ϑ among the generators of R^{32}, these being the trisecants conjugate to the points where ϑ is met by its own tangents.

If a point O lies on two tangents of ϑ, then the conjugate point O' must lie on the two satellite chords that are conjugate to these tangents. Hence *the nodal curve of S^{16} and the nodal curve of R^{32} are conjugate curves.* Known

* Edge, *Journal London Math. Soc.*, 11 (1936), 198.

formulae* show that the nodal curve of S^{16} is of order $\nu = 96$ and meets ϑ in $\iota = 96$ points; hence the nodal curve of R^{32} is of order $\nu' = 3\nu - \iota = 192$, while the number of its intersections with ϑ is $\iota' = 8\nu - 3\iota = 480$.

If a point O lies on three tangents of ϑ, then the conjugate point O' must lie on the three satellite chords that are conjugate to these tangents. Hence, since S^{16} is known to have 304 triple points, R^{32} also has 304 triple points.

11. The fact that a chord has two intersections with ϑ is equivalent to the fact that its conjugate chord meets R^8 in two points which are not on ϑ. Since a tangent is a chord whose two intersections with ϑ coincide, it follows that a satellite chord is such that the two points, apart from its intersections with ϑ, in which it meets R^8 coincide. Hence *every generator of* R^{32} *touches* R^8. Moreover, the chords which meet any trisecant t of ϑ are conjugate to the chords that pass through the point P of ϑ to which t is conjugate; since the only tangent among these chords is the tangent at P, it follows that t has only one intersection with R^{32} apart from its three intersections with ϑ. These three points absorb thirty of the intersections of t and R^{32}; the remaining two must coincide. Hence *every generator of* R^8 *touches* R^{32}. The scrolls R^8 and R^{32} therefore touch along a curve. Their complete intersection consists of ϑ counted thirty times, twenty-four trisecants, and their curve of contact counted twice; the order of this curve of contact is therefore $\frac{1}{2}(256-180-24) = 26$, and it is, like R^8 and R^{32}, of genus 3. The two scrolls are both inscribed in the developable, of class 14, generated by the planes which join the points of ϑ to their conjugate trisecants.

12. If AB is the satellite chord of P, the chords PA and PB are conjugate to one another; hence the scroll, generated by chords that join pairs of points of ϑ such that one point of the pair lies on the satellite chord of the other, is a self-conjugate surface.

The multiplicity of ϑ on this scroll is the number of generators that pass through P. Such a generator must either join P to one of the two intersections of ϑ with the satellite chord of P, or else must join P to a point of ϑ whose satellite chord passes through P. The former alternative gives

* Cayley, *Papers*, 8 (1871), 72–91; the order ν is Cayley's x. The 96 intersections of the nodal curve of S^{16} with ϑ consist of the 48 stalls of ϑ and of the 24 points where ϑ is met by its own tangents; these latter points are to be counted twice, as the nodal curve has cusps there: *cf.* Cayley, 84–85. Observe that a plane section of R^{32}, having six points of multiplicity 10 and 192 nodes, is of genus 3, as it ought to be.

the two generators PA and PB. Concerning the latter alternative, we remark that, if Q is such that its satellite chord passes through P, then the cone (P) passes through Q, and conversely. Hence, since (P) has two intersections with ϑ at P itself, there are ten points Q; or, alternatively, the fact that there are ten points Q follows from the fact that ten tangents can be drawn to a plane quartic δ from a point p of itself. It follows that ϑ is of multiplicity 12 on the scroll.

Since the scroll is self-conjugate it follows at once that it is of order 48, We thus have a self-conjugate scroll R^{48}, and it is clearly a combinantal covariant of the net of quadrics.

Among the generators of R^{48} are the tangents of ϑ at the twenty-four points, corresponding to the inflections of δ, where quadrics of the net have stationary contact.

There are also trisecants among the generators of R^{48}. It has been shown* that there are eight planes each of which contains three trisecants of ϑ; the trisecants EMN, FNL, GLM that lie in such a plane form a triangle LMN whose vertices lie on ϑ, the trisecants conjugate to the vertices being the opposite sides of the triangle. The satellite chords of L, M, N are FG, GE, EF respectively, so that the three trisecants are generators of R^{48}. The eight tritangent planes of R^8 each contain three generators of R^{48}.

The twenty-eight self-conjugate lines are also generators of R^{48}; for each of these lines has two intersections with ϑ, and the satellite chord of either of these two points is the tangent at the other.

13. R^{48} may be defined also as the locus of chords of ϑ that are generators of cones belonging to the net.

A line which is a generator of a quadric of the net is cut in involution by the quadrics, and is the join of a pair of conjugate points, namely the double points of the involution; such lines belong to a cubic complex. The chords of ϑ form a congruence of order 7 and class 15; those chords of ϑ that belong to the complex therefore generate a scroll of order $3(7+15) = 66$. This scroll must include R^{48}; it therefore consists of R^{48} and another scroll R^{18}; R^{18} may be defined as the locus of chords of ϑ that are generators of quadrics of the net which are *not* cones. The generators of R^{18} that pass through a point P of ϑ are the lines which join P to the intersections of ϑ with the trisecant conjugate to P; hence ϑ is a triple curve on R^{18}. The surface conjugate to R^{18} is therefore another scroll, R^{30}, generated by chords of ϑ, on which ϑ is a curve of multiplicity 9.

* Edge, *Proc. London Math. Soc.* (2), 34 (1932), 492–525 (513–514).

The twenty-four trisecants of ϑ that lie in the tritangent planes of R^8 are generators of R^{18} as well as of R^{48}. There are no tangents of ϑ among the generators of R^{18}. The forty-eight tangents of ϑ that belong to the cubic complex must therefore all be generators of R^{48}; they consist, however, of the tangents of ϑ at the twenty-four points where quadrics of the net have stationary contact, each counted twice.

14. The curve ϑ, being of genus 3, has seven apparent double points; through an arbitrary point O there pass seven chords of ϑ, and the seven chords that are conjugate to these must be the seven chords that pass through the point O' conjugate to O. If, then, two of the seven chords through O coincide, two of the seven chords through O' must also coincide; hence the locus of a point which is such that two of the seven chords of ϑ passing through it coincide is a self-conjugate locus.

The projection of ϑ from an arbitrary point is a plane sextic with seven nodes, but if two of the chords of ϑ passing through the point coincide, the sextic has five ordinary nodes and a tacnode. The point of projection then lies on a chord of ϑ which is such that the tangents of ϑ at its two intersections with the chord lie in a plane; such a chord has been called* a *torsal chord* of ϑ, and the projection of ϑ from *any* point of the torsal chord has a tacnode. The self-conjugate locus with which we are concerned is therefore the scroll generated by the torsal chords of ϑ; it is a developable surface, and called by Cayley† the *nexal torse* of ϑ. Incidentally, *if a chord of ϑ is torsal, its conjugate chord is also torsal.*

The order of the nexal torse is, from known formulae‡, 48; since it is a self-conjugate surface, it must have ϑ as a curve of multiplicity 12, one-quarter of the order of the surface. We have thus obtained a surface S^{48} that is self-conjugate, and it is also a combinantal covariant.

Among the generators of S^{48} are the tangents of ϑ at its forty-eight stalls, and also those twenty-four trisecants that are conjugate to the points where ϑ is touched by its trisecants. The forty-eight chords of ϑ which are conjugate to the tangents at the stalls are both satellite chords and torsal chords, and so are common generators of R^{32} and S^{48}.

If two torsal chords of ϑ intersect, then the two torsal chords conjugate to these also intersect, and their point of intersection is conjugate to the intersection of the first two. Hence the nodal curve of S^{48} is self-conjugate.

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* Baker, *Principles of geometry*, 6 (Cambridge, 1933), 17.

† *Loc. cit.*, 73.

‡ Cayley, *ibid.* The order of the nexal torse is Cayley's q'.

Further, if three torsal chords are concurrent, the three conjugate torsal chords must also be concurrent; the triple points of the nexal torse are therefore conjugate to one another in pairs.

15. In conclusion, we have obtained, apart from R^8, the following combinantal covariants of three quadric surfaces:

$$F^{16};\quad \begin{matrix} S^{16}; & R^{18}; \\ R^{32}; & R^{30}; \end{matrix}\quad R^{48};\quad S^{48}.$$

They consist of two conjugate pairs and of three self-conjugate covariants.

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ON THE SYLOW SYSTEMS OF A SOLUBLE GROUP†

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1. *Summary.* The present note is a continuation and in a sense the completion of two previous ones, "A note on soluble groups", *Journal London Math. Soc.*, 3 (1928), 98–105, and " A characteristic property of soluble groups", *ibid.*, 12 (1937), 198–200, which will be referred to as (I) and (II), respectively.

Let G be a group of order $p_1^{a_1} p_2^{a_2} \dots p^{a_r}$, where the p's are different primes. Two subgroups are said to be *complementary* in G if their *meet* (or common part) is 1 and their *product* is G. A subgroup of index $p_i^{a_i}$ in G will be called a *Sylow p_i-complement* of G. Thus, in the sense defined, every Sylow p_i-complement of G is complementary to every Sylow p_i-subgroup of G.

G always has at least one Sylow p_i-subgroup; and the main result of (II) was to show that G is a soluble group if and only if it also has (for each of the r primes p_i) at least one Sylow p_i-complement. Thus it is quite natural to base a structural theory of soluble groups on the properties of Sylow complements.

Let G be soluble; and let S_i be any arbitrarily chosen Sylow p_i-complement of G. Then the r subgroups

$$S_1, \ S_2, \ \dots, \ S_r \tag{1}$$

will be called a *complete set of Sylow complements* of G.

Σ being any subset of the set of r suffixes 1, 2, ..., r, let S_Σ be the meet of the corresponding subset of the set of subgroups (1). When Σ is the null set, we define S_Σ to be G.

† A preliminary notice of the main theorems of the present paper appeared in the note, "Solvable groups", *Bull. American Math. Soc.*, 42 (1936), 642–643.

The system of 2^r subgroups S_Σ so obtained will be called a *Sylow system* of G, or more precisely, the Sylow system determined by the complete set of Sylow complements (1).

The main object of the paper is to extend the three classical theorems about Sylow subgroups to the Sylow systems of a soluble group. Of these three results, the *existence theorem*, that every soluble group has at least one Sylow system, follows from what has already been said. There remain the *conjugacy theorem*: *any two Sylow systems of G are conjugate in G*; and the analogue of Speiser's Satz 68†: *any given Sylow system of a subgroup H of G consists of the meets of H with the members of a suitably chosen Sylow system of G.*

The arrangement is as follows. In § 2, the nature of the Sylow systems is described in greater detail, and it is shown how the new results form a natural generalization and completion of the "Sylowian" results first proved in (I), which have exactly the same relation to S-subgroups as the theorems now to be given have to Sylow systems. (An *S-subgroup is a* subgroup whose order and index are coprime.)

The two main theorems are proved in §§ 4 and 5; while § 3 is devoted to showing that *every system of permutable S-subgroups of a soluble group G belongs to some Sylow system of G.*

We may recall that two subgroups H and K are said to be *permutable* if their product HK (in the sense of Frobenius: that is, the set of all elements of the form PQ, with P in H and Q in K) is the same as the product KH in the opposite order. If this is the case, then HK is a subgroup, and is then, in fact, the *join* (or least subgroup containing both) of H and K. Conversely, if HK is a subgroup, then H and K are permutable.

If we denote the meet and join of H and K by M and J, then (using brackets to denote orders) we always have

$$(H)(K) \leqslant (J)(M).$$

When (J) is finite, the equality

$$(H)(K) = (J)(M)$$

is both necessary and sufficient for the permutability of H and K.

Finally, in § 6, we use the results outlined above to prove that the number of automorphisms of a soluble group G must always divide ϖq, where ϖ is the number of distinct Sylow systems of G, and q is the number of automorphisms of the direct product of the Sylow subgroups of G (which is a group of the same order as G itself, but having $\varpi = 1$).

† *Theorie der Gruppen von endlicher Ordnung*, (1927), 65.

2. *Sylow systems.* If the suffix set Σ is $i_1, i_2, \ldots, i_\rho$, then S_Σ is the meet of the ρ subgroups $S_{i_1}, S_{i_2}, \ldots, S_{i_\rho}$, whose indices in G are coprime in pairs. Hence the index of their meet S_Σ is equal to the product of the indices of the ρ Sylow complements S_{i_λ}. Thus the members S_Σ of a Sylow system are all S-subgroups of G, and a given Sylow system contains exactly one member from each of the 2^r distinct classes of conjugate S-subgroups which occur in G.

If Σ' is any other suffix set, let $\Sigma \vee \Sigma'$ denote the sum set of Σ and Σ', and let $\Sigma \wedge \Sigma'$ denote their common part. Then we have

$$S_{\Sigma \vee \Sigma'} = S_\Sigma \wedge S_{\Sigma'}, \tag{1}$$

where $H \wedge K$ is the meet of the subgroups H and K. This is obvious from the definition of the S_Σ. But also

$$S_{\Sigma \wedge \Sigma'} = S_\Sigma S_{\Sigma'}. \tag{2}$$

For if we use brackets to denote orders, then the product $S_\Sigma S_{\Sigma'}$ contains exactly $(S_\Sigma)(S_{\Sigma'})/(S_\Sigma \wedge S_{\Sigma'})$ distinct elements of G; and by (1) and what we have seen about the indices of the S_Σ, this fraction is equal to the order of $S_{\Sigma \wedge \Sigma'}$. Since both S_Σ and $S_{\Sigma'}$, and hence their product too, belong to $S_{\Sigma \wedge \Sigma'}$, (2) follows.

Thus any two members S_Σ and $S_{\Sigma'}$ of a Sylow system are permutable; and their product (or, what is the same since they are permutable, their join) also belongs to the system. In other words, a Sylow system is a subgroup lattice, that is, a system of subgroups closed with respect to the two operations "meet" and "join". Moreover, we have a 1-1 correspondence $S_\Sigma \longleftrightarrow \Sigma$ between the members of the Sylow system and the set of all subsets of a set of r things, and this correspondence merely interchanges the two operations $\vee$ and $\wedge$. In the usual terminology, therefore, a Sylow system is a Boolean lattice.

Summing up, we may describe *a Sylow system of G as a Boolean lattice of permutable S-subgroups of G, which contains just one member from each of its 2^r classes of conjugate S-subgroups.*

The converse is evident, and it is clear that we might equally well have defined a Sylow system in terms of Sylow subgroups rather than Sylow complements. Let S_i' be a Sylow p_i-subgroup of G, and let

$$S_i' S_j' = S_j' S_i' \quad (i, j = 1, 2, \ldots, r). \tag{3}$$

Then the r subgroups

$$S_1', S_2', \ldots, S_r' \tag{4}$$

will be called *a complete set of permutable Sylow subgroups of G.*

The 2^r products of the subgroups (4) obviously form a Sylow system of G; and since every Sylow system, as we have seen, contains a complete set of permutable Sylow subgroups, every Sylow system can be obtained in this way.

The hypothesis (3), however, is essential. For two Sylow subgroups of different orders (unlike Sylow complements) are not necessarily permutable except in the trivial case $r = 2$. For this reason the definition actually given in terms of Sylow complements seemed preferable.

In any case, the existence in a group G of a complete set of permutable Sylow subgroups is a necessary and sufficient condition for the solubility of G, for it is equivalent to the existence of a complete set of Sylow complements.

We shall need to make appeal to two theorems first proved in (I), viz.

(2.2) *any two S-subgroups of the same order of a soluble group G are conjugate in G*; and

(2.3) *any subgroup of G whose order divides the order m of an S-subgroup of G is contained in some S-subgroup of order m.*

An equivalent form of (2.3), only apparently more general, is

(2.3)′ *any S-subgroup of a subgroup H of G is the meet of H with a suitably chosen S-subgroup of G, whose order involves exactly the same primes.*

These are results of the same Sylowian type as the two main theorems of the present paper; and from what has been said above, it is clear that the latter are a natural generalization of the former, stating as they do the corresponding facts for a Sylow system, in place of a single S-subgroup.

We may remark that a Sylow system can also be defined as *a maximal set of permutable S-subgroups*, maximal in the sense that no further S-subgroup can be adjoined to the system without its losing the property of permutability. This is a consequence of the theorem of inclusion to be proved in § 3, together with the fact that two distinct S-subgroups of the same order cannot be permutable.

3. *Inclusion theorem.* Let $\mathfrak{T}$ be any set of permutable S-subgroups of the soluble group G. We have to show that $\mathfrak{T}$ can be included in some Sylow system $\mathfrak{S}$ of G.

Let J_i be the join of all those members of $\mathfrak{T}$ whose orders are prime to p_i, and if there are none of these let $J_i = 1$. In any case, since the members

of $\mathfrak{T}$ are all permutable, the order of J_i is also prime to p_i. By Theorem (2.3), we can therefore choose, for each i, a Sylow p_i-complement S_i of G containing J_i. Let $\mathfrak{S}$ be the Sylow system of G determined by the complete set of Sylow complements $S_1, S_2, \ldots, S_r$.

Let T be any member of $\mathfrak{T}$, and let its index in G involve the primes $p_{i_1}, p_{i_2}, \ldots, p_{i_\rho}$ and no others. Writing Σ for the suffix set $i_1, i_2, \ldots, i_\rho$, we show that $T = S_\Sigma$.

For T is an S-subgroup of G, so that its order is prime to p_{i_λ}. Hence T is contained in J_{i_λ} and therefore also in S_{i_λ} for each $\lambda = 1, 2, \ldots, \rho$. Thus T is contained in S_Σ. But it has the same order as S_Σ, and so coincides with it. This concludes the proof.

4. *Conjugacy theorem.* Let $\mathfrak{S}$ and $\mathfrak{S}^*$ be any two Sylow systems of G. We suppose them defined, respectively, by the two complete sets of Sylow complements

$$S_1, S_2, \ldots, S_r \tag{1}$$

and

$$S_1^*, S_2^*, \ldots, S_r^*, \tag{2}$$

where S_i and S_i^* are both of index $p_i^{a_i}$ in G.

Let the two sets (1) and (2) have just ρ terms in common. If $\rho = r$, then $\mathfrak{S} = \mathfrak{S}^*$ and there is nothing to prove. So let $\rho < r$. Then we show first that there exists in G an element Q with the property that the two Sylow systems $Q^{-1}\mathfrak{S}Q$ and $\mathfrak{S}^*$ have just $\rho+1$ Sylow complements in common.

For let $S_i \neq S_i^*$. And let S_i' be the Sylow p_i-subgroup of the system $\mathfrak{S}$. Then $S_i S_i' = G$, and so we obtain all the conjugates of S_i in G by transforming it with suitably chosen elements from S_i'. In particular, since S_i^* is conjugate to S_i by Theorem (2.2), we can choose an element Q from S_i' such that $Q^{-1} S_i Q = S_i^*$. Now S_i' is the meet of the $r-1$ terms of (1) other than S_i. Hence Q is contained in S_j for all $j \neq i$; so that, for all such j, we have $Q^{-1} S_j Q = S_j$. This proves the result that $Q^{-1}\mathfrak{S}Q$ and $\mathfrak{S}^*$ have just $\rho+1$ Sylow complements in common, namely, the ρ terms common to (1) and (2), together with S_i^*.

Repeating this argument $r-\rho$ times, we see that there exists in G an element P with the property that $P^{-1}\mathfrak{S}P$ and $\mathfrak{S}^*$ have r Sylow complements in common; in other words, we must have

$$P^{-1} S_i P = S_i^*$$

for all i. But this implies that

$$P^{-1} S_\Sigma P = S_\Sigma{}^*$$

for every suffix set Σ; or, symbolically,

$$P^{-1} \mathfrak{S} P = \mathfrak{S}^*,$$

the result required.

5. *The Sylow systems of a subgroup.* Let H be any subgroup of the soluble group G, and let $\mathfrak{T}$ be any Sylow system of H.

Of the r primes p_i which divide the order of G, we may suppose that only the first s enter into the order of H. (s can be equal to r.) Then let

$$T_1, T_2, \ldots, T_s \tag{1}$$

be the Sylow complements of H contained in $\mathfrak{T}$, T_i being of index a power of p_i in H.

By Theorem (2.3), for each $i \leqslant s$, we can choose a Sylow p_i-complement S_i of G containing T_i; and, for each $i > s$, we can choose such an S_i containing H. Then the complete set of Sylow complements of G so chosen defines a Sylow system $\mathfrak{S}$ of G.

Let Σ_1 be any subset of 1, 2, ..., s; and let Σ_2 be any subset of $s+1$, $s+2$, ..., r. Then we shall show that, in our previous notation,

$$H \wedge S_{\Sigma_1 \vee \Sigma_2} = T_{\Sigma_1}. \tag{3}$$

For the left of (3) certainly contains T_{Σ_1}, by definition of the S_i. On the other hand, its order divides both (H) and $(S_{\Sigma_1 \vee \Sigma_2})$; and hence cannot exceed (T_{Σ_1}), which is the greatest common divisor of (H) and $(S_{\Sigma_1 \vee \Sigma_2})$. This proves (3), and the subgroup theorem follows.

6. *Automorphisms.* Let A be the group of automorphisms of the soluble group G, and I its group of inner automorphisms. If $\mathfrak{S}$ is any Sylow system of G, let $A_\mathfrak{S}$ and $I_\mathfrak{S}$ denote the greatest subgroups of A and I, respectively, whose elements all leave $\mathfrak{S}$ invariant. Then, since any two Sylow systems are conjugate, we have

$$A = I A_\mathfrak{S}. \tag{1}$$

Also, obviously,

$$I_\mathfrak{S} = I \wedge A_\mathfrak{S}. \tag{2}$$

Let θ be any element of $A_{\mathfrak{S}}^{\mathfrak{H}}$, and let S_1', S_2', ..., S_r' be the complete set of permutable Sylow subgroups of G which belongs to $\mathfrak{S}$. If θ_i denotes the automorphism of S_i' which is obtained on transforming it by θ, then θ itself is completely determined by its r "components" $\theta_1, \theta_2, \ldots, \theta_r$; for every element of G can be expressed as a product of the form $P_1 P_2 \ldots P_r$, with P_i in S_i'.

Hence $A_{\mathfrak{S}}$ *is isomorphic with a subgroup of the direct product*

$$A_1 \times A_2 \times \ldots \times A_r,$$

where A_i is the group of automorphisms of a Sylow p_i-subgroup of G.

From the index relation $(A : A_{\mathfrak{S}}) = (I : I_{\mathfrak{S}}) = \varpi$, we deduce that *the order of A divides*

$$\varpi a_1 a_2 \ldots a_r, \tag{3}$$

where a_i is the order of A_i, and ϖ is the number of distinct Sylow systems of G.

It is well known that, if the order of S_i' is $p_i^{\alpha_i}$, then a_i divides the number

$$\phi(p_i^{\alpha_i}) = (p_i^{\alpha_i} - 1)(p_i^{\alpha_i} - p) \ldots (p_i^{\alpha_i} - p_i^{\alpha_i - 1}).$$

Also ϖ divides the order of G. Writing

$$g = p_1^{\alpha_1} \ldots p_r^{\alpha_r},$$

and

$$\phi(g) = \prod_{i=1}^{r} \phi(p_i^{\alpha_i}),$$

we obtain the result† that the number of automorphisms of a soluble group of order g divides $g\phi(g)$.

According to the amount of information available about the Sylow subgroups of G, this result may be improved in the direction of (3). For example‡, if it is known that S_i' can be generated by d elements, the factor $\phi(p_i^{\alpha_i})$ may be replaced by

$$\phi(p^d) p^{d(\alpha_i - d)}.$$

On the other hand, for an improvement of (3) itself, further information about the relations between Sylow subgroups of different orders is necessary. For instance, if the order of the group of automorphisms of an

† Obtained by another method in the paper by Garrett Birkhoff and Philip Hall, "On the order of groups of automorphisms", *Trans. Amer. Math. Soc.*, 39 (1936), 496.

‡ P. Hall, "A contribution to the theory of groups of prime-power order", *Proc. London Math. Soc.* (2), 36 (1933), 37.

S-subgroup,

$$S_\Sigma' = \prod_{i \text{ in } \Sigma} S_i',$$

is known to be a_Σ, we may replace the factors $\prod_{i \text{ in } \Sigma} a_i$ of (3) by a_Σ/ϖ', where ϖ' is the number of Sylow systems of S_Σ'.

We conclude by remarking that *if the order t of an automorphism θ of G is prime to the order of G, then θ must leave invariant at least one Sylow system of G.* For let J be the join of θ and G (regarded as a subgroup of the holomorph of G, for example). Then J is soluble, and $\{\theta\}$ is an S-subgroup of J. But if N is the normaliser in J of any Sylow system of G, the conjugacy theorem gives $NG = J$. Thus t divides the order of N, and N must contain an S-subgroup of order t. This is clearly an S-subgroup of J, and hence is conjugate to $\{\theta\}$; so that θ must belong to one of the conjugates of N, and thus leaves invariant some Sylow system of G.

This argument applies equally well if θ is replaced by a group of automorphisms T whose order is prime to that of G, provided that T is soluble.

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SOME REMARKABLE APPROXIMATE FORMULAE ARISING IN BALLISTICS

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1. Consider the vertical trajectory of a particle in a medium whose resistance varies as the n-th power of the velocity. Let V be the initial velocity (upwards), and let

$$kg\left(\frac{v}{V}\right)^n$$

be the retardation due to the medium when the velocity is v. A large k means a large resistance; the value $k=1$ corresponds to equality of the retardations due to gravity and initial resistance. Let T be the total time of flight (up and down), H the maximum height, and $\bar{H}$ the mean height

$$\bar{H}=\frac{1}{T}\int_0^T y\,dt.$$

Let

$$\rho=\rho(k)=\rho_n(k)=\frac{H}{\frac{1}{8}gT^2},\quad \sigma=\sigma(k)=\sigma_n(k)=\frac{\bar{H}}{\frac{2}{3}H}.$$

On grounds of dynamical similarity these numbers are functions of n and k only. When $k=0$, corresponding to motion *in vacuo*, $\rho=\sigma=1$.

2. When $n=2$ there is a natural transformation $k=\tan^2\theta$, and we have the explicit formulae

$$\rho(\theta)=\rho_2(k)=\frac{8\log\sec\theta}{\{\theta+\log\tan(\frac{1}{2}\theta+\frac{1}{4}\pi)\}^2}, \tag{1}$$

$$\sigma(\theta)=\sigma_2(k)=\tfrac{3}{2}-\tfrac{3}{2}\,\frac{\int_0^\theta(1+\sec t)\log\sec t\,dt}{\{\theta+\log\tan(\frac{1}{2}\theta+\frac{1}{4}\pi)\}\log\sec\theta}. \tag{2}$$

It has often been remarked that for the quadratic law the relation $H = \frac{1}{8}gT^2$ remains approximately true up to fairly large values of k; in other words, the function $\rho(\theta)$ of (1) remains sensibly equal to 1 for a considerable range of θ. We have, for example‡,

$$\theta = 26^\circ 30' \quad 45^\circ \quad 78^\circ 40'$$

$$\rho(\theta) = \cdot 9998 \quad \cdot 998 \quad \cdot 96.$$

For $\bar{H}$ the approximation $\sigma = 1$ is even better, by a factor of about 3§.

When $n \leqslant 2$ the vertical trajectory subject to a resistance μv^n becomes infinite when the initial velocity does (μ being kept fixed); when $n > 2$ the trajectory remains finite; when $n = 2$ the infinity is "logarithmic" and all the formulae are bound up with logarithms. It is therefore natural to suppose that $n = 2$ is a critical value of n. But the results for $n = 3$‖ turn out to be at least as good.

Few mathematicians can resist striking numerical approximations, even in a trivial context, and at this point one's curiosity becomes seriously aroused. The present paper describes some of the results which emerge when we indulge our curiosity to the end.

3. The first thing to explore is evidently the expansion (for general n) of ρ and σ in ascending powers of k: the expansions are certainly valid for small k. The results of expanding as far as k^3 are:

$$\rho = 1 + A_n k^2 - \frac{2n}{n+2} A_n k^3 + \ldots,$$

$$\sigma = 1 + \frac{n}{2n+3} A_n k^2 - \frac{2n}{n+2} \frac{n}{2n+3} A_n k^3 + \ldots,$$

where

$$A_n = \frac{n^2 - n - 3}{(n+1)(n+2)^2(2n+1)}.$$

This reveals, first, that ρ and σ possess a common critical value of n, given by

$$n = \nu = \tfrac{1}{2}\{\sqrt{(13)} + 1\} = 2{\cdot}30277\ 56377\ldots,$$

‡ For fuller details see Table II.

§ See Table II.

‖ For which the formulae are fairly explicit.

the positive root of

$$\nu^2-\nu-3=0,$$

and that for this critical value the expansions of ρ and σ have no term in k below k^4; secondly, that (by an apparent coincidence) the linear combination

$$\tau(k)=\tau_n(k)=\frac{2n+3}{n+3}\sigma-\frac{n}{n+3}\rho$$

has this last property *whatever the value of* n.

For $1-\rho_\nu$ and $1-\sigma_\nu$ we have the following numerical values.

TABLE I.

k	$\frac{1}{4}$	1	10	10^6	∞
$1-\rho_\nu(k)$	$8\cdot4\times10^{-8}$	$7\cdot3\times10^{-6}$	$7\cdot4\times10^{-4}$	$5\cdot1\times10^{-2}$	$8\cdot4\times10^{-2}$
$1-\sigma_\nu(k)$	$3\cdot2\times10^{-8}$	$2\cdot7\times10^{-6}$	$2\cdot7\times10^{-4}$	$1\cdot8\times10^{-2}$	$3\cdot0\times10^{-2}$

Their behaviour is seen to be of a different order from that of ρ_2 and σ_2. The function τ_n may be expected to show comparable behaviour for any reasonable n.

The following further developments for values of n other than 2 may be noted. For $\tau_n(k)$ there are two critical values of n, the positive roots of a sextic equation; for each of them the coefficient of k^4, and with it that of k^5, vanishes. One of these values is numerically 2·03501 4573 ..., the nearness of which to 2 partly explains the remarkable behaviour we shall find in $\tau_2(k)$. For $n=\nu$ the linear combination τ is not the critical one; there is another, $\omega(k)$ say, for which the coefficients of k^4 and k^5 vanish. The behaviour of this $\omega(k)$ and of the critical $\tau_n(k)$ is more extreme than anything recorded in the present paper‡. The numerical calculations dealing with these questions are incomplete, and a statement of the full results must be postponed.

In what follows we shall be exclusively concerned with the case $n=2$, which presents some features peculiar to itself.

‡ For example $\omega(k)=1+\gamma_3k^6+\ldots$, where $\gamma_3=3\cdot2\times10^{-8}$.

4. The numerical behaviour of $\tau(\theta) = \tau_2(k)$ is given in the following table, along with that of $\rho(\theta)$ and $\sigma(\theta)$, and may be left to speak for itself‡.

TABLE II.

k	θ	$1-\rho_2(k)$	$1-\sigma_2(k)$	$\tau_2(k)-1$
10^{-2}	$5°\ 42\frac{1}{2}'$	$4{\cdot}11\times10^{-7}$	$1{\cdot}17\times10^{-7}$	$6{\cdot}13\times10^{-15}$
$\frac{1}{5}$	$24°\ 5\frac{3}{4}'$	$1{\cdot}40\times10^{-4}$	$3{\cdot}96\times10^{-5}$	$6{\cdot}97\times10^{-10}$
$\frac{1}{2}$	$35°\ 16'$	$6{\cdot}84\times10^{-4}$	$1{\cdot}96\times10^{-4}$	$1{\cdot}70\times10^{-8}$
1	$45°$	$1{\cdot}99\times10^{-3}$	$5{\cdot}69\times10^{-4}$	$1{\cdot}45\times10^{-7}$
2	$54°\ 44'$	$4{\cdot}98\times10^{-3}$	$1{\cdot}42\times10^{-3}$	$9{\cdot}00\times10^{-7}$
5	$65°\ 54\frac{1}{2}'$	$1{\cdot}30\times10^{-2}$	$3{\cdot}71\times10^{-3}$	$6{\cdot}16\times10^{-6}$
25	$78°\ 41\frac{1}{2}'$	$4{\cdot}07\times10^{-2}$	$1{\cdot}16\times10^{-2}$	$5{\cdot}83\times10^{-5}$
10^2	$84°\ 17\frac{1}{2}'$	$7{\cdot}58\times10^{-2}$	$2{\cdot}15\times10^{-2}$	$2{\cdot}23\times10^{-4}$
10^4	$89°\ 25\frac{1}{2}'$	$2{\cdot}17\times10^{-1}$	$6{\cdot}07\times10^{-2}$	$1{\cdot}78\times10^{-3}$
10^8	$89°\ 59\frac{2}{3}'$	$4{\cdot}40\times10^{-1}$	$1{\cdot}20\times10^{-1}$	$8{\cdot}29\times10^{-3}$
10^{40}		$8{\cdot}42\times10^{-1}$	$2{\cdot}16\times10^{-1}$	$3{\cdot}48\times10^{-2}$
∞	$90°$	1	$\frac{1}{4}$	$\frac{1}{20}$

5. The expansions of $\rho(\theta)$, $\sigma(\theta)$ in powers of θ begin

$$\rho = 1-\tfrac{1}{240}\theta^4+\ldots, \quad \sigma = 1-\tfrac{1}{840}\theta^4+\ldots. \tag{3}$$

‡ The reader may be reminded that ρ and σ are the ratios of important ballistic quantities to drastically simple approximations for them; also that the numerators and denominators in the trigonometrical expressions (1) and (2) increase from 0 to ∞ as θ increases from 0 to π. τ is, of course, an artificial construction, but it is no worse than a linear combination of ρ and σ.

For $\tau(=\frac{7}{5}\sigma-\frac{2}{5}\rho)$ we have‡

$$\tau = 1+\tfrac{1}{1584000}\theta^8+\dots. \tag{4}$$

The right variable, however, to adopt in connection with ρ, σ, and τ is not θ or $\tan\theta$, but $x=\tan\frac{1}{2}\theta$. When we write $\rho(x)$, etc., instead of $\rho(\theta)$, etc., the formulae take the following shape. Let

$$P(x)=\tfrac{1}{2}\log\frac{1+x^2}{1-x^2}=x^2+\frac{x^6}{3}+\frac{x^{10}}{5}+\dots,$$

$$Q(x)=\tfrac{1}{2}\arctan x+\tfrac{1}{4}\log\frac{1+x}{1-x}=x+\frac{x^5}{5}+\frac{x^9}{9}+\dots,$$

$$R(x)=\tfrac{3}{2}\int_0^x\frac{1}{1-\xi^4}\log\frac{1+\xi^2}{1-\xi^2}\,d\xi=3\left(\frac{x^3}{3}+(1+\tfrac{1}{3})\frac{x^7}{7}+(1+\tfrac{1}{3}+\tfrac{1}{5})\frac{x^{11}}{11}+\dots\right).$$

Then

$$1-\rho(x)=\frac{Q^2-P}{Q^2},\quad 1-\sigma(x)=\frac{R-PQ}{2PQ},\quad \tau(x)-1=\frac{11PQ^2-7QR-4P^2}{10PQ^2},$$

and the expansions of $1-\rho$, $1-\sigma$, $\tau-1$ proceed in powers of x^4, or y, say§:

$$1-\rho(x)=a_1y+a_2y^2+a_3y^3+\dots,$$

$$1-\sigma(x)=b_1y+b_2y^2+b_3y^3+\dots,$$

$$\tau(x)-1=c_2y^2+c_3y^3+\dots.$$

The expressions in terms of P, Q, R give complete insight into the behaviour of ρ and σ. If we write

$$p=\frac{P}{x^2}=1+\frac{y}{3}+\frac{y^2}{5}+\dots,\quad q=\frac{Q}{x}=1+\frac{y}{5}+\frac{y^2}{9}+\dots,$$

we have, *e.g.*,

$$1-\rho=\frac{q^2-p}{q^2}.$$

Now the denominator q^2 increases with x, but only very slowly; and if we consider, *e.g.*, the range $0\leqslant x\leqslant\frac{3}{4}$, q^2 lies between 1 and 1·17, and $1-\rho$ and q^2-p do not differ appreciably; in any case the latter is the smaller. Now

$$q^2-p=(\tfrac{2}{5}-\tfrac{1}{3})x^4+\dots,$$

and the first term is easily seen to dominate the series. The "smallness" of q^2-p, such as it is, depends firstly on structure, in that it is a series in powers of x^4; secondly on the fact that $\frac{2}{5}-\frac{1}{3}$ is the reciprocal of the number

‡ This follows from Table IV below: a direct calculation would be very heavy.

§ The radius of convergence is 1 in each case, and the range (0, 1) of x or y corresponds to the complete range $(0, \infty)$ of k.

15, which is moderately large: when $k = 1$ the reciprocal of x^4 is 34, and 34 multiplied by 15 is 510, which is felt as large in the context. The behaviour of σ is similarly transparent, but the same cannot be said of τ.

It is interesting to note, in passing, that since pq^2 does not vary by more than a factor 1·32 over the range $0 \leqslant x \leqslant \frac{3}{4}$ (say), the smallness of $\tau - 1$ involves the smallness, over this range, of the combination

$$T(x) = \tfrac{11}{10} PQ^2 - \tfrac{7}{10} QR - \tfrac{4}{10} P^2$$

of the three more or less "logarithmic" series P, Q, R. We have, in fact,

TABLE III.

x	$\frac{1}{10}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{9}{10}$	$\frac{99}{100}$	1
$T(x)$	$1{\cdot}6 \times 10^{-16}$	$9{\cdot}9 \times 10^{-12}$	$4{\cdot}0 \times 10^{-8}$	$1{\cdot}6 \times 10^{-5}$	$3{\cdot}0 \times 10^{-4}$	$1{\cdot}2 \times 10^{-2}$	∞

The early coefficients a, b, c are:

TABLE IV.

n	1	2	3	4	5
a_n	$\frac{1}{15}$	$\frac{8}{225}$	·0237	·0176	·0139
b_n	$\frac{2}{105}$	·0100	·0067	·0049	·0039
c_n	0	$\frac{2}{12375}$	·000170	·000160	·000157

These figures suggest that all the a's, b's, and c's are positive and that the a's and b's are decreasing. This conjecture for τ (at any rate) is a new surprise, and another challenge difficult to resist. The results are, in point of fact, true, and are probably the most interesting feature of the quadratic case. Before we proceed to the proof, one or two remarks are perhaps called for.

(i) If the coefficients a and b do not strike one as being particularly small, and if (4) seems sadly diminished, the reason is that (*e.g.*) y^2 is, for small k, not approximately k^4 or θ^8, but $\frac{1}{25}\frac{1}{6}\theta^8$, and (*e.g.*) for $k = 1$, y^2 is as small as ·0009 (y^2/k^4 decreases as k increases).

(ii) The smallness of the c_n (say) is not in the least due to c_n being a high order infinitesimal for large n: the coincidences producing it are essentially *numerical*. It is easily seen, in fact, that $c_n \sim An^{-1}(\log n)^{-2}$, where

$$A = \frac{16}{35} - \frac{3}{5}\left(\frac{1}{1\cdot 2} + \frac{1}{5\cdot 6} + \frac{1}{9\cdot 10} + \ldots\right) = \cdot 12\ldots.$$

We can prove similarly that not only the a_n and b_n, but also the c_n, are *ultimately* decreasing (as well as positive).

(iii) If c_n is proved positive for $n > n_0$, the question of its being always positive would be settled by computation of the first n_0 values. But any straightforward argument would lead to an n_0 quite impossibly large for the purpose. The whole question, such as it is, is one of a *numerical* state of things.

(iv) Since the c_n's are positive and $\tau = \frac{2}{2}\frac{1}{0}$ when $x = 1$, it follows that

$$|\tau(x) - 1| \leqslant \tfrac{1}{2}\tfrac{1}{0}$$

for all x satisfying $|x| \leqslant 1$, real or complex. If we write $x = e^{\frac{1}{2}\pi i}\xi$, $\tau(x) = \tau^*(\xi)$, we have

$$\tau^*(\xi) = 1 + c_2\xi^8 - c_3\xi^{12} + c_4\xi^{16}\ldots.$$

$\tau^*(\xi)$ and $\tau(\xi)$ are much the same for positive ξ not too near $\xi = 1$. For $\xi = 1$, however, we find that

$$\tau^*(1) = 1\cdot 00011.$$

We have also (with the obvious notation)

$$\rho^*(1) = 1\cdot 045, \quad \sigma^*(1) = 1\cdot 018.$$

6. We come now to the proof that the a's, b's, and c's are positive, and the a's and b's decreasing. To prove that a series (in $y = x^4$)

$$f(y) = \sum_1^\infty \lambda_n y^n$$

has decreasing coefficients is equivalent to proving that either

$$\left(1 - \frac{1}{y}\right) f(y) + \lambda_1 = \sum_1^\infty (\lambda_n - \lambda_{n+1}) y^n, \quad \text{or} \quad x\frac{d}{dx}\left(\left(1 - \frac{1}{y}\right) f(y)\right)$$

has positive coefficients. Subject to the appropriate behaviour of the "constants of integration", a series has positive coefficients if its first or higher derivative has. The detailed procedure tends to become fairly systematic with experience, but though the final arguments can be presented fairly briefly the details gave a good deal of trouble to arrive at.

7. *Notation.* Let us denote by $\mathcal{P}$ any series $A_0+A_1y+\ldots$ in y with non-negative coefficients, and by $\mathcal{P}_n$ a series whose coefficients from y^n onwards are non-negative. We write sometimes $f \succ g$ or $g \prec f$ for $f = g+\mathcal{P}$. A bracket followed by a * implies that the series for the contents of the bracket is a $\mathcal{P}$. A bracket followed by a † implies that the series for the contents of the bracket is a $\mathcal{P}_1$.

Differentiations are always with respect to x, not y, and d/dx will be denoted by D.

Let $S = Q^2$, and let

$$Q_1 = Q_1(x) = \int_0^x Q(\xi)\,d\xi, \qquad S_1 = S_1(x) = \int_0^x S(\xi)\,d\xi = \int_0^x Q^2(\xi)\,d\xi.$$

Let $G = G(x)$ denote the geometric series in x^4,

$$G = 1+y+y^2+\ldots = 1+x^4+x^8+\ldots = \frac{1}{1-y} = \frac{1}{1-x^4}.$$

We have the identities

$$DQ = G, \quad DP = 2xG, \quad DR = 3PG,$$

$$P = 2(Qx-Q_1), \quad R = 3(xQ^2-2QQ_1+S_1);$$

and with the help of these we can express our calculations, except for an occasional P (or p) in a denominator, in terms of Q and its associates Q_1 and S_1 $\left(=\int Q^2\,dx\right)$.

It is convenient also to use the notation $p = x^{-2}P$, $q = x^{-1}Q$, $q_1 = x^{-2}Q_1$, $s_1 = x^{-3}S_1$; the small letter functions are power series in y with constant terms 1, 1, $\frac{1}{2}$, $\frac{1}{3}$ respectively. We make considerable use of the constant

$$\kappa = Q_1(1) = q_1(1) = \frac{1}{1.2}+\frac{1}{5.6}+\frac{1}{9.10}+\ldots.$$

Its value lies between ·56 and ·57.

8. We begin with a number of subsidiary results. They are stated mostly in terms of small letter functions; such a result implies a corres-

sponding result for large letter functions, which need not be stated as well.

We have first a group of results which are immediately evident from the series, or are trivial in some other way.

(I) $\quad \frac{1}{2} \prec q_1 \prec \frac{1}{2}q, \quad \frac{1}{3} \prec s_1 \prec \frac{1}{3}q \prec \frac{1}{3}q^2, \quad \frac{1}{3}+\frac{1}{6}q-q_1 = \mathcal{P}.$

Next:

(II) *The series for*

$$q^n (n = 1, 2, 3, 4, 5),\ qq_1,\ pq,\ pq^2,\ pq^3,\ p^2,\ p^2q,\ 2q^2-3qq_1$$

have decreasing coefficients.

We select the more difficult cases, leaving one or two easier ones to the reader.

(i) We have $x^5 D\left((1-x^{-4})\,Q^n\right) = -nx\,Q^{n-1}+4Q^n.$

If $n \leqslant 4$ this is evidently a $\mathcal{P}$ (since $Q \succ x$).

If $n = 5$ it is $4(x+X)^5-5x(x+X)^4$, where $Q-x = X$; and this, expanded in x and X, is of the form $-x^5+x^{13}\mathcal{P}$. Thus in any case $x^5 D\left((1-y^{-1})\,Q^n\right)$ has at most one negative coefficient, and it follows (for slightly different reasons in different cases) that Q^n has decreasing coefficients (when $n \leqslant 5$).

(ii) For qq_1 we observe that

$$x^6 D\left((1-x^{-4})\,QQ_1\right) = xQ_1(4Q-x)+x^6(1-x^{-4})\,Q^2 = \mathcal{P}+\mathcal{P}_2 = \mathcal{P}_2,$$

in virtue of (i). Combining this knowledge with calculation of the early coefficients we find that $(1-x^{-4})\,xQQ_1$ has only its first coefficient negative, and the desired result follows.

(iii) We have (after rearrangement)

$$x^4 D\left((1-x^{-4})\,PQ^3\right) = -2x^4+x^2P+(Q-x)(6Px-6x^3)$$
$$+(Q-x)^2(9P-6x^2)+(Q-x)^3(4Px^{-1}-2x) = \mathcal{P}_2,$$

whence, by calculation, $pq^3(1-y^{-1}) = -1+y\mathcal{P}$.

The remaining cases are easier.

(III) $\quad \frac{G}{q^n}\,(n = 1, 2, 3, 4, 5), \quad \frac{G}{pq^n}\,(n = 1, 2, 3), \quad \frac{G}{p^2q}, \quad \frac{G}{qq_1}, \quad$ *are* $\mathcal{P}$'s.

Consider (*e.g.*) $G/(pq^3)$. By (II), $pq^3(1-x^4) = 1-y\mathcal{P}$. Hence

$$\frac{G}{pq^3} = \frac{1}{pq^3(1-x^4)} = \frac{1}{1-y\mathcal{P}} = \mathcal{P}.$$

The other expressions are $\mathcal{P}$'s for the same reason.

(IV) $$1-\frac{8q_1^3}{q}, \quad 2-\frac{1}{q_1}, \quad \tfrac{1}{2}-\frac{q_1}{q} \quad \textit{are} \quad \mathcal{P}'s\ddagger.$$

$$-D\left(\frac{Q_1^3}{Q}\right) = \frac{Q_1^3 G}{Q^2} - 3Q_1^2 \succ \tfrac{1}{8}xQ^3 - 3Q_1^2,$$

by (III). Hence

$$-xD^2\left(\frac{Q_1^3}{Q}\right) \succ \tfrac{3}{8}x^2Q^2G - 6xQQ_1 \succ x^4\left(\tfrac{3}{8}G - 6q(\tfrac{1}{3}+\tfrac{1}{6}q)\right) = \mathcal{P}_4.$$

Hence $-q_1^3/q = \mathcal{P}_3$, and the first result follows by calculation.

Next, $$x^3 D\left(-\frac{1}{Q_1}\right) = \frac{x^3 Q}{Q_1^2},$$

which is a $\mathcal{P}$, since

$$xD\left(\log\frac{x^3Q}{4Q_1^2}\right) = \frac{G}{qq_1}\left(q_1 + (3qq_1 - 2q^2)(1-y)\right)$$

is a $\mathcal{P}$ on account of (II) and (III). This gives the second result. The third is easier.

(V) We shall often use the following principle for a product of G by another factor.

If $f(y) \succ a_0+a_1y+\ldots+a_my^m$, *then*

$$Gf = (a_0+a_1+\ldots+a_m)\,G + \mathcal{P}_m.$$

In fact $Gf-(a_0+a_1+\ldots+a_m)\,G \succ -\sum_1^m a_r G(1-y^r)$, and this is a polynomial in y of degree $m-1$.

We use also the facts that, if $\kappa = q_1(1) = Q_1(1)$, then

$$(\kappa-q_1)\,G = \mathcal{P}, \quad \left(\frac{1}{q_1}-\frac{1}{\kappa}\right)G = \mathcal{P}.$$

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‡ Roughly speaking, things like q_1/q, which tend to 0 as $x\to1$, have expansions like $\frac{1}{2}-A_1y-A_2y^2-\ldots$; at any rate the coefficients are negative after the first term or two.

If $$q_1 = \sum_0^\infty a_m y^m, \quad 1/q_1 = 2 - \sum_1^\infty \beta_m y^m,$$

we have $$\Sigma a_m = \kappa, \quad \Sigma \beta_m = 2 - 1/\kappa.$$

Hence, $$(\kappa - q_1) G = \sum_0^\infty a_m G(1-y^m) = \Sigma a_m \mathcal{P} = \mathcal{P},$$

$$\left(\frac{1}{q_1} - \frac{1}{\kappa}\right) G = \sum_1^\infty \beta_m G(1-y^m) = \Sigma \beta_m \mathcal{P} = \mathcal{P},$$

the β's being positive, by (IV).

9. We now come to the main proofs‡.

(i) $$xD(1-\rho) = \frac{2Gx}{Q^3}(Qx - 2Q_1) = \frac{2G}{q^3}(q - 2q_1) = \mathcal{P},$$

by (III). Hence $1-\rho$ is a $\mathcal{P}$.

(ii) *Proof that* $1-\rho$ *has decreasing coefficients.*

$$x^5 D\{(1-x^{-4})(1-\rho)\} = 4\left(1 - \frac{P}{Q^2}\right) - \frac{2x}{Q^3}(Qx - 2Q_1)$$

$$= 4 - \frac{8x}{Q} + \frac{8Q_1 - 2x^2}{Q^2} + \frac{4Q_1 x}{Q^3} = J,$$

say. Also $$xDJ = \frac{4Gx^3}{Q^3} K,$$

where $$K = (2q - 3q_1 - q_1 x^4) + \left(1 - \frac{3q_1}{q}\right)\dagger.$$

By inspection, the first term is a $\mathcal{P}_2$. Hence K is a $\mathcal{P}_2$ and, by calculation, a $\mathcal{P}$. The result follows.

10. We have actually constructed a proof that $1-\sigma$ has decreasing coefficients, but it is more trouble than the result is worth, and we should need to set out a number of subsidiary results. We therefore suppress it, and proceed to the proof of the most interesting of the results, viz. that the coefficients of τ are positive.

‡ Since identities can be checked by rule of thumb, while the results of suppressing $\mathcal{P}$'s cannot, we work throughout with identities, arranging them to suit the arguments, but suppressing the proofs.

We have the following identities.

$$(1) \qquad D\left(x^2 D\left(\frac{P^2 Q}{xG} D\tau\right)\right) = \tfrac{2}{35} \frac{Gx^6}{Q^3} (x^4 G\Phi + \Psi),$$

$$(2) \quad \Phi = (32-42q_1)q^3 + 42s_1 q_1 q - 160q_1{}^2 q + 80q_1{}^2 + 128q_1{}^3 - 21q_1 s_1 - 96\frac{q_1{}^3}{q},$$

$$(3) \quad \Psi = \tfrac{21}{2}q^4 + (16-21q_1)q^3 + \tfrac{21}{2}s_1 q^2 - 80q_1 q^2 + 64q_1{}^2 q - 21q_1 s_1 + 80q_1{}^2 - 96\frac{q_1{}^3}{q},$$

$$(4) \qquad \frac{1}{x^4} D(x^5 \Phi) = EG + F,$$

$$(5) \qquad E = (96-126q_1)q^2 - 160q_1{}^2 + 42s_1 q_1,$$

$$(6) \quad F = -42q^4 - \tfrac{175}{6}q^2 + \left(\tfrac{169}{6}(q-1)^3 + \tfrac{169}{6} + \tfrac{752}{3}q_1{}^2(q-1) + 42q^3(q_1 - \tfrac{1}{2})\right)^* + q(160q_1 + 112q_1{}^2 - \tfrac{169}{2})^* + 21s_1 q(2q - 2q_1 - 1)^* + (341q^2 + 128q_1{}^2)(\tfrac{1}{3} + \tfrac{1}{6}q - q_1)^* + 96(q_1{}^3 q^{-2} G)^*.$$

Now

$$(7) \quad D(x^4 E) = ax^2 QG - 126Q^3 - 320Q_1 Q + \{D(42x^4 s_1 q_1)\}^* + 252Q\{(\kappa x^2 - Q_1)G\}^*,$$

where $\kappa = Q_1(1)$ and $a = 192 - 252\kappa$ [the last term on the right-hand side of (7) being a * in virtue of (V)]. Rejecting *'s and observing that $a > 48$, we find‡ that $D(x^4 E) = x^3 \mathcal{P}_2$ and $E = \mathcal{P}_3$. By calculation,

$$E \succ (5{\cdot}3)y + (4{\cdot}3)y^2 + (3{\cdot}8)y^3 + (3{\cdot}3)y^4 + (2{\cdot}8)y^5 + (2{\cdot}5)y^6 + (2{\cdot}2)y^7,$$

and so, by (V),

$$GE = 23G + \mathcal{P}_7.$$

Hence, from (6),

$$GE + F = 23G - 42q^4 - 30q^2 + \mathcal{P}_7 = \mathcal{P}_7.$$

Hence Φ is a $\mathcal{P}_7$; calculation gives

$$(8) \quad y\Phi = -\tfrac{3}{2}y - \tfrac{1}{30}y^2 + (\cdot 32)y^3 + (\cdot 48)y^4 + (\cdot 56)y^5 + (\cdot 59)y^6 + (\cdot 60)y^7 + (\cdot 61)y^8 + \mathcal{P},$$

‡ The coefficients of Q^3 and $Q_1 Q$ being decreasing.

and so

$$yG\Phi = -\tfrac{3}{2}y-(1{\cdot}54)\,y^2-(1{\cdot}22)\,y^3-(\cdot74)\,y^4-(\cdot18)\,y^5+\mathcal{P}. \tag{9}$$

We have next

$$\frac{1}{x^3}D(x^4\Psi) = HG+K, \tag{10}$$

where

$$H = 42q^3+(48-63q_1)\,q^2+21s_1q-160q_1q+64{q_1}^2, \tag{11}$$

$$\begin{aligned} K = {} & -\tfrac{21}{2}q^4-\tfrac{529}{9}q^3+[\tfrac{709}{18}q^2+\tfrac{656}{9}q+7q_1]^* \\ & +(21q^3+107q^2+160q)(q_1-\tfrac{1}{2})^*+\tfrac{7}{2}(q^2+2q)(q-3s_1)^* \\ & +7q_1(3s_1-1)^*+64q\{(\tfrac{1}{3}+\tfrac{1}{6}q)^2-{q_1}^2\}^*+\{-96x^{-3}D(x^4{q_1}^3q^{-1})\}\dagger. \end{aligned} \tag{12}$$

Substituting from $q_1 \succ \frac{1}{3}+\frac{1}{6}q$ in the negative terms of H, we have crudely $H \succ -\frac{63}{2}q^3-\frac{160}{3}q$, which is easily seen to be a $\mathcal{P}_1$. By calculation,

$$H \succ \tfrac{3}{2}+13y+12y^2,$$

and so

$$HG = 25G+\mathcal{P}_2.$$

Hence, from (12) and (10),

$$HG+K = 25G-\tfrac{21}{2}q^4-\tfrac{529}{9}q^3+\mathcal{P}_2 = \mathcal{P}_4,$$

$$\Psi = \mathcal{P}_4.$$

Calculation now gives

$$\Psi = \tfrac{3}{2}y+(1{\cdot}9)\,y^2+(2{\cdot}3)\,y^3+(2{\cdot}3)\,y^4+(2{\cdot}3)\,y^5+\mathcal{P}. \tag{13}$$

By combination with (9), $yG\Phi+\Psi = \mathcal{S}$; hence the right side of (1) is a $\mathcal{P}$, and so successively (constants of integration being zero),

$$\frac{P^2Q}{G}(D\tau) = \mathcal{P}, \quad xD\tau = \left(\frac{x^5G}{P^2Q}\right)\mathcal{S} = \mathcal{P},$$

$$\tau = \mathcal{P}.$$

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NOTE ON A PAIR OF SIMULTANEOUS DIFFERENTIAL EQUATIONS INVOLVING TWO PARAMETERS

By R. A. Newing.

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The two simultaneous equations

$$\frac{d}{d\xi}\left\{(\xi^2-1)\frac{dX}{d\xi}\right\}+\left[-\Lambda\xi^2+2R\xi-\frac{m^2}{\xi^2-1}-\mu\right]X=0 \quad (1\leqslant\xi\leqslant\infty), \qquad \text{(I)}$$

$$\frac{d}{d\eta}\left\{(1-\eta^2)\frac{dY}{d\eta}\right\}+\left[\Lambda\eta^2-\frac{m^2}{1-\eta^2}+\mu\right]Y=0 \quad (-1\leqslant\eta\leqslant1), \qquad \text{(II)}$$

have been discussed in a previous paper*. The equations, which occur in the wave mechanics of the hydrogen molecular ion, were originally discussed by A. H. Wilson† and have since been studied by W. G. Baber and H. R. Hassé‡. In these equations R is a positive constant, m may be zero or an integer, and Λ and μ are parameters which may take on values for which X and Y and their derivatives are finite, single-valued and continuous in their respective intervals. X and $dX/d\xi$ tend to zero as ξ tends to infinity.

It has been shown that μ may be regarded as a function of Λ defined by equation (II). Equation (I) then becomes an equation involving a single parameter, and it has been proved that solutions of the required type exist for certain values Λ_n of Λ. Now, from the nature of the physical problem in which these equations occur, it is necessary to show that some at least of the characteristic values of Λ are positive; in particular it is necessary to establish the fact that there exists a *positive* value Λ_0 of Λ for which X and Y have no zeros in the open intervals $(1, \infty)$ and $(-1, 1)$ respectively, for the case $m=0$. In the previous paper the parameter Λ was called λ^2 to stress the fact that positive values were required; but there was nothing in this paper to ensure that the characteristic values of Λ, which were shown to exist, were actually positive. It was in fact supposed that Λ could

* W. H. McCrea and R. A. Newing, *Proc. London Math. Soc.* (2), 37 (1933), 520–534.

† A. H. Wilson, *Proc. Royal Soc.* (A), 118 (1928), 617–635, 635–647.

‡ W. G. Baber and H. R. Hassé, *Proc. Camb. Phil. Soc.*, 31 (1935), 564–581.

assume negative values*. Now Wilson's original criticism of the wave mechanical theory† was that, although it was possible to obtain negative characteristic values for Λ, the existence of positive values seemed doubtful. It is thus particularly important to establish the existence of positive characteristic values for Λ.

1. Equations (I) and (II) are of the general form

$$\frac{d}{dx}\left\{p(x)\frac{dy}{dx}\right\}+q(x)y+\Big(\Lambda A(x)+\mu B(x)\Big)y=0.$$

For any particular value of Λ the solution of an equation of this type, subject to the given boundary conditions, becomes a simple boundary value problem, and the equation defines a set of characteristic values μ_n of μ. When Λ is given some other value, there will be defined, in general, another set of characteristic values μ_n', and by variation of Λ it will be seen that the equation defines a set of "parameter functions" $\mu_n(\Lambda)$. The functions $\mu_n(\Lambda)$ may be represented graphically and will be referred to as *parameter curves*. The parameter curves defined by equations (I) and (II) may be studied in the manner of R. G. D. Richardson‡.

Consider first the curves defined by equation (II). It follows readily from this equation§ that

$$\frac{d\mu}{d\Lambda}=-\int_{-1}^{1}\eta^2\,Y^2\,d\eta\Big/\int_{-1}^{1}Y^2\,d\eta,$$

and therefore $-1<d\mu/d\Lambda<0$, since the integrals are essentially positive and cannot vanish except in the trivial case $Y\equiv 0$. $\mu(\Lambda)$ is therefore a monotonic decreasing function of Λ. The intersections of the parameter curves with the straight lines $\mu=\kappa\Lambda$ will be obtained by replacing $\Lambda\eta^2+\mu$ in equation (II) by $\Lambda(\eta^2+\kappa)$. For any given value of κ, equation (II) now becomes an equation of the Sturm-Liouville type

$$\frac{d}{dx}\left\{k(x)\frac{dy}{dx}\right\}+(\Lambda g-l)y=0,$$

involving the single parameter Λ. Now, if boundary conditions are given at the end points a, b and if the functions k, g, and l are all positive in the

* McCrea and Newing, *loc. cit.*, 530. It was stated that "$g\to-\infty$ as $\mu\to\infty$", and this is used to prove that $-G^*/K^*\to\infty$ as $\mu\to\infty$. Now since $g=\Lambda x^2+\mu-m^2/(1-x^2)$, it follows that Λ must take on negative values.

† Wilson, *loc. cit.*, 641.

‡ R. G. D. Richardson, *Trans. American Math. Soc.*, 13 (1912), 22; *Math. Annalen* 73 (1912), 289.

§ *Cf.* McCrea and Newing, *loc. cit.*, 529.

interval (a, b), it is known that there will be *one* value of Λ associated with the solution y_n having n zeros in (a, b). If g changes sign in (a, b) there will be two sets of characteristic values, Λ_n^+ and Λ_n^-, associated with y_n. These results are usually given for ordinary end points a, b. In the present case -1 and 1 are singular points of equation (II), but from an application of the results of our first paper* it may be seen that the above results remain valid for our present case. It follows, therefore, that the lines $\mu = \kappa\Lambda$ will intersect any particular parameter curve in one or in two points according as $\eta^2+\kappa$ remains positive, or changes sign, in the interval -1, 1. Thus

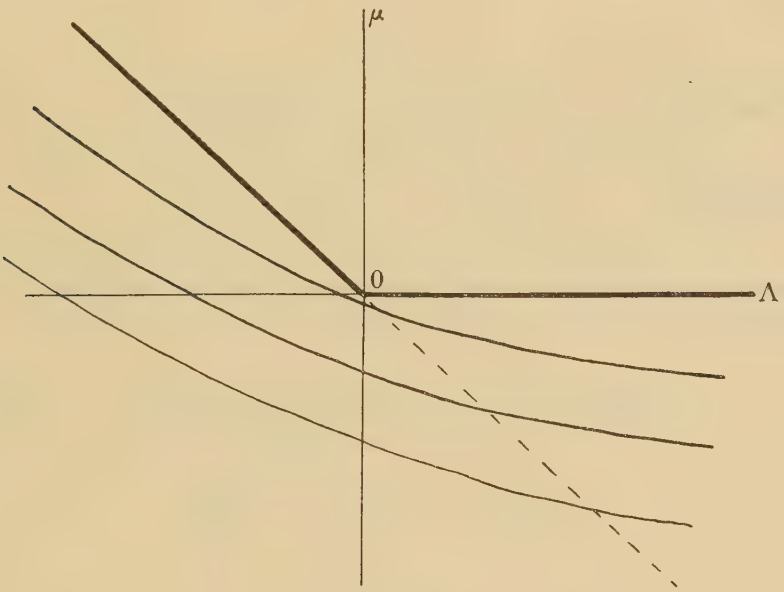

Fig. 1 (*a*).

lines through the origin cut the parameter curves twice if their slope is between 0 and -1; otherwise there is one point of intersection. Now, in considering the case of any particular value of κ for which $\eta^2+\kappa$ remains positive, it is known from the general theory that Λ increases with the number of zeros occurring in the solution Y in the given interval. It follows, therefore, that the distance of the parameter curves from the origin increases with the number of zeros in Y.

There are two possible general forms 1 (*a*) and 1 (*b*) for the parameter curves and it is necessary to distinguish between them.

* *Loc. cit.*

It follows from equation (II) that

$$\Lambda \int_{-1}^{1} \eta^2 Y^2 d\eta + \mu \int_{-1}^{1} Y^2 d\eta = \int_{-1}^{1} \frac{m^2}{1-\eta^2} Y^2 d\eta + \int_{-1}^{1} (1-\eta^2) \left(\frac{dY}{d\eta}\right)^2 d\eta,$$

from which it is obvious that Λ and μ cannot both be negative except for the trivial case $Y = 0$. That case 1 (*b*) is the general form of the parameter curves defined by equation (II) is confirmed by a study of the inter-

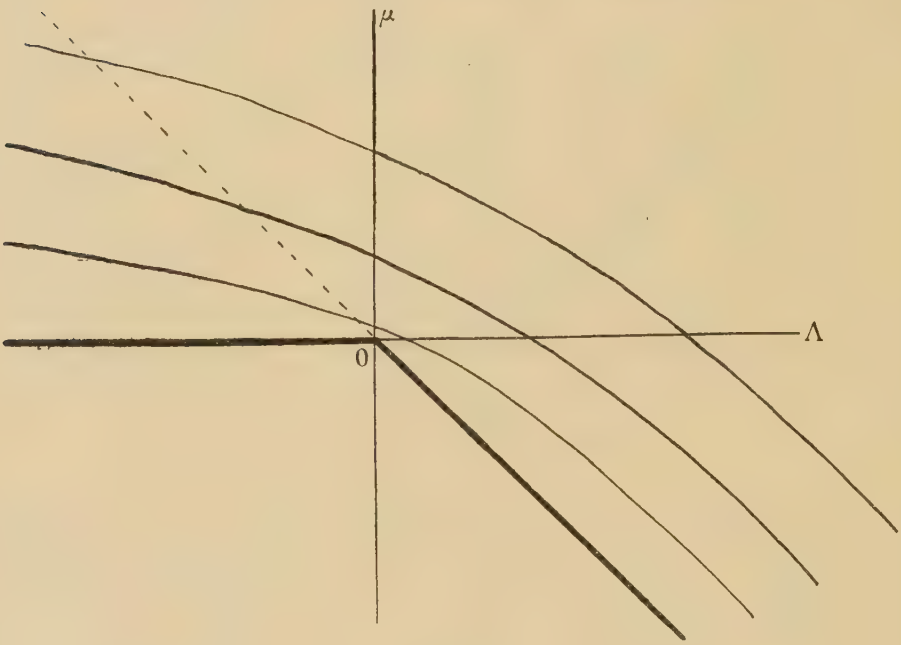

Fig. 1 (*b*).

sections of the line $\mu = \kappa\Lambda$ with the parameter curves when κ assumes one of the limiting values 0 and -1.

$$\Lambda \int_{-1}^{1} (\eta^2 + \kappa) Y^2 d\eta > 0.$$

Consider the case $\Lambda > 0$, $\kappa = -1+\epsilon$. Then $\eta^2+\kappa$ is positive only for an infinitesimal part of the interval $(-1, 1)$, and since the integral is itself positive, it must be infinitesimal. Therefore Λ must increase without limit as $\epsilon \to 0$. In a similar manner it follows that, when Λ is negative and $\kappa = 0-\epsilon$, Λ decreases without limit as $\epsilon \to 0$. If, when $\kappa = -1+\epsilon$, we

take $\Lambda < 0$, then the integral

$$\int_{-1}^{1} (\eta^2+\kappa)\, Y^2 d\eta < 0,$$

and this can occur without the integral being infinitesimal. Thus there is no necessity for an infinite Λ in this case.

Particular attention must be paid to the parameter curve corresponding to the solution Y_0 of equation (II) for which $m = 0$, and which has no zeros in the interval $-1 < \eta < 1$. In this case

$$\int_{-1}^{1} (\Lambda\eta^2+\mu)\, Y_0 d\eta = 0,$$

and since Y_0 does not change sign in the interval $(-1, 1)$, either $\mu = \Lambda = 0$, or μ and Λ are of opposite signs. It follows readily that $0 > \mu/\Lambda > -1$, and it may be concluded that the parameter curve passes through the origin and is included between the lines $\mu = 0$ and $\mu = -\Lambda$. This geometrical result may be expressed very simply in an analytical form. On account of the vanishing of the integral, $\Lambda\eta^2+\mu$ must change sign in $(-1, 1)$, since Y_0 remains of one sign. Therefore, for any particular values of Λ and μ, η_0 must exist such that $-1 < \eta_0 < 1$, and $\Lambda\eta_0{}^2+\mu = 0$.

2. It is now necessary to study the parameter curves derived from equation (I). In this case it may readily be proved* that

$$\frac{d\mu}{d\Lambda} = -\int_1^\infty \xi^2 X^2 d\xi \Big/ \int_1^\infty X^2 d\xi,$$

whence

$$-\infty < \frac{d\mu}{d\Lambda} < -1.$$

As in § 1, $\Lambda\xi^2+\mu$ may be written as $\Lambda(\xi^2+\kappa)$, and equation (I) becomes an equation of the Sturm-Liouville type for any particular value of κ. In the standard notation† we have $-g = \xi^2+\kappa$, and $l = -2R\xi+m^2/(\xi^2-1)$. g changes sign in $(1, \infty)$ when $-\infty < \kappa < -1$, and is essentially negative when $\kappa > -1$. In this case, however, it is not possible to state at once the nature of the intersections of the lines $\mu = \kappa\Lambda$ with the parameter curves; for the usual theory is given for positive l. In the present case l is either negative (*i.e.* when $m = 0$) or changes sign in $(1, \infty)$. We consider

* *Cf.* McCrea and Newing, *loc. cit.*, 529.

† E. L. Ince, *Ordinary differential equations* (1927), § 10.33.

first the particular set of parameter curves for which $m = 0$. The problem is to determine the nature of the characteristic values λ_n of λ for which the Sturm-Liouville equation possesses solutions y_n having n zeros in (a, b) and which satisfy given boundary conditions at a and b, when $k > 0$, $l < 0$, and g may have either sign, or may change sign in (a, b).

(i) $g > 0$. The solution y of the Sturm-Liouville equation certainly does not oscillate in (a, b) if $\Lambda \leqslant -A$, where A is the maximum value of $|l|/g$ in the interval (a, b). As Λ increases from $-A$, y begins to oscillate in (a, b), and the number of oscillations increases indefinitely as $\Lambda \to \infty$.

(ii) $g < 0$. There are no oscillations in (a, b) if $\Lambda \geqslant B$, where B is the maximum value of $|l/g|$ in (a, b). As Λ *decreases* from B to $-\infty$ the number of oscillations increases indefinitely from zero.

(iii) g changes sign in (a, b). Then sub-intervals exist such that $g > 0$ in (a', b') and $g < 0$ in (a'', b''). Then, if A' and B'' are defined as above for the two intervals respectively, the number of oscillations in (a', b') increases indefinitely from zero as Λ increases from $-A'$ to $+\infty$; the number of oscillations in (a'', b'') will increase indefinitely from zero as Λ decreases from B'' to $-\infty$. In this case an interval (Λ_1, Λ_2) of Λ, $-A' < \Lambda_1 \leqslant 0 \leqslant \Lambda_2 < B''$, *may* exist for which y oscillates either in (a', b') or in (a'', b''); as $|\Lambda|$ is arbitrarily increased the number of oscillations in (a, b) increases indefinitely. Sufficient conditions may readily be found for y to oscillate at least r times in (a, b) when Λ lies in (Λ_1, Λ_2). It would not then be possible to find Λ_n such that y oscillates n times in (a, b) if $n < r$.

Using the results of the previous paper we may show that the results (i), (ii), and (iii), obtained for ordinary end points a, b, may be applied to our present case. Now consider the Sturm-Liouville equation obtained by writing $\kappa\Lambda$ for μ in equation (I); g changes sign in $(1, \infty)$ when $-\infty < \kappa < -1$, and an interval (Λ_1, Λ_2) of Λ *may* exist for which X has at least r zeros in $(1, \infty)$; it is not then possible to find values of Λ for which X has less than r zeros in $(1, \infty)$. Thus for values of κ in the open interval $(-\infty, -1)$ the lines $\mu = \kappa\Lambda$ do not intersect the first r parameter curves. It follows that, unlike the case of equation (II), it is not possible in the present case to state that the parameter curves are excluded from a definite region of the $\mu\Lambda$ plane. It may happen in this case that the two possible systems of parameter curves, 2 (*a*) and 2 (*b*), extend on either side of $\mu 0_\tau$ and $\mu' 0_{\tau'}$ respectively.

Now, when $\kappa > -1$, $g < 0$ in $(1, \infty)$; and, for a given value of κ, the number of zeros in X increases as Λ decreases, and the lines $\mu = \kappa\Lambda$ intersect each parameter curve once. It follows that the parameter curves, which may be on the right-hand side of $\mu 0\tau$ or of $\mu' 0\tau'$ when X has a small number of zeros in $(1, \infty)$, moves to the left towards $\mu 0\tau$ or $\mu' 0\tau'$ as the number of zeros increases, and may actually cross over to the region of the $\mu\Lambda$ plane on the left of these lines. Let it be supposed that the first r parameter

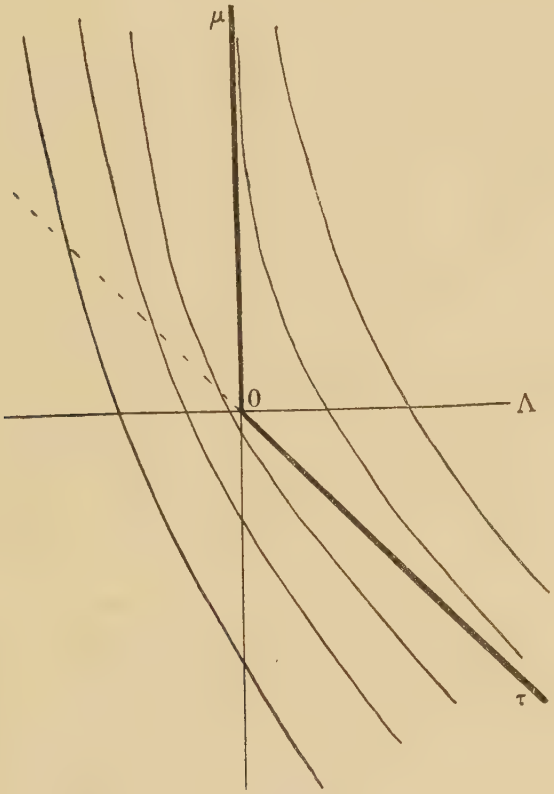

Fig. 2(*a*).

curves lie to the right of the origin. Then in case 2 (*a*) the lines $\mu = \kappa\Lambda$, $-\infty < \kappa < -1$, intersect all except the first r parameter curves, and this is an illustration of the possibility referred to in (iii) above. In case 2(*b*), however, the first r curves are all intersected by these lines, and it will follow from (iii) that all the curves must be intersected by the lines. In this latter case, therefore, *the parameter curves must be confined to the right-hand side of* $\mu' 0\tau'$.

In order to establish that, for the case $m \neq 0$, the possible forms for the parameter curves are also given by 2 (*a*) and 2 (*b*), it is only necessary to consider a fourth modification of the conditions usually associated with the Sturm-Liouville equation.

(iv) l changes sign in (a, b). Then sub-intervals exist such that $l > 0$ in (α, β), and $l < 0$ in (α', β'). The usual theory may be applied to

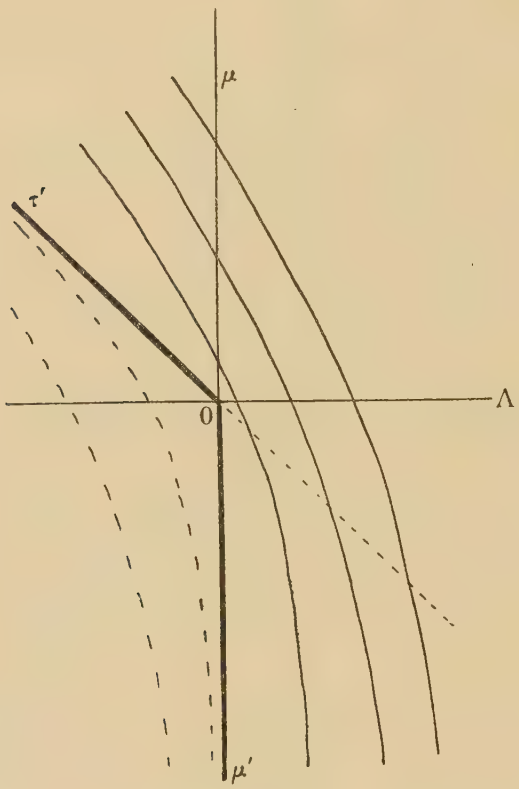

Fig. 2(*b*).

(α, β), and the results (i), (ii), and (iii) to (α', β'). In particular it will follow that an interval (Λ_1', Λ_2') of Λ, $\Lambda_1' < 0 < \Lambda_2'$, *may* exist for which there are at least r zeros in (α', β'), and therefore *a fortiori* in (a, b). It may then be concluded, as for the case $m = 0$, that some of the parameter curves may not be intersected by the lines $\mu = \kappa\Lambda$ when $-\infty < \kappa < -1$.

It is now necessary to distinguish between the two possibilities represented by 2 (*a*) and 2 (*b*).

It follows from equation (I) that

$$\int_1^\infty (\Lambda\xi^2+\mu)X^2 d\xi - 2R\int_1^\infty \xi X^2 d\xi = -m^2\int_1^\infty \frac{X^2}{\xi^2-1}\,d\xi - \int_1^\infty (\xi^2-1)\left(\frac{dX}{d\xi}\right)^2 d\xi,$$

from which it follows that the left-hand side of the equation is essentially negative. Thus, when $\mu = -\Lambda$, it is possible for Λ to be finitely positive or arbitrarily negative; Λ cannot be arbitrarily positive. When $\Lambda = 0$, μ may be finitely positive or arbitrarily negative; it cannot be arbitrarily positive. We thus obtain information concerning the intersections of the lines $\mu = -\Lambda$ and $\Lambda = 0$ with the parameter curves. This information is inconsistent with 2 (a) and we conclude that the required type of parameter curves is given by 2 (b), provided that the curves are confined to the right-hand side of $\mu' 0\tau'$. This conclusion is confirmed by considering the special curve for which there are no zeros in X, and $m = 0$. In this case, since

$$\int_1^\infty (-\Lambda\xi^2-\mu+2R\xi)X_0 d\xi = 0,$$

and X_0 does not change sign in $(1, \infty)$, then, for any particular values of Λ and μ, ξ_0 must exist such that $1 < \xi_0 < \infty$, $-\Lambda\xi_0^2+2R\xi_0-\mu = 0$.

It is at once obvious that no part of the curve can lie in the third quadrant. The curve crosses the axis $\mu = 0$ when $\Lambda = 2R/\xi_0$, the axis $\Lambda = 0$ when $\mu = 2R\xi_0$, and the line $\mu = -\Lambda$ when $\Lambda = 2R\xi_0/(\xi_0^2-1)$.

We conclude that the required parameter curves are represented by 2 (b) and lie between this first parameter curve and $\mu' 0\tau'$.

3. In §§ 1 and 2 it has been possible to determine the general nature of the two sets of parameter curves defined respectively by equations (II) and (I). The points of intersection of these two sets of curves define those values of μ and Λ for which equations (I) and (II) can be solved simultaneously, subject to the given boundary conditions.

It has been established that the nature of the parameter curves defined by equation (II) is given by Fig. 1 (b) and the parameter curve associated with equation (I) for $m = 0$ when X has no zeros in $(1, \infty)$ has been shown to lie to the right of the origin. It will be noted that it is now possible to assert that intersections of the two systems of curves occur for at least some positive values of Λ, whether the parameter curves of equation (I) be represented by Fig. 2 (a) or by Fig. 2 (b). But it has been established that the appropriate set of curves for equation (I) is given by Fig. 2 (b), and it is possible to obtain more detailed information concerning the intersections

of the two systems of curves. The two sets of curves are represented diagramatically in Fig. 3, the curves being numbered 0, 1, 2, 3, ... according to the number of zeros occurring in X or Y. Let $C_n^m(\mathrm{I})$ be the parameter curve associated with the solution X_n of equation (I) which has n zeros in $(1, \infty)$, and let $C_n^m(\mathrm{II})$ and Y_n be defined in a similar manner. Then the following results may be noted concerning the intersections of the two sets of curves.

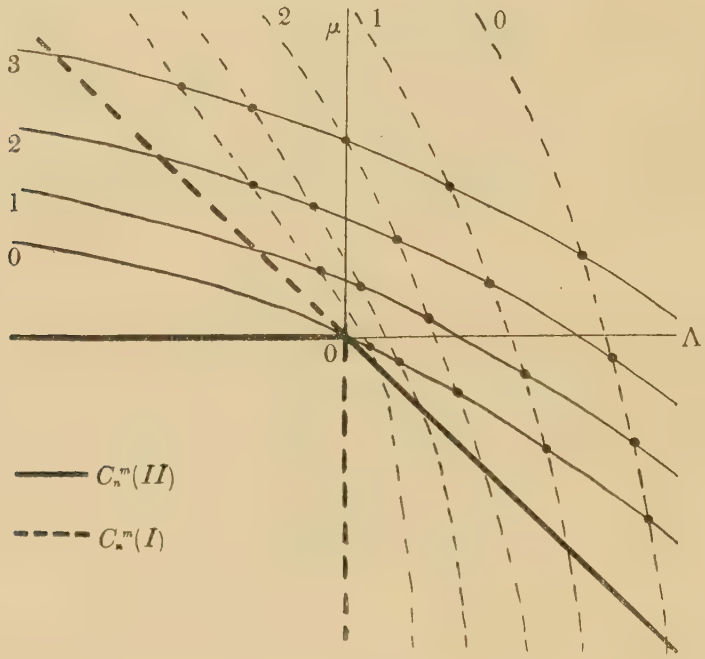

Fig. 3.

(i) Intersections between the two systems occur for both positive and negative values of Λ.

(ii) The points of intersection of the curve $C_0^0(\mathrm{II})$ with the curves $C_n^m(\mathrm{I})$ are all in the fourth quadrant, and therefore occur for positive values of Λ.

(ii$a$) The Λ coordinate of the point of intersection of the curves $C_0^0(\mathrm{I})$ and $C_0^0(\mathrm{II})$ is positive.

(iii) The Λ coordinates of the points of intersection of the curves $C_n^m(\mathrm{II})$ with a fixed curve $C_s^m(\mathrm{I})$ decrease as n increases.

(iv) The Λ coordinates of the points of intersection of the curves $C_n^m(\mathrm{I})$ with a fixed curve $C_t^m(\mathrm{II})$ decrease as n increases.

These results are in accordance with those to be expected from the nature of the physical problem with which equations (I) and (II) are associated.

It may be noted that the Λ coordinate of the point of intersection of C_0^0(I) and C_0^0(II) may be shown to be positive, independently of the preceding geometrical discussion. For any point on C_0^0(I), ξ_0 must exist, $1 < \xi_0 < \infty$, such that $\Lambda\xi_0^2 + \mu - 2R\xi_0 = 0$. For any point on C_0^0(II), η_0 must exist, $-1 < \eta_0 < 1$, such that $\Lambda\eta_0^2 + \mu = 0$.

ξ_0 and η_0 depend on the position on the curves. Now, if (Λ_0, μ_0) is a point common to C_0^0(I) and C_0^0(II), then $\Lambda_0 = 2R\xi_0/(\xi_0^2 - \eta_0^2)$; this shows that there is a unique point of intersection, and that Λ_0 is positive.

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ON THE CONVERGENCE OF SERIES OF POLYNOMIALS

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1. *Introduction.*

A general theory of series of the type

$$A_0 p_0(z)+A_1 p_1(z)+\ldots, \tag{1.1}$$

where $p_0(z)$, $p_1(z)$, ... is a given set of polynomials, has been developed by J. M. Whittaker*. The polynomials are assumed to form a *basic set*, *i.e.* they are such that an arbitrary polynomial can be expressed in one and only one way as a finite linear combination of the given polynomials. This certainly is the case if $p_n(z)$ is of degree n, but of course this condition is not necessary.

If the equations expressing $1, z, z^2, \ldots$ in terms of the polynomials of the basic set are represented in general by

$$z^n = \pi_{n0} p_0(z)+\pi_{n1} p_1(z)+\ldots, \tag{1.2}$$

and if $f(z)$ is a function regular at $z=0$, then the series (1.1) can be written

$$p_0(z)\,\Pi_0 f(0)+p_1(z)\,\Pi_1 f(0)+\ldots, \tag{1.3}$$

where

$$\Pi_n f(0) = \pi_{0n} f(0)+\pi_{1n} f'(0)+\frac{1}{2!}\pi_{2n} f''(0)+\ldots.$$

In general (1.3) is an infinite series, though (1.3) is a finite series when $f(z)$ is a polynomial†. (1.3) is called the *basic series* associated with the

* *Interpolatory function theory* (Cambridge Tract, 1935), 1. We are not concerned here with series in which the polynomials employed depend on the function expanded. Such series have been studied frequently, as, for example, by Walsh in *Interpolation and approximation* (American Math. Soc. Colloquium Publications, 1935).

† Whittaker, *op. cit.*, 7.

polynomials and Whittaker* has shown that it represents all integral functions of sufficiently low order.

Many particular cases of series of polynomials have been exhaustively studied. Taylor's series is the simplest case, and other familiar examples are the expansions of interpolation theory and series of the polynomials of Legendre, Laguerre, Hermite, and Bernoulli. In all this work the particular properties of the polynomials studied are exploited to the full. In another type of convergence theorem studied by Whittaker† the hypotheses concern only the degree of the polynomials and the order of magnitude of their coefficients. It is shown, for example, that if

$$p_i(z) = p_{i0} + p_{i1}z + \ldots + p_{i,i-1}z^{i-1} + z^i \quad (i = 0, 1, 2, \ldots)$$

is a set of polynomials whose coefficients satisfy the inequality $|p_{ij}| \leqslant M$ and if $f(z)$ is regular in $|z| < R$, where $R > 1 + M$, then the basic series converges absolutely to $f(z)$ in $|z| < R$.

The purpose of the present work is to show that it is possible to make a certain amount of progress with hypotheses relating only to the degree of the polynomials. From this point of view the special cases mentioned above are indistinguishable, since in all of them the n-th polynomial is of degree n. Such series are called *simple* series.

It will be convenient to use the following terminology. The *circle of regularity* of a function is the circle with centre at the origin which passes through the nearest singularity, and its radius is the *radius of regularity*. A basic series *represents* $f(z)$ *in* $|z| = R$ if it converges to $f(z)$ uniformly in any smaller circle with centre at the origin; and *represents* $f(z)$ *in and on* $|z| = R$ if it converges to $f(z)$ uniformly in $|z| \leqslant R$. In the latter case R may be zero, the basic series being then convergent at the origin only.

Whittaker introduced the function

$$\omega_n(R) = \sum_{i=0}^{\infty} |\pi_{ni}| M_i(R), \tag{1.5}$$

where

$$M_i(R) = \max_{|z|=R} |p_i(z)|,$$

and proved the following result:

LEMMA 1. *If* $f(z) = \Sigma a_n z^n$ *is such that*

$$\sum_{n=0}^{\infty} |a_n| \omega_n(R) \tag{1.6}$$

converges, the basic series (1.3) *represents* $f(z)$ *in and on* $|z| = R$.

* Whittaker, *op. cit.*, 11.

† Whittaker, *op. cit.*, 15.

Let N_n denote the number of non-zero coefficients π_{ni} in (1.2). Then our first theorem is as follows:

THEOREM 1. *Let $p_0(z)$, $p_1(z)$, ... be a basic set of polynomials such that*

$$N_n^{1/n} \to 1 \quad as \quad n \to \infty. \tag{1.7}$$

Then a necessary and sufficient condition that the series (1.3) *should represent every function in its circle of regularity is that*

$$\{\omega_n(R)\}^{1/n} \to R \quad as \quad n \to \infty \text{ for every } R \geqslant 0. \tag{1.8}$$

It appears that, when $N_n^{1/n} \to 1$, the function $\omega_n(R)$, crude though it looks, is an accurate guide to the convergence properties of the series.

The condition (1.7) is sufficiently wide to cover all ordinary cases but the general case can also be treated. (1.8) is replaced by a similar but more complicated condition and it is shown by means of an example that this is not always the same as (1.8). Theorem 1 therefore becomes false if (1.7) is omitted. An example is also given to show that the above condition is sometimes essentially the same as (1.8) even though (1.7) may not be satisfied.

It is a striking fact that in the following proofs we have much greater difficulty in proving the necessity of the conditions than in proving their sufficiency. A very similar thing occurs in the discussion of the Taylor series $\sum_0^\infty a_n z^n$. The radius of convergence ρ is defined by

$$\overline{\lim_{n\to\infty}} |a_n|^{1/n} = \frac{1}{\rho}.$$

The condition $R < \rho$ is sufficient for the convergence of the Taylor series in $|z| \leqslant R$ but this condition is, in general not necessary, since there are Taylor series with radius of convergence ρ which converge at all points of the circle of convergence. If, however, we want *every* Taylor series with radius of convergence ρ to converge in $|z| \leqslant R$ then $R < \rho$ is both necessary and sufficient. Generalizations of this result to basic series are given in the following work.

In addition to N_n we introduce a number D_n defined to be the degree of the polynomial of highest degree in (1.2) and consider the condition

$$\frac{D_n}{n} \to 1 \quad \text{as} \quad n \to \infty, \tag{1.9}$$

The conditions (1.7) and (1.9) are not strictly comparable, but, roughly speaking, (1.9) is much more restrictive. In the case of a simple series

$$D_n = n, \quad 1 \leqslant N_n \leqslant n+1, \tag{1.10}$$

so that (1.7), (1.9) are both satisfied. Considerations of a simple nature show that (1.9) implies (1.7).

In cases where (1.9) is satisfied the results throw light on the possible types of convergence of the series. A note of explanation is called for. Consider

(i) the polynomials $1, z, z^2, \ldots,$

(ii) the polynomials $1, 1+z, 1+z^2, 1+z^3, \ldots.$

(iii) the Bernoulli polynomials, $B_0(z), B_1(z), \ldots.$

In case (i) the basic series represents every function in its circle of regularity. The basic series in case (ii) has the same property if the radius of regularity is greater than unity, but it need not represent in any circle a function whose radius of regularity is less than unity. This is a common type of convergence. Consider a previous example. The basic series associated with the polynomials

$$p_n(z) = p_{n0} + p_{n1} z + \ldots + p_{n\,n-1} z^{n-1} + z^n,$$

where $|p_{ij}| \leqslant M$, represents in its circle of regularity every function whose radius of regularity exceeds $1+M$ but need not represent in any circle a function that does not satisfy this condition*.

In case (iii) the basic series either converges in the whole plane or else only at isolated points, and so is only suitable for representing integral functions.

It is easy to conceive other plausible types of convergence. For example, we have seen in case (ii) that there may be a lower limit to the effectiveness of a simple series, since it may represent only functions whose circle of regularity encloses some fixed circle; and we observe that, in the example given, the series also has the property of representing every such function in its whole circle of regularity. The question arises immediately whether there is any necessary connection between these properties. The following result shows that the second property is implied by the first.

* Whittaker, *op. cit.*, 15.

THEOREM 2. *If a simple series represents in and on a fixed circle of radius greater than zero every function which is regular in and on this circle, then it represents every such function in its whole circle of regularity.*

Again, can there be an upper limit to the effectiveness of a simple series? This question must be answered in the negative as the following result shows:

THEOREM 3. *If a simple series represents in its circle of regularity every function whose radius of regularity is less than a constant λ, then it represents every function in its circle of regularity.*

In this and the preceding theorem we have taken the series to be simple, since this is the important case. More generally the theorems apply to any basic series for which (1.9) is satisfied.

2. LEMMA 2. *If $N_n^{1/n} \to 1$ as $n \to \infty$, and for any $R \geqslant 0$ a basic series represents in and on $|z| = R$ every function which is regular in and on this circle, then*

$$\{\omega_n(R)\}^{1/n} \to R \quad as \quad n \to \infty. \tag{2.1}$$

From the definition of $\omega_n(R)$ it follows that

$$\omega_n(R) \geqslant \max_{|z|=R} \left| \sum_i \pi_{ni} p_i(z) \right|$$

$$= R^n.$$

Therefore
$$\varliminf_{n\to\infty} \{\omega_n(R)\}^{1/n} \geqslant R. \tag{2.2}$$

It is now sufficient to show that

$$\varlimsup_{n\to\infty} \{\omega_n(R)\}^{1/n} \leqslant R.$$

Suppose that
$$\varlimsup_{n\to\infty} \{\omega_n(R)\}^{1/n} = R_1 > R. \tag{2.3}$$

Let
$$R < \rho < R_1.$$

The sum by rows of the double series

$$\sum_{n,i} \frac{|\pi_{ni}| M_i(R)}{\rho^n}$$

is divergent.

Hence the double series is itself divergent. It follows that either

$$(a) \quad \sum_{n=0}^{\infty} \frac{|\pi_{ni}|}{\rho^n} \text{ is divergent for some } i,$$

or $$(b) \quad \sum_{n=0}^{\infty} \frac{|\pi_{ni}|}{\rho^n} \text{ converges to } S_i \text{ for all } i,$$

but $\sum_{i=0}^{\infty} M_i(R)\, S_i$ is divergent.

Case (a). We then have

$$\overline{\lim_{n\to\infty}} \left| \frac{\pi_{ni}}{\rho^n} \right|^{1/n} \geqslant 1$$

for some i.

Hence, if $r > 1$, $$\overline{\lim_{n\to\infty}} \left| \frac{r^n \pi_{ni}}{\rho^n} \right|^{1/n} \geqslant r > 1,$$

so that $\sum_{n=0}^{\infty} \left(\frac{r}{\rho}\right)^n \pi_{ni}$ is divergent. But this series is $\Pi_i g(0)$, where $g(z) = \sum_{0}^{\infty} \left(\frac{r}{\rho}\right)^n z^n$. Thus the basic series could not represent the function $\sum_{0}^{\infty} \left(\frac{r}{\rho}\right)^n z^n$, which is regular at $z = 0$.

Hence the state of affairs in (a) is impossible.

Case (b). We have

$$\overline{\lim_{n\to\infty}} \{\omega_n(R)\}^{1/n} = R_1 > R.$$

Hence, if $N_n^{1/n} \to 1$, we must have

$$\overline{\lim_{n\to\infty}} \left\{ \max_i |\pi_{ni} M_i(R)| \right\}^{1/n} = R_1.$$

It follows that we can choose a sequence

$$n_1, \ n_2, \ \ldots \tag{2.4}$$

such that

$$\left\{ \max_i |\pi_{ni} M_i(R)| \right\}^{1/n} > R_2 \quad (\rho < R_2 < R_1; \ n = n_1, n_2, \ldots).$$

Let $$|\pi_{n_k p_k}| M_{p_k}(R) = \max_i |\pi_{ni} M_i(R)|.$$

Then $$|\pi_{n_k p_k}| M_{p_k}(R) > R_2^{n_k} \quad (k = 1, 2, \ldots). \tag{2.5}$$

If all the terms $\pi_{n_k p_k}$ are in a finite number of columns, an infinite number must be in one of these columns, say the first. Then $|\pi_{n_k 0}| M_0(R) > R_2^{n_k}$ for an infinity of k; and since $R_2 > \rho$,

$$\overline{\lim_{n\to\infty}} \left| \frac{\pi_{n0}}{\rho^n} M_0(R) \right|^{1/n} > 1,$$

whence $$\overline{\lim_{n\to\infty}} \left| \frac{\pi_{n0}}{\rho^n} \right|^{1/n} > 1.$$

This has been shown to be impossible under (a).

If the terms $\pi_{n_k p_k}$ are not all in a finite number of columns, p_k must take infinitely large values. We can select a subsequence

$$\pi_{m_1 q_1}, \quad \pi_{m_2 q_2}, \quad \ldots \tag{2.6}$$

such that $m_1 < m_2 < \ldots$, $q_1 < q_2 < \ldots$, and such that m_r, q_r both tend to infinity with r. This can be done by omitting from the sequence (2.14) any term in which the corresponding p_k is less than or equal to some previous p_k. If we divide the matrix $\Pi = \{\pi_{ij}\}$ into four parts by drawing lines through a given row and column there will be members of (2.6) in the lower right-hand part.

Now $\sum_{n=0}^{\infty} \frac{|\pi_{ni}|}{\rho^n}$ is convergent for every i by hypothesis; hence, given $\epsilon_i > 0$, there is an integer M_i such that

$$\sum_{n=M_i}^{\infty} \frac{|\pi_{ni}|}{\rho^n} < \epsilon_i.$$

Given a member $\pi_{m_s q_s}$ of the sequence (2.6), we can find an integer $\mu_s > m_s$ such that

$$\sum_{n=\mu_s}^{\infty} \frac{|\pi_{nq_s}|}{\rho^n} < \tfrac{1}{2} \frac{|\pi_{m_s q_s}|}{\rho^{m_s}}.$$

Draw a "row-line" through $\pi_{\mu_s q_s}$ and a "column-line" such that all non-zero elements in the first m_s+1 rows of Π are in the top left-hand part. Let $\pi_{c_2 b_2}$ be the first member of (2.6) which appears in the lower right-hand part of this division of Π. Let $\pi_{c_3 b_3}$ be that member of (2.6) which is

obtained from $\pi_{c_2 b_2}$ by exactly the same process as that by which $\pi_{c_2 b_2}$ has just been obtained from $\pi_{m_s q_s} = \pi_{c_1 b_1}$. By proceeding in this way, we find a sequence

$$\pi_{c_1 b_1}, \quad \pi_{c_2 b_2}, \quad \ldots. \tag{2.7}$$

Let $$f(z) = \sum_0^\infty a_n z^n,$$

where
$$\left.\begin{aligned} |a_n| &= \frac{1}{\rho^n} \quad (n = c_1, c_2, \ldots), \\ a_n &= 0 \qquad \text{otherwise.} \end{aligned}\right\} \tag{2.8}$$

The sequence (2.7) has been chosen so that $\sum_{n=0}^{\infty} a_n \pi_{n b_t}$ consists of the term $a_{c_t} \pi_{c_t b_t}$ together with a remainder whose modulus is less than $\frac{1}{2}|a_{c_t} \pi_{c_t b_t}|$. Therefore

$$\begin{aligned} \left| \sum_{n=0}^{\infty} a_n \pi_{n b_t} \right| &> \tfrac{1}{2} |a_{c_t} \pi_{c_t b_t}| \\ &> \tfrac{1}{2} \frac{1}{M_{b_t}(R)} \left(\frac{R_2}{\rho}\right)^{c_t}, \text{ by (2.5), (2.8),} \end{aligned}$$

i.e.,
$$|\Pi_{b_t} f(0)| M_{b_t}(R) > \tfrac{1}{2} \left(\frac{R_2}{\rho}\right)^{c_t} \to \infty. \tag{2.9}$$

Uniform convergence of the basic series would therefore be impossible. We have therefore proved that (2.3) is false. Hence

$$\overline{\lim_{n \to \infty}} \{\omega_n(R)\}^{1/n} = R.$$

From (2.2) it now follows that

$$\{\omega_n(R)\}^{1/n} \to R \quad \text{as} \quad n \to \infty.$$

LEMMA 3. *If, for any $R \geqslant 0$, the condition (2.1) is satisfied, the basic series represents in and on $|z| = R$ any function which is regular in and on this circle.*

For, if $\sum_0^\infty a_n z^n$ is any such function, its radius of regularity is greater than R, and $\sum_0^\infty |a_n z^n|$ is convergent at any point inside the circle of regularity.

Suppose that the radius of regularity is $R+2\epsilon$ $(\epsilon>0)$; then from (2.1) we have

$$\omega_n(R) < (R+\epsilon)^n \quad (n \geqslant n_0).$$

Hence $\sum_0^\infty |a_n|\,\omega_n(R)$ is convergent by comparison with the convergent series

$$\sum_0^\infty |a_n|\,(R+\epsilon)^n.$$

It now follows from Lemma 1 that the basic series represents $\sum_0^\infty a_n z^n$ in and on $|z|=R$.

Proof of Theorem 1. The condition (1.8) is necessary. For if it is false, then, for some $R_0 \geqslant 0$,

$$\overline{\lim_{n\to\infty}}\ \{\omega_n(R)\}^{1/n} > R_0.$$

By Lemma 2, there is a function regular in and on $|z|=R_0$, and so in $|z|<R_1$, where $R_1>R_0$, which the corresponding basic series fails to represent on $|z|=R_0$. This function is therefore not represented in its circle of regularity by its basic series.

The sufficiency of (1.8) follows immediately from Lemma 3.

COROLLARY. *If* $N_n^{1/n}\to 1$, *then a necessary and sufficient condition that the basic series should represent in* $|z|\leqslant R_0(\rho>R_0)$ *and in* $|z|<\rho$ $(\rho\leqslant R_0)$, *all functions regular at* $z=0$ *is that*

$$\{\omega_n(R)\}^{1/n}\to R \quad (0\leqslant R\leqslant R_0),$$

where ρ *is the radius of convergence of the Taylor series of the function about the origin.*

3. *The general case.*

If the condition (1.7) is not satisfied, it is found that the function $\omega_n(R)$ is not sufficiently powerful to deal with the problem. In its place we consider the expression $F_n(R)$, which denotes the maximum for all integers i, j, of

$$\max_{|z|=R} |\pi_{ni}p_i(z)+\ldots+\pi_{nj}p_j(z)|\ldots. \tag{3.1}$$

The theorem can be stated as follows:

THEOREM 4. *A necessary and sufficient condition that a basic series should represent every function in its circle of regularity is that*

$$\{F_n(R)\}^{1/n}\to R \text{ as } n\to\infty \text{ for every } R\geqslant 0. \tag{3.2}$$

From (3.1) it follows immediately that, for all integral values of h, k, and for $|z| \leqslant R$,

$$\left| \sum_{i=h}^{k} \pi_{ni} p_i(z) \right| \leqslant F_n(R). \tag{3.3}$$

Also,

$$F_n(R) \geqslant \max_{|z|=R} \left| \sum_{i=0}^{\infty} \pi_{ni} p_i(z) \right|$$

$$= R^n;$$

so that

$$\varlimsup_{n\to\infty} \{F_n(R)\}^{1/n} \geqslant R. \tag{3.4}$$

Let the Taylor series $\sum_0^\infty a_n z^n$ of a function $f(z)$ have radius of convergence ρ. Then (3.2) is sufficient, for, if it is satisfied,

$$\varlimsup_{n\to\infty} |a_n F_n(R)|^{1/n} = \frac{R}{\rho} < 1 \quad (R < \rho);$$

i.e., $\sum_{n=0}^{\infty} |a_n| F_n(R)$ is convergent.

Hence, given $\epsilon > 0$, we can find an integer M such that

$$\sum_{n=M}^{\infty} |a_n| F_n(R) < \epsilon. \tag{3.5}$$

Draw a "row-line" through the $(M+1)$-th row of $\Pi = \{\pi_{ij}\}$ and a "column-line" such that all non-zero elements in the first $M+1$ rows of Π are in the top left-hand part. This can be done since Π is row-finite. Let π_{M, i_0} be the element in the $(M+1)$-th row through which this "column-line" passes.

Choose h, $k \geqslant i_0$; then from (3.3) we have,

$$\sum_{n=M}^{\infty} \left| \sum_{i=h}^{k} a_n \pi_{ni} p_i(z) \right| \leqslant \sum_{n=M}^{\infty} |a_n| F_n(R)$$

$$< \epsilon, \quad \text{by} \quad (3.5). \tag{3.6}$$

Now

$$\sum_{n=M}^{\infty} \left| \sum_{i=h}^{k} a_n \pi_{ni} p_i(z) \right| \geqslant \left| \sum_{i=h}^{k} p_i(z) \sum_{n=M}^{\infty} a_n \pi_{ni} \right|$$

$$= \left| \sum_{i=h}^{k} p_i(z) \left\{ \Pi_i f(0) - \sum_{r=0}^{M-1} a_r \pi_{r,i} \right\} \right|.$$

But h has been chosen so that

$$\sum_{r=0}^{M-1} a_r \pi_{r,i} = 0 \quad (i \geqslant h).$$

Therefore

$$\sum_{n=M}^{\infty} \left| \sum_{i=h}^{k} a_n \pi_{ni} p_i(z) \right| \geqslant | p_h(z) \Pi_h f(0) + \ldots + p_k(z) \Pi_k f(0) |$$

and so, by (3.6), the expression on the right is less than ϵ. Thus the basic series

$$p_0(z) \Pi_0 f(0) + p_1(z) \Pi_1 f(0) + \ldots$$

is uniformly convergent and it is easy to see that its sum is $f(z)$, so that (3.2) is sufficient.

In this theorem the essential difficulty is more evenly divided between the necessity and sufficiency proofs than in the previous theorem. To prove the necessity of (3.2) suppose that

$$\overline{\lim_{n\to\infty}} \{F_n(R)\}^{1/n} = R_1 > R \quad \text{for some } R. \tag{3.7}$$

Then either

$$(c) \qquad \overline{\lim_{n\to\infty}} \left\{ \max_i | \pi_{ni} M_i(R) | \right\}^{1/n} = R_1' > R,$$

or

$$(d) \qquad \overline{\lim_{n\to\infty}} \left\{ \max_i | \pi_{ni} M_i(R) | \right\}^{1/n} \leqslant R.$$

Case (*c*). Let $R < \rho < R_1'$. We can choose a sequence $n_1, n_2, \ldots$ such that

$$\left\{ \max_i | \pi_{ni} M_i(R) | \right\}^{1/n} > R_2 \quad (\rho < R_2 < R_1').$$

(2.5)–(2.9) now follow immediately.

Thus the basic series cannot be uniformly convergent. Hence (*c*) is impossible.

Case (*d*). Let $R < \rho < R_1$. Then

$$\overline{\lim_{n\to\infty}} \left\{ \frac{1}{\rho^n} \max_i | \pi_{ni} M_i(R) | \right\}^{1/n} < 1.$$

Hence

$$\sum_{n=0}^{\infty} \frac{1}{\rho^n} \max | \pi_{ni} M_i(R) |$$

is convergent; *i.e.* given any $\epsilon > 0$, we can find an integer M so that

$$\sum_{n=M}^{\infty} \frac{1}{\rho^n} \max_i |\pi_{ni} M_i(R)| < \epsilon. \tag{3.8}$$

Define $\phi_n(R)$, h_n, k_n, by the equations

$$\left.\begin{aligned} \phi_n(R) &= \max_{i,\, j>n} \left\{ \max_{|z|=R} |\pi_{ni}\, p_i(z) + \ldots + \pi_{nj}\, p_j(z)| \right\} \\ &= \max_{|z|=R} |\pi_{nh_n} p_{h_n}(z) + \ldots + \pi_{nk_n} p_{k_n}(z)| \end{aligned}\right\}, \tag{3.9}$$

where π_{nh_n}, π_{nk_n} are non-zero elements of Π and h_n, $k_n > n$. Then

$$F_n(R) \leqslant \sum_{i=0}^{n} |\pi_{ni}| M_i(R) + \phi_n(R), \tag{3.10}$$

and

$$F_n(R) \geqslant \phi_n(R). \tag{3.11}$$

From (3.7), (3.11) we have

$$\overline{\lim_{n\to\infty}} \{\phi_n(R)\}^{1/n} \leqslant R_1. \tag{3.12}$$

Suppose that

$$\overline{\lim_{n\to\infty}} \{\phi_n(R)\}^{1/n} < R_1.$$

Then from (3.7) and (3.10) we have

$$\overline{\lim_{n\to\infty}} \left\{ \sum_{i=0}^{n} |\pi_{ni}| M_i(R) \right\}^{1/n} \geqslant R_1.$$

Therefore

$$\overline{\lim_{n\to\infty}} \left\{ \max_i |\pi_{ni} M_i(R)| \right\}^{1/n} \geqslant R_1 > R.$$

This contradiction of (*d*) implies that

$$\overline{\lim_{n\to\infty}} \{\phi_n(R)\}^{1/n} = R_1.$$

We can choose a sequence

$$n_1,\ n_2,\ \ldots, \tag{3.13}$$

such that

$$\{\phi_n(R)\}^{1/n} > R_2 \qquad (\rho < R_2 < R_1;\ n = n_1, n_2 \ldots),$$

i.e.,

$$\phi_{n_k}(R) > R_2^{n_k} \quad (k = 1, 2, \ldots). \tag{3.14}$$

Given any member n_s of (3.13), we can find an integer $\mu_s' > n_s$ so that

$$\sum_{n=\mu_s}^{\infty} \frac{1}{\rho^n} \max_i |\pi_{ni} M_i(R)| < \frac{\frac{1}{2}\phi_{n_s}(R)}{\rho^{n_s}(k_{n_s} - h_{n_s} + 1)} \quad \text{by} \quad (3.8).$$

Hence
$$\sum_{n=\mu_s}^{\infty} \frac{1}{\rho^n} \sum_{i=h_{n_s}}^{k_n} |\pi_{ni}| M_i(R) < \frac{\frac{1}{2}\phi_{n_s}(R)}{\rho^{n_s}}.$$

Draw a "row-line" through the (μ_s+1)-th row of Π and a "column-line" so that all non-zero elements in the first μ_s+1 rows of Π are in the top left-hand portion. Suppose that the latter line is drawn through the (λ_s+1)-th column.

From the sequence (3.13) select another sequence

$$m_1, \ m_2, \ \ldots, \tag{3.15}$$

by taking as the next member to n_s the first member of (3.13) greater than both μ_s and λ_s and continuing the process as in forming (2.7).

Let
$$\left.\begin{aligned} |a_n| &= \frac{1}{\rho^n} \quad (n = m_1, m_2, \ldots), \\ a_n &= 0 \quad \text{otherwise.} \end{aligned}\right\} \tag{3.16}$$

The sequence (3.15) has been chosen so that

$$\sum_{n=0}^{\infty} |a_n| \max_{|z|=R} \left| \sum_{i=h_{m_t}}^{k_{m_t}} \pi_{ni} p_i(z) \right|$$

consists of the term $|a_{m_t}| \phi_{m_t}(R)$ together with a remainder whose modulus is less than $\frac{1}{2}|a_{m_t}| \phi_{m_t}(R)$.

It is now clear that

$$\max_{|z|=R} \left| \sum_{n=0}^{\infty} a_n \sum_{i=h_{m_t}}^{k_{m_t}} \pi_{ni} p_i(z) \right| \geqslant \tfrac{1}{2}|a_{m_t}| \phi_{m_t}(R)$$

$$> \tfrac{1}{2}\left(\frac{R_2}{\rho}\right)^{m_t} \quad \text{by (3.14), (3.16);}$$

i.e.,
$$\max_{|z|=R} \left| \sum_{i=h_{m_t}}^{k_{m_t}} p_i(z) \Pi_i f(0) \right| > \tfrac{1}{2}\left(\frac{R_2}{\rho}\right)^{m_t} \to \infty.$$

Uniform convergence of the basic series would therefore be impossible. Hence (*d*) is impossible; thus (3.7) is false.

It follows from (3.4) that

$$\{F_n(R)\}^{1/n} \to R \quad \text{as} \quad n \to \infty.$$

Hence (3.2) is necessary. This proves the theorem.

Example 1. The following example shows that the condition

$$\{F_n(R)\}^{1/n} \to R \quad \text{as} \quad n \to \infty \quad (0 \leqslant R \leqslant R_0)$$

does not always imply the condition

$$\{\omega_n(R)\}^{1/n} \to R \quad \text{as} \quad n \to \infty \quad (0 \leqslant R \leqslant R_0).$$

Choose a sequence $n_1, n_2, \ldots$ of integers such that

$$n_{t+1} = 2^{n_t+1} - n_t + 2$$

and write

$$p_n(z) = z^n \quad (n = 0, 1, 2, \ldots, n_1 - 1),$$

$$p_{n_t+s}(z) = z^{2^{n_t}+\frac{1}{2}s+1} \quad (s = 0, 2, 4, \ldots, 2^{n_t+1} - 2n_t),$$

$$p_{n_t+s}(z) = -(z^{2^{n_t}-\frac{1}{2}(s-3)} + z^{2^{n_t}+\frac{1}{2}(s+1)} - z^{2^{n_t}-\frac{1}{2}(s-1)})$$

$$(s = 3, 5, 7, \ldots, 2^{n_t+1} - 2n_t + 1),$$

$$p_{n_t+1}(z) = -(z^{2^{n_t}+1} - z^{2^{n_t}}).$$

The set of polynomials

$$p_{n_t}(z), \; p_{n_t+1}(z), \; \ldots, \; p_{n_{t+1}-1}(z) \quad (t = 1, 2, \ldots)$$

and

$$p_0(z), \; p_1(z), \; \ldots, \; p_{n_1-1}(z),$$

is clearly basic* and

$$\begin{aligned} z^{n_t} \equiv \; & z^{2^{n_t}+1} - (z^{2^{n_t}+1} - z^{2^{n_t}}) + z^{2^{n_t}+2} \\ & - (z^{2^{n_t}} + z^{2^{n_t}+2} - z^{2^{n_t}-1}) + z^{2^{n_t}+3} \\ & - \ldots \quad \ldots \quad \ldots \quad + z^{2^{n_t+1}-n_t+1} \\ & - (z^{n_t+1} + z^{2^{n_t+1}-n_t+1} - z^{n_t}). \end{aligned}$$

z^{n_t+s}, where s is any integer less than or equal to $2^{n_t} - n_t$, is obtained by

* Whittaker, *op. cit.*, 5, 1.

omitting the last $2s$ groups in the above representation of z^{n_t}. If s is integral and $2^{n_t}-n_t < s < n_{t+1}-n_t$ then z^{n_t+s} is a polynomial of the basic set. It is now clear that, for any n, $F_n(R)$ is less than the sum of four terms of the form R^{n+m} $(m \geqslant 0)$.

It follows that, for $R \leqslant 1$, $F_n(R) \leqslant 4R^n$. Hence

$$\overline{\lim_{n\to\infty}} \{F_n(R)\}^{1/n} \leqslant R.$$

But $\omega_n(1) \geqslant 2^n - n$, if n belongs to the sequence $n_1, n_2, \ldots$; therefore

$$\overline{\lim_{n\to\infty}} \{\omega_n(1)\}^{1/n} \geqslant 2.$$

Example 2. There are cases in which $\overline{\lim_{n\to\infty}} N_n^{1/n} > 1$ and

$$\lim_{n\to\infty} \{F_n(R)\}^{1/n} = \lim_{n\to\infty} \{\omega_n(R)\}^{1/n} = R$$

for all R.

Choose a sequence $n_1, n_2, \ldots$ of integers such that

$$n_{t+1} = 2^{n_t}+1 \quad (t = 1, 2, \ldots).$$

Let $$a_t = (n_t)^{2^{n_t}}, \quad b_t = (2^{n_t})^{2^{n_t}},$$

and put $$p_{n_t}(z) = a_t(z^{n_t+1}+z^{n_t+2}+\ldots+z^{2^{n_t}})+b_t z^{n_t},$$

$$p_n(z) = a_t z^n \quad (n_t < n \leqslant 2^{n_t}).$$

The polynomials defined above for $t = 1, 2, \ldots$ together with the polynomials $p_n(z) = z^n$ $(n = 0, 1, \ldots, n_1 - 1)$ form a basic set in which

$$\overline{\lim_{n\to\infty}} N_n^{1/n} = 2;$$

for

$$z^{n_t} = \frac{1}{b_t}\left[\{a_t(z^{n_t+1}+z^{n_t+2}+\ldots+z^{2^{n_t}})+b_t z^{n_t}\}-a_t z^{n_t+1}-a_t z^{n_t+2}-\ldots-a_t z^{2^{n_t}}\right].$$

Given any R, then for all t such that $n_t \geqslant R$,

$$F_{n_t}(R) \leqslant \omega_{n_t}(R) \leqslant \frac{a_t}{b_t} 2^{n_t+1}(n_t)^{2^{n_t}} + R^{n_t}$$

$$< 2R^{n_t}$$

for t sufficiently great.

For all remaining n,

$$F_n(R) = \omega_n(R) = R^n.$$

Hence $$\lim_{n\to\infty} \{F_n(R)\}^{1/n} = \lim_{n\to\infty} \{\omega_n(R)\}^{1/n} = R$$

for all R.

4. *The condition* $D_n/n \to 1$.

LEMMA 4. *If* (1.9) *is satisfied, then*

$$\frac{1}{r} \overline{\lim_{n\to\infty}} \{\omega_n(r)\}^{1/n} \leqslant \frac{1}{R} \overline{\lim_{n\to\infty}} \{\omega_n(R)\}^{1/n} \quad (r \geqslant R > 0). \quad (4.1)$$

For, if $p_t(z)$ is the polynomial of degree D_n in the representation of z^n in terms of the basic polynomials, then,

$$M_t(r) \leqslant |p_{t0}| + |p_{t1}| r + \ldots + |p_{tD_n}| r^{D_n}$$

$$\leqslant (|p_{t0}| + |p_{t1}| R + \ldots + |p_{tD_n}| R^{D_n}) \left(\frac{r}{R}\right)^{D_n} \quad (r \geqslant R > 0).$$

Now $$|p_{ts}| = \frac{1}{2\pi} \left| \int_{|w|=R} \frac{p_t(w)\, dw}{w^{s+1}} \right| \leqslant \frac{M_t(R)}{R^s}.$$

Hence $$M_t(r) \leqslant (D_n + 1) M_t(R) \left(\frac{r}{R}\right)^{D_n},$$

so that $$\omega_n(r) \leqslant (D_n + 1) \left(\frac{r}{R}\right)^{D_n} \omega_n(R),$$

or $$\frac{\omega_n(r)}{r^n} \leqslant (D_n + 1) \left(\frac{r}{R}\right)^{D_n - n} \frac{\omega_n(R)}{R^n}.$$

Since $D_n/n \to 1$, (4.1) follows at once.

Proof of Theorem 2. Since the basic series is assumed *simple*, (1.9) [and hence (4.1)] is satisfied. If the hypothesis of Lemma 2 is satisfied for some $R > 0$, *i.e.* if the basic series represents in and on $|z| = R$ every function which is regular in and on this circle then (4.1), (2.1) give

$$\frac{1}{r} \{\omega_n(r)\}^{1/n} \to 1 \quad \text{as} \quad n \to \infty \quad (r \geqslant R).$$

The truth of the theorem now follows easily from the last result and

Lemma 3. For, if in and on $|z| = R$ the basic series represents every function which is regular in and on this circle, it represents any such function in and on $|z| = r_1$ $(r_1 \geqslant R)$ and so in its whole circle of regularity.

Proof of Theorem 3. The hypothesis of the theorem implies that $\{\omega_n(R)\}^{1/n} \to R$ as $n \to \infty$ $(0 \leqslant R < \lambda)$; for, if for R_0 $(0 \leqslant R_0 < \lambda)$,

$$\overline{\lim}\, \{\omega_n(R_0)\}^{1/n} > R_0 \quad \text{as} \quad n \to \infty,$$

then, by Lemma 2, there is a function regular in and on $|z| = R_0$, and so in $|z| < R_1$, where $\lambda > R_1 > R_0$, and the corresponding basic series fails to represent this on $|z| = R_0$. This function would not be represented in its circle of regularity by its basic series, contrary to our hypothesis.

From (4.1), (2.2), we have

$$\{\omega_n(R)\}^{1/n} \to R \quad \text{as} \quad n \to \infty \quad (R \geqslant 0)$$

and the truth of the theorem follows from Theorem 1.

The condition $D_n/n \to 1$ is only a sufficient condition, for consider the following basic set:

$$p_0(z) = 1, \quad p_{2n}(z) = z^{2n} \ (n \geqslant 1), \quad p_{2n+1}(z) = \{(2n+1)z\}^{2n+1} + z^{4n+2} \ (n \geqslant 0).$$

Now
$$z^{2n+1} = (2n+1)^{-(2n+1)} [\{(2n+1)z\}^{2n+1} + z^{4n+2} - z^{4n+2}]$$
$$= (2n+1)^{-(2n+1)} \{p_{2n+1}(z) - p_{4n+2}(z)\}.$$

Therefore
$$\omega_{2n+1}(R) \leqslant R^{2n+1} + 2(2n+1)^{-2n+1} R^{4n+2},$$

so that
$$\lim_{n\to\infty} \{\omega_{2n+1}(R)\}^{1/(2n+1)} \leqslant R \quad \text{for all } R.$$

Also
$$\omega_{2n}(R) = R^{2n} \quad \text{for all } R,$$

whence
$$\lim_{n\to\infty} \{\omega_{2n}(R)\}^{1/2n} = R \quad \text{for all } R.$$

For this basic set, the equality $\lim_{n\to\infty} \{\omega_2(R)\}^{1/n} = R$, is true for all R, by (2.2), although

$$\overline{\lim_{n\to\infty}} \frac{D_n}{n} = \lim_{n\to\infty} \frac{4n+2}{2n+1} = 2.$$

Theorem 2 does not imply that if a function has a radius of regularity less than or equal to R, then the basic series represents this function in its whole circle of regularity. In fact, it may not be represented in any circle at all.

Consider the basic set $p_0(z) = 1$, $p_n(z) = 1+z^n$ $(n \geqslant 1)$ so that

$$z^n = -1+(1+z^n) = -p_0(z)+p_n(z).$$

If $|z| = R \geqslant 1$, then $\omega_n(R) \leqslant 3R^n$; so that

$$\overline{\lim_{n\to\infty}} \{\omega_n(R)\}^{1/n} \leqslant R \quad (R \geqslant 1).$$

Hence, the basic series represents in the whole circle of regularity any function which is regular in and on the unit circle.

But, if $R < 1$, then $\omega_n(R) \geqslant 1$, and

$$\underline{\lim_{n\to\infty}} \{\omega_n(R)\}^{1/n} \geqslant 1 \quad (R < 1);$$

so that the basic series (which is a *simple* series) need not represent in any circle functions whose circles of regularity lie inside the unit circle.

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ON TRIGONOMETRICAL SERIES WITH MIXED CONDITIONS

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1. *Introduction.*

In this paper we determine values of A_n to satisfy the equations

$$\cos m\theta = \sum_{n=0}^{\infty} A_n \cos n\theta \quad (0 < \theta < \tfrac{1}{2}\pi),$$

$$-\sin m\theta = \sum_{n=1}^{\infty} A_n \sin n\theta \quad (\tfrac{1}{2}\pi < \theta < \pi),$$

where m is a positive integer and we show that the representation in this form is valid.

The particular case where $m = 1$ is applied in § 5 of the paper to the solution of two problems in electrostatics.

2. *The solution for even values of* $m\,(= 2k)$.

Let $A_{2n,k}$ be a set of constants such that

$$\sum_{n=0}^{\infty} |A_{2n,k}| < \infty. \tag{1}$$

Consider the function

$$f(\theta) = \cos 2k\theta - \sum_{n=0}^{\infty} A_{2n,k} \cos 2n\theta, \tag{2}$$

defined as the sum of the (absolutely and) uniformly convergent trigonometrical series on the right; it is continuous and (2) is its Fourier series in the range $-\pi < \theta < \pi$.

Define $$f^*(\theta) = f(\theta) \qquad (-\tfrac{1}{2}\pi < \theta < \tfrac{1}{2}\pi),$$

$$= -f(\theta) \quad (-\pi < \theta < -\tfrac{1}{2}\pi \text{ and } \tfrac{1}{2}\pi < \theta < \pi).$$

† I am much indebted to Dr. E. H. Linfoot and to Prof. G. Temple for helpful suggestions.

Then $f^*(\theta)$ is continuous in $-\pi < \theta < \pi$ except possibly at $\pm\frac{1}{2}\pi$, and so possesses a Fourier series over this range, which is of the form

$$\sum_{n=0}^{\infty} \alpha_{2n+1,k} \cos(2n+1)\theta.$$

This Fourier series converges (by the Riemann localisation theorem) at every point inside the range $-\frac{1}{2}\pi < \theta < \frac{1}{2}\pi$ to the sum $f^*(\theta)$ $[=f(\theta)]$. Hence, at every point in the range $-\frac{1}{2}\pi < \theta < \frac{1}{2}\pi$,

$$\sum_{n=0}^{\infty} \alpha_{2n+1,k} \cos(2n+1)\theta = \cos 2k\theta - \sum_{n=0}^{\infty} A_{2n,k} \cos 2n\theta, \tag{3}$$

where the actual values of the $\alpha_{2n+1,k}$ are given by

$$\begin{aligned} \alpha_{2n+1,k} &= \frac{1}{\pi} \int_{-\pi}^{+\pi} f^*(\theta) \cos(2n+1)\theta \, d\theta \\ &= \frac{4}{\pi} \int_{0}^{\frac{1}{2}\pi} f(\theta) \cos(2n+1)\theta \, d\theta. \end{aligned} \tag{4}$$

Similarly we define the function

$$g(\theta) = -\sin 2k\theta - \sum_{n=1}^{\infty} A_{2n,k} \sin 2n\theta$$

and the modified function

$$\begin{aligned} g^*(\theta) &= -g(\theta) \quad (-\tfrac{1}{2}\pi < \theta < \tfrac{1}{2}\pi), \\ &= g(\theta) \qquad (-\pi < 0 < -\tfrac{1}{2}\pi, \quad \tfrac{1}{2}\pi < \theta < \pi). \end{aligned}$$

It follows that $g^*(\theta)$ possesses a Fourier series in the range $-\pi < \theta < \pi$ of the form

$$\sum_{n=0}^{\infty} \beta_{2n+1,k} \sin(2n+1)\theta,$$

which, by the same argument as before, converges to $g(\theta)$ in $-\pi < \theta < -\frac{1}{2}\pi$ and $\frac{1}{2}\pi < \theta < \pi$. Hence for these values of θ

$$\sum_{n=0}^{\infty} \beta_{2n+1,k} \sin(2n+1)\theta = -\sin 2k\theta - \sum_{n=1}^{\infty} A_{2n,k} \sin 2n\theta, \tag{5}$$

where the actual values of $\beta_{2n+1,k}$ are given by

$$\beta_{2n+1,k} = \frac{4}{\pi} \int_{\frac{1}{2}\pi}^{\pi} g(\theta) \sin(2n+1)\theta \, d\theta. \tag{6}$$

If now it is possible, by a suitable choice of the $A_{2n,k}$, subject to the condition (1), to secure that

$$\alpha_{2n+1,k} = \beta_{2n+1,k} \quad (n = 0, 1, 2, \ldots),$$

and also $$\sum_{n=0}^{\infty} |\alpha_{2n+1,k}| < \infty, \tag{7}$$

then, by taking $$A_{2n+1,k} = \alpha_{2n+1,k} = \beta_{2n+1,k}, \tag{8}$$

equations (3) and (5) become

$$\left.\begin{aligned} \cos 2k\theta - \sum_{n=0}^{\infty} A_{2n,k} \cos 2n\theta &= \sum_{n=0}^{\infty} A_{2n+1,k} \cos(2n+1)\theta \quad (0 < \theta < \tfrac{1}{2}\pi), \\ -\sin 2k\theta - \sum_{n=1}^{\infty} A_{2n,k} \sin 2n\theta &= \sum_{n=0}^{\infty} A_{2n+1,k} \sin(2n+1)\theta \quad (\tfrac{1}{2}\pi < \theta < \pi), \end{aligned}\right\} \tag{9}$$

i.e.
$$\cos 2k\theta = \sum_{n=0}^{\infty} A_{n,k} \cos n\theta \quad (0 < \theta < \tfrac{1}{2}\pi), \tag{10}$$

$$-\sin 2k\theta = \sum_{n=1}^{\infty} A_{n,k} \sin n\theta \quad (\tfrac{1}{2}\pi < \theta < \pi), \tag{11}$$

where $$A_{2r+1,k} = \frac{4}{\pi} \int_0^{\frac{1}{2}\pi} f(\theta) \cos(2r+1)\theta \, d\theta$$

and $$A_{2r+1,k} = \frac{4}{\pi} \int_{\frac{1}{2}\pi}^{\pi} g(\theta) \sin(2r+1)\theta \, d\theta.$$

Subtracting and adding these equations, we obtain

$$\frac{(-1)^{k+1}}{2k+2r+1} = \sum_{n=0}^{\infty} \frac{(-1)^n A_{2n,k}}{2n-2r-1}, \tag{12}$$

and $$\tfrac{1}{2}\pi A_{2r+1,k} = \frac{(-1)^{r+k+1}}{2k-2r-1} - \sum_{n=0}^{\infty} \frac{(-1)^{n+r} A_{2n,k}}{2r+2n+1} \tag{13}$$

$$(r = 0, 1, 2, \ldots).$$

We must solve the equations (12) and (13) subject to conditions (1) and (7). First consider the set of equations

$$\frac{(-1)^{k+1}}{2k+2r+1} = \sum_{n=0}^{s} \frac{(-1)^n A_{2n,k}}{2n-2r-1},$$

for the values $r = 0, 1, 2, \ldots s$.

This set of equations was solved for the values $s = 0, 1, 2, \ldots, 6$ and from the results the solution in the general case was obtained. The deductions from this solution are later verified. The solution obtained is given by

$$A_{2n,k} = (-1)^{k+n} \frac{1 \cdot 3 \ldots (2s-2n+1) \cdot 1 \cdot 3 \ldots (2n+1)(k+s)! \, 1 \cdot 3 \ldots (2k-1)}{(s-n)! \, n! \, (2n+1)(k-1)! \, (k+n) \cdot 1 \cdot 3 \ldots (2k+2s+1)}$$

$$(n = 0, 1, 2, \ldots, s). \tag{14}$$

Denote $\dfrac{1.3.5\ldots(2n-1)}{2.4.6\ldots 2n}$ by $[n]$ and let $[0]=1$; then, on making $s\to\infty$ in equation (14), we obtain formally

$$A_{2n,k}=(-1)^{n+k}[n][k]\frac{2k}{2k+2n}. \tag{15}$$

Since $A_{2n,k}=O(n^{-\frac{3}{2}})$ when $n\to\infty$, it follows that condition (1) is satisfied. We must now verify that these values of $A_{2n,k}$ really are solutions of equation (12), *i.e.* that

$$\sum_{n=0}^{\infty}\frac{[n][k]\,2k}{(2r-2n+1)(2k+2n)}=\frac{1}{2k+2r+1} \tag{16}$$

$$(r=0,\ 1,\ 2,\ \ldots).$$

We have

$$\sum_{n=0}^{\infty}\frac{[n][k]\,2k}{(2r-2n+1)(2k+2n)}=\frac{2k[k]}{2k+2r+1}\left\{\sum_{n=0}^{\infty}\frac{[n]}{2k+2n}+\sum_{n=0}^{\infty}\frac{[n]}{2r-2n+1}\right\}, \tag{17}$$

since both the series on the right-hand side are absolutely convergent.

Now

$$(1-z^2)^{-\frac{1}{2}}z^{2k-1}=\sum_{n=0}^{\infty}[n]z^{2n+2k-1}\quad(|z|<1),$$

so that

$$\sum_{n=0}^{\infty}\frac{[n]}{2n+2k}=\int_0^1(1-z^2)^{-\frac{1}{2}}z^{2k-1}\,dz=\frac{1}{2k[k]}\quad(k=1,\ 2,\ 3,\ \ldots). \tag{18}$$

Again

$$(1-z^2)^{-\frac{1}{2}}z^{-2r-2}=\sum_{n=0}^{\infty}[n]z^{2n-2r-2}\quad(|z|<1),$$

so that

$$\sum_{n=r+1}^{\infty}[n]z^{2n-2r-2}=(1-z^2)^{-\frac{1}{2}}z^{-2r-2}-\sum_{n=0}^{r}[n]z^{2n-2r-2}.$$

Each side of this equation is analytic within and on the closed contour formed by that half of the circle $|z|=1$ which lies above the real axis, and the portion $-1<z<1$ of the real axis, except at the points $z=\pm1$. Indenting the contour at these points and making the size of the indentations tend to zero, we easily find that

$$\int_{-1}^{+1}\left\{\sum_{n=r+1}^{\infty}[n]x^{2n-2r-2}\right\}dx$$

$$=-\int_0^{\pi}\left\{(1-e^{2i\theta})^{-\frac{1}{2}}e^{-2(r+1)i\theta}-\sum_{n=0}^{r}[n]e^{2(n-r-1)i\theta}\right\}ie^{i\theta}\,d\theta,$$

i.e.

$$2\sum_{n=0}^{\infty}\frac{[n]}{2n-2r-1}=-\int_0^{\pi}(-2i\sin\theta)^{-\frac{1}{2}}e^{-(2r+\frac{3}{2})i\theta}i\,d\theta$$

$$=2^{-\frac{1}{2}}i^{\frac{3}{2}}\int_0^{\pi}\{\cos(2r+\tfrac{3}{2})\theta-i\sin(2r+\tfrac{3}{2})\theta\}(\sin\theta)^{-\frac{1}{2}}d\theta. \tag{19}$$

By means of reduction formulae it can be shown that

$$\int_0^{\pi} \begin{matrix}\cos\\ \sin\end{matrix} (2r+\tfrac{3}{2})\theta\,(\sin\theta)^{-\frac{1}{2}}\,d\theta = 0.$$

Equation (19) then gives

$$\sum_{n=0}^{\infty} \frac{[n]}{2n-2r-1} = 0 \quad (r = 0,\ 1,\ 2,\ \ldots). \tag{20}$$

From equations (17), (18), and (20) it follows that equation (16) is verified and the solution of equations (12), given by equation (15), is valid. It now remains to find $A_{2n+1,\,k}$ from equations (13) and (15). On substituting the values of $A_{2n,\,k}$ from (15) in (13), we obtain

$$\begin{aligned}\tfrac{1}{2}\pi A_{2r+1,\,k} &= \frac{(-1)^{r+k+1}}{2k-2r-1} - \sum_{n=0}^{\infty} \frac{(-1)^{r+k}\,[n][k]\,2k}{(2k+2n)(2r+2n+1)} \\ &= \frac{(-1)^{r+k+1}}{2k-2r-1} + (-1)^{r+k}\,\frac{2k[k]}{2k-2r-1} \\ &\qquad \times \left\{ \sum_{n=0}^{\infty} \frac{[n]}{2k+2n} - \sum_{n=0}^{\infty} \frac{[n]}{2r+2n+1} \right\}. \end{aligned} \tag{21}$$

But from equation (18)

$$\sum_{n=0}^{\infty} \frac{[n]}{2k+2n} = \frac{1}{2k[k]} \quad (k = 1,\ 2,\ 3,\ \ldots),$$

and in a similar way we find that

$$\sum_{n=0}^{\infty} \frac{[n]}{2r+2n+1} = \tfrac{1}{2}\pi[r] \quad (r = 0,\ 1,\ 2,\ \ldots). \tag{22}$$

These two results with equation (21) give

$$A_{2n+1,\,k} = (-1)^{n+k}\,\frac{[n][k]\,2k}{2n-2k+1}. \tag{23}$$

Since $A_{2n+1,\,k} = O(n^{-\frac{3}{2}})$ when $n \to \infty$, condition (7) is satisfied. The solution is now completely given by equations (15) and (23).

3. *The solution for odd values of* $m\,(= 2k+1)$.

In a similar way we can find the values of the constants $A_{n,\,k}$ such that

$$\cos(2k+1)\theta = \sum_{n=0}^{\infty} A_{n,\,k} \cos n\theta \quad (0 < \theta < \tfrac{1}{2}\pi), \tag{24}$$

$$-\sin(2k+1)\theta = \sum_{n=1}^{\infty} A_{n,\,k} \sin n\theta \quad (\tfrac{1}{2}\pi < \theta < \pi). \tag{25}$$

The solution depends on the same set of linear equations and the coefficients are

$$A_{2n+1,\,k} = (-1)^{k+n+1}\,\frac{[n][k](2k+1)}{2k+2n+2}, \tag{26}$$

$$A_{2n,\,k} = (-1)^{k+n+1}\,\frac{[n][k](2k+1)}{2n-2k-1}. \tag{27}$$

4. *An extension of the results.*

Suppose that the function $V(\theta)$ is defined in the range $0 < \theta < \frac{1}{2}\pi$ and that the function $U(\theta)$ is defined in the range $\frac{1}{2}\pi < \theta < \pi$. Then the object of this section of the paper is to show that, under certain conditions, the coefficients A_n can be found such that

$$V(\theta) = \sum_0^\infty A_n \cos n\theta \quad (0 < \theta < \tfrac{1}{2}\pi), \tag{28}$$

$$U(\theta) = \sum_1^\infty A_n \sin n\theta \quad (\tfrac{1}{2}\pi < \theta < \pi). \tag{29}$$

Let us suppose first of all that $V(\theta)$ and $U(\theta)$ are defined by equations of the forms

$$V(\theta) = \sum_{k=0}^{N} 2\alpha_k \cos 2k\theta \quad (0 < \theta < \tfrac{1}{2}\pi),$$

$$U(\theta) = \sum_{k=1}^{N} 2\beta_k \sin 2k\theta \quad (\tfrac{1}{2}\pi < \theta < \pi).$$

These may be written

$$V(\theta) = \sum_{k=0}^{N} (\alpha_k+\beta_k) \cos 2k\theta + \sum_{k=0}^{N} (\alpha_k-\beta_k) \cos 2k\theta \quad (0 < \theta < \tfrac{1}{2}\pi),$$

$$U(\theta) = \sum_{k=1}^{N} (\alpha_k+\beta_k) \sin 2k\theta - \sum_{k=1}^{N} (\alpha_k-\beta_k) \sin 2k\theta \quad (\tfrac{1}{2}\pi < \theta < \pi),$$

where $\beta_0 = \alpha_0$; and, on using equations (10) and (11), we obtain

$$V(\theta) = \sum_{k=0}^{N} (\alpha_k+\beta_k) \cos 2k\theta + \sum_{n=0}^{\infty} \sum_{k=0}^{N} (\alpha_k-\beta_k) A_{n,\,k} \cos n\theta \quad (0 < \theta < \tfrac{1}{2}\pi),$$

$$U(\theta) = \sum_{k=1}^{N} (\alpha_k+\beta_k) \sin 2k\theta + \sum_{n=1}^{\infty} \sum_{k=1}^{N} (\alpha_k-\beta_k) A_{n,\,k} \sin n\theta \quad (\tfrac{1}{2}\pi < \theta < \pi).$$

These are expansions of the forms (28) and (29).

The corresponding result with N replaced by ∞ will certainly hold if the series $\sum_{n,\,k} (\alpha_k-\beta_k) A_{n,\,k}$ is absolutely convergent. Now

$$\sum_{n=0}^{\infty} |A_{2n,\,k}| = 2k[k] \sum_{n=0}^{\infty} \frac{[n]}{2k+2n},$$

and by equation (18) this becomes

$$\sum_{n=0}^{\infty} |A_{2n,k}| = 1.$$

Again

$$\sum_{n=0}^{\infty} |A_{2n+1,k}| = 2k[k]\left\{\sum_{n=0}^{k-1} \frac{[n]}{2k-1-2n} + \sum_{n=k}^{\infty} \frac{[n]}{2n-2k+1}\right\},$$

but, by equation (20),

$$\sum_{n=0}^{k-1} \frac{[n]}{2k-1-2n} = \sum_{n=k}^{\infty} \frac{[n]}{2n-2k+1},$$

so that

$$\begin{aligned}\sum_{n=0}^{\infty} |A_{2n+1,k}| &= 4k[k] \sum_{n=k}^{\infty} \frac{[n]}{2n-2k+1} \\ &= 4k[k] \sum_{n=0}^{\infty} \frac{[n+k]}{2n+1} \\ &< 4k[k] \sum_{n=0}^{\infty} \frac{[n]}{2n+1},\end{aligned}$$

and, on using equation (22), we obtain

$$\sum_{n=0}^{\infty} |A_{2n+1,k}| < 2\pi k[k].$$

It follows that sufficient conditions for the validity of the interchange of the order of summation are that the series

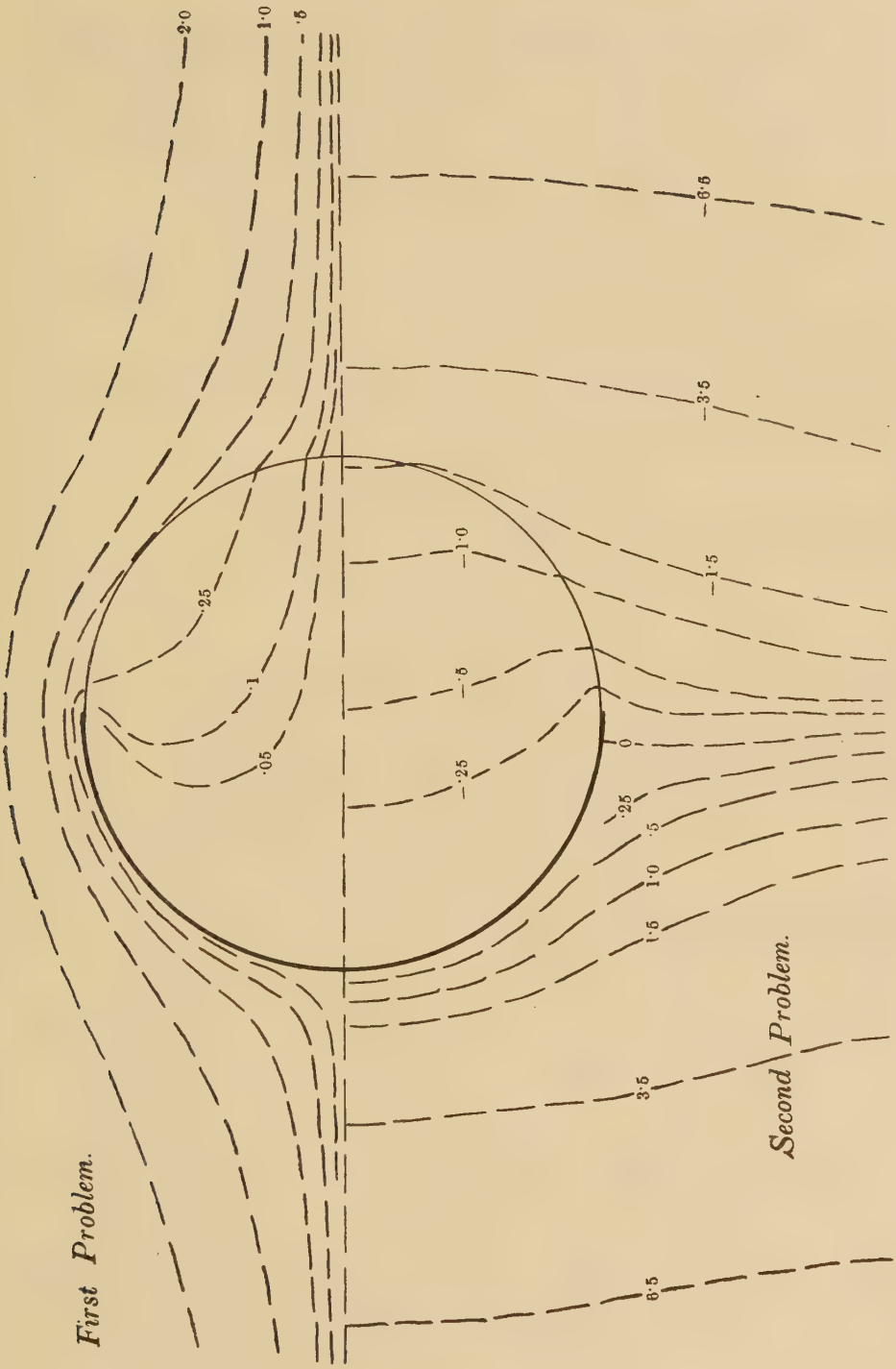
$$\sum_{k=1}^{\infty} |\alpha_k - \beta_k| \quad \text{and} \quad \sum_{k=1}^{\infty} k[k]|\alpha_k - \beta_k|$$

should be convergent. The second condition includes the first and is itself equivalent to the convergence of

$$\sum_{k=1}^{\infty} k^{\frac{1}{2}}|\alpha_k - \beta_k|.$$

5. *An application.*

To illustrate the method of application of the foregoing results, two problems of two-dimensional electrostatics have been solved. These problems are of the simplest type to which the method applies, but the solution of more complex problems can be obtained in a similar way. The diagram shows the cross-section of a circular cylinder of dielectric of specific inductive capacity K, the left-hand half of which is covered by a thin conducting layer connected to earth. Taking polar coordinates in



Equipotential lines. The values of $7U/2$ are marked on each line.

the plane of the cross-section with the pole at the centre; the cylinder occupies the region $r < 1$, and, in the first problem, the conducting layer is defined by $r = 1$, $\frac{1}{2}\pi < \theta < 3\pi/2$. The system is placed in a uniform electric field of which the undisturbed potential is $-Vr\sin\theta$. If the potential at points outside the cylinder is U_0 and that inside is U_i, we may write

$$U_0 = -Vr\sin\theta + \sum_{n=1}^{\infty} C_n r^{-n}\sin n\theta, \tag{30}$$

$$U_i = -Vr\sin\theta + \sum_{n=1}^{\infty} C_n r^{n}\sin n\theta. \tag{31}$$

These values make the potential continuous across the surface $r = 1$ and U_0 takes the appropriate values when r is large. Since the conductor is at zero potential we must have

$$\sum_{n=1}^{\infty} C_n \sin n\theta = V\sin\theta \quad (\tfrac{1}{2}\pi < \theta < \pi). \tag{32}$$

On the boundary of the dielectric we have

$$K\frac{\partial U_i}{\partial r} = \frac{\partial U_0}{\partial r},$$

i.e.
$$(1+K)\sum_{n=1}^{\infty} C_n n\sin n\theta = (K-1)V\sin\theta \quad (0 < \theta < \tfrac{1}{2}\pi).$$

When the values of C_n have been determined it will be found that this series is uniformly convergent so that the condition is equivalent to

$$\sum_{n=0}^{\infty} C_n \cos n\theta = \frac{K-1}{K+1} V\cos\theta \quad (0 < \theta < \tfrac{1}{2}\pi) \tag{33}$$

where C_0 is the constant of integration.

We may rewrite equations (33) and (32) in the forms

$$\sum_{n=0}^{\infty} C_n \cos n\theta = \frac{K}{K+1} V\cos\theta - \frac{1}{K+1} V\cos\theta \quad (0 < \theta < \tfrac{1}{2}\pi), \tag{34}$$

$$\sum_{n=1}^{\infty} C_n \sin n\theta = \frac{K}{K+1} V\sin\theta + \frac{1}{K+1} V\sin\theta \quad (\tfrac{1}{2}\pi < \theta < \pi). \tag{35}$$

If we put $k = 0$ in equations (24) and (25) and substitute the results in equations (34) and (35) we obtain

$$\sum_{n=0}^{\infty} C_n \cos n\theta = \frac{K}{K+1} V\cos\theta - \frac{1}{K+1}\sum_{n=0}^{\infty} A_{n,0}\cos n\theta \quad (0 < \theta < \tfrac{1}{2}\pi),$$

$$\sum_{n=1}^{\infty} C_n \sin n\theta = \frac{K}{K+1} V\sin\theta - \frac{1}{K+1}\sum_{n=1}^{\infty} A_{n,0}\sin n\theta \quad (\tfrac{1}{2}\pi < \theta < \pi),$$

where the coefficients $A_{n,0}$ are given by equations (26) and (27). It now follows that the coefficients C_n of equations (30) and (31) are given by

$$C_1 = \frac{(2k+1)V}{2(V+1)}, \quad C_2 = -\frac{V}{2(V+1)},$$

$$C_{2n-1} = -C_{2n} = (-1)^{n+1}\frac{V}{K+1}\frac{1.3....(2n-3)}{2.4...2n} \quad (n = 2, 3, 4, ...).$$

With the help of the series so formed, some of the equipotential lines have been drawn in the upper half of the diagram, for the value $K = 6$.

In the second problem we take the conducting layer to be defined by $r = 1$, $-\frac{1}{2}\pi < \theta < \frac{1}{2}\pi$ and the undisturbed potential to be $Vr\cos\theta$+constant. The potentials outside and inside are now given by

$$U_0 = Vr\cos\theta + \sum_{n=0}^{\infty} D_n r^{-n}\cos n\theta,$$

$$U_i = Vr\cos\theta + \sum_{n=0}^{\infty} D_n r^{n}\cos n\theta.$$

Proceeding in the same way as before we find that

$$D_0 = -\frac{V}{K+1}, \quad D_1 = \frac{(1-2K)V}{2(1+K)}, \quad D_2 = -\frac{V}{2(K+1)},$$

$$D_{2n-1} = -D_{2n} = (-1)^{n+1}\frac{V}{K+1}\frac{1.3....(2n-3)}{2.4...2n} \quad (n = 2, 3, 4, ...).$$

Some of the equipotential lines for this problem have been drawn in the lower half of the diagram, again for the value $K = 6$.

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RELATIVISTIC HYDRODYNAMICS

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PART I. *Introduction.*

1. *The problem.*

Although the energy-tensor for a perfect fluid in general relativity and the partial differential equations satisfied by it have been known for a long time, there does not appear to have been any systematic attempt to develop

a hydrodynamical theory in general relativity analogous to classical hydrodynamics†. Such a theory would hardly find useful physical application in the discussion of terrestrial phenomena, but should occupy a basic position in the treatment of rotating masses of stellar dimensions or in the discussion of very large-scale phenomena, in which a scattered distribution of stars and nebulae is replaced by a fictitious continuous medium, treated as a perfect fluid.

The relativistic theory of Schwarzschild for the interior of the sun belongs to hydrodynamics, and so does the theory of the expanding universe. But for the most part the hydrodynamical aspects of these problems are very simple, owing to assumptions regarding symmetry or statical character.

Classical hydrodynamics splits sharply into two divisions—*kinematics* and *kinetics*, the first dealing with the geometry of the motion of a continuous medium, and the second with the consequences of Newton's laws of motion. Although in general relativity the geometry of space-time is conditioned by the distribution of matter in it, and that distribution depends on the dynamical laws embodied in the field equations, it is possible to split our subject along similar lines.

By *kinematics* in relativistic hydrodynamics we understand the geometry of a congruence of world-lines of flow, or stream-lines, in a Riemannian space of four dimensions, whose metric is general, not conditioned in any particular way by the presence of matter. The field equations are, in fact, entirely left out of account. By *kinetics* we understand the theory consequent on the introduction of the field equations.

But kinetics in relativistic hydrodynamics may itself be subdivided into two parts. The field equations,

$$R_{ij} - \tfrac{1}{2} g_{ij} R - \lambda g_{ij} = -\kappa T_{ij}, \tag{1.1}$$

are consistent with, and imply, the equations of conservation,

$$T^{ij}_{\cdots,j} = 0. \tag{1.2}$$

We may study the consequences of (1.2) before introducing (1.1). This procedure has certain advantages. The theory consequent on the equations of conservation is simpler than that which involves consideration of the field equations. Moreover there is reason to believe that the equations of conservation are more fundamental than the field equations themselves.

† Some general results have been developed by L. P. Eisenhart, *Trans. American Math. Soc.*, 26 (1924), 205–220; this paper will be referred to as E. A few of the results of the present paper were given in E, as will be indicated in the appropriate places.

Indeed the field equations were deliberately constructed to satisfy the equations of conservation, and, when modified by the addition of the cosmological term, were still left in a form consistent with the equations of conservation. In the present paper we shall make no use of the field equations (1.1), basing the arguments entirely on the equations of conservation (1.2).

In Parts II, III, and IV the space-time forming the background for the theory is to be regarded as a general Riemannian space of four dimensions with an indefinite line-element with signature $+++-$. A surprising number of concepts familiar in classical hydrodynamics permit easy generalization, as a survey of the Table of Contents shows.

In Part V space-time is regarded as flat. The most remarkable result is that concerning the velocity of propagation of a small irrotational disturbance in a compressible fluid possessing a pressure-density equation; this velocity is found to tend to infinity as the fluid tends to incompressibility, a result which appears to indicate a lower limit to the compressibility of a compressible fluid, since it should not be possible to obtain a signal with a velocity exceeding that of light.

2. *Notation and fundamental auxiliary formulae.*

Latin suffixes having the range 1, 2, 3, 4 (and Greek suffixes the range 1, 2, 3), the fundamental form in space-time is denoted by $g_{ij}dx^i dx^j$ with the usual summation convention. Covariant derivatives are denoted by commas.

The indicator ϵ of a vector v^i is a number chosen equal to ± 1 to make $\epsilon g_{ij}v^i v^j$ positive. The signature of the fundamental form is taken to be† $+++-$; a space-like vector has therefore an indicator $+1$, and a time-like vector an indicator -1.

A sub-manifold (curve, 2-space or 3-space) of space-time is said to be *space-like* if at every point of the sub-manifold every infinitesimal displacement in the sub-manifold is space-like. It is obvious that every sub-manifold of a space-like sub-manifold is itself space-like. Every vector lying in a space-like sub-manifold has an indicator $+1$. A sub-manifold

† Many writers prefer to use the signature $---+$, but $+++-$ has some advantages. Formal simplicity may sometimes be increased by introducing imaginary coordinates to give the formal signature $++++$; the signature $+++-$ only requires one imaginary coordinate to do this, whereas $---+$ requires three. Moreover, probably the most complete compendium of relativistic formulae available is that given by Pauli, *Encyk. d. math. Wiss.*, V, 19 (1920), 539–775, and there the signature employed is $+++-$.

is said to be *time-like* if at every point it contains a time-like displacement.

Since they will be required several times in the work, two convenient forms† of the generalized Green-Stokes theorem in space-time will be noted here. Let V_2 be a 2-dimensional region, bounded by a closed curve V_1, on which a positive sense is assigned. Let $t^i_{(1)}$, $t^i_{(2)}$, $t^i_{(3)}$, $t^i_{(4)}$ be an orthogonal tetrad of unit vectors, defined at each point of V_2. The tetrad has the orientation of the parametric lines of the coordinates x^i; $t^i_{(1)}$, $t^i_{(2)}$ are tangent to V_2, and have in V_2 the same orientation as the positive sense of V_1, and the *outward* normal to V_1 in V_2; $t^i_{(3)}$, $t^i_{(4)}$ are, of course, normal to V_2, but they are otherwise arbitrary, except for orientation.

Let v_i be an arbitrary vector field in space-time. Then

$$\oint_{V_1} v_i dx^i = \int_{V_2} (v_{i,j} - v_{j,i})\, t^i_{(1)} t^j_{(2)}\, dS, \tag{2.1}$$

and

$$\oint_{V_1} v_i dx^i = -\int_{V_2} v_{i,j}\, \eta^{ijkl} t_{(3)k} t_{(4)l}\, \epsilon_3 \epsilon_4\, dS, \tag{2.2}$$

where ϵ_3, ϵ_4 are the indicators of $t^i_{(3)}$, $t^i_{(4)}$ respectively. The integrals round V_1 are taken in the positive sense, dS is a positive element of area, and

$$\eta^{ijkl} = -\epsilon_{ijkl}/(-g)^{\frac{1}{2}}, \tag{2.3}$$

ϵ_{ijkl} being the usual numerical permutation symbol and g the determinant of the fundamental tensor. The positive value of the radical is understood, here and throughout.

We note here for reference the Frenet formulae for a time-like curve, the signs being in accordance with the convention that all curvatures shall be positive‡. The unit tangent vector being λ^i, the first, second, and third unit normals being $\nu^i_{(1)}$, $\nu^i_{(2)}$, $\nu^i_{(3)}$, and the corresponding curvatures being $k_{(1)}$, $k_{(2)}$, $k_{(3)}$, we have

$$\begin{cases} \delta\lambda^i/\delta s = k_{(1)}\nu^i_{(1)}, \\ \delta\nu^i_{(1)}/\delta s = k_{(2)}\nu^i_{(2)} + k_{(1)}\lambda^i, \\ \delta\nu^i_{(2)}/\delta s = k_{(3)}\nu^i_{(3)} - k_{(2)}\nu^i_{(1)}, \\ \delta\nu^i_{(3)}/\delta s = -k_{(3)}\nu^i_{(2)}, \end{cases} \tag{2.4}$$

$\delta/\delta s$ indicating the absolute derivative, here and throughout the paper.

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† J. L. Synge, *Proc. Roy. Soc.* (A), 157 (1936), 434–443, equations (2.20), (2.21).

‡ J. L. Synge, *Proc. Int. Math. Cong.*, 1 (Toronto, 1924), 857–862.

PART II. *Kinematics.*

3. *Kinematical definitions.*

The history of a fluid defines in Riemannian space-time a congruence of time-like curves, the stream-lines†. Their derivation from the energy tensor is deferred to § 6, since the following kinematical theory belongs to the geometry of a congruence of time-like curves, and does not, in itself, involve the concepts of density and pressure. The stream-lines may be described by the contravariant components $\lambda^i (= dx^i/ds)$ of their unit tangent vectors, pointing into the future, considered as functions of the four space-time coordinates x^i. We call λ^i the *velocity-vector*. We note the identities

$$\lambda_i \lambda^i = -1, \quad \lambda_{i,j} \lambda^i = 0. \tag{3.1}$$

The *kinematical flow* along any curve in space-time, joining points A and B, is defined as the integral

$$\int_A^B \lambda_i dx^i, \tag{3.2}$$

taken along the curve, and the *kinematical circulation* in any closed curve or circuit is defined as

$$C = \oint \lambda_i dx^i, \tag{3.3}$$

the integral being taken round the circuit in an assigned sense.

The skew-symmetric *kinematical vorticity-tensor* is defined as

$$\omega_{ij} = \tfrac{1}{2}(\lambda_{j,i} - \lambda_{i,j}). \tag{3.4}$$

We note that, on account of (3.1),

$$2\omega_{ij} \lambda^j = -\lambda_{i,j} \lambda^j = -\delta\lambda_i/\delta s, \tag{3.5}$$

the absolute derivative being taken along the stream-line. Hence we have this formal expression of a well-known result:

THEOREM I. *A stream-line is a geodesic if, and only if, the equations*

$$\omega_{ij} \lambda^j = 0 \tag{3.6}$$

are satisfied along it.

† In classical hydrodynamics, stream-lines are of little importance except in the case of steady motion. But our present stream-lines are curves in space-time, and their physical importance is not limited by any such restriction.

With the notation of (2.3), the *conjugate kinematical vorticity-tensor* is defined as

$$\omega^{*ij} = \tfrac{1}{2}\eta^{ijkl}\omega_{kl}, \tag{3.7}$$

and the *kinematical vorticity-vector* is defined as

$$\omega^i = \omega^{*ij}\lambda_j = \tfrac{1}{4}\eta^{ijkl}\lambda_j(\lambda_{l,k} - \lambda_{k,l}). \tag{3.8}$$

It is obvious that this vector is perpendicular to the stream-line.

We say that a motion is *irrotational* in a certain region if throughout that region the kinematical vorticity-vector vanishes. But $\omega^i = 0$ are precisely the conditions of integrability of the total differential equation $\lambda_i dx^i = 0$. Hence we have this result:

THEOREM II. *A motion is irrotational in a certain region of space-time if, and only if, the stream-lines form a normal congruence in that region*†.

The *expansion* in a fluid is defined as the rate of increase of 3-volume of the normal cross-section of an infinitesimal tube of stream-lines, per unit 3-volume of normal cross-section. Hence, by a well-known formula‡, the expansion θ is given by the divergence of the velocity-vector,

$$\theta = \frac{1}{\sigma}\frac{d\sigma}{ds} = \lambda^i{}_{,i}, \tag{3.9}$$

σ being the infinitesimal 3-volume of normal cross-section.

4. *Theorems concerning kinematical circulation.*

Let Γ be a circuit, with assigned positive sense, bounding a 2-dimensional region V_2 which lies wholly in the portion of space-time occupied by by the fluid. Then, by (3.3) and (2.1), the kinematical circulation in Γ is

$$C = \oint_\Gamma \lambda_i dx^i = -2\int_{V_2} \omega_{ij} t^i_{(1)} t^j_{(2)}\, dS, \tag{4.1}$$

or by (2.2)

$$C = 2\int_{V_2} \omega^{*ij} t_{(3)i}\, t_{(4)j}\, \epsilon_3\, \epsilon_4\, dS. \tag{4.2}$$

Let us first consider the case where V_2 is composed of stream-lines. We may take $t^i_{(1)} = \lambda^i$, the velocity-vector; then $t^i_{(2)}$ is the unit tangent

† It might appear natural to define irrotational motion by the conditions $\omega_{ij} = 0$, but these would be too stringent, implying not only the normality of the congruence of stream-lines, but also their geodesic character.

‡ *Cf.* A. J. McConnell, *Annali di Matematica* 6 (1928–29), 224; J. L. Synge, *Trans. Roy. Soc. Canada*, III, 28 (1934), 171.

vector to the orthogonal trajectories in V_2 of the stream-lines. By (4.1) and (3.5) we have

$$C = -2\int_{V_2} \omega_{ij}\lambda^i t^j_{(2)}\, dS = -\int_{V_2} \frac{\delta\lambda_j}{\delta s}\, t^j_{(2)}\, dS. \tag{4.3}$$

By (2.4) this may also be written

$$C = -\int_{V_2} k_{(1)}\nu_{(1)j}\, t^j_{(2)}\, dS. \tag{4.4}$$

The following results are then obvious:

THEOREM III. *The kinematical circulation in any reducible† circuit drawn on a 2-space composed of stream-lines is zero if the stream-lines are geodesics.*

THEOREM IV. *If the kinematical circulation is zero in all circuits drawn on 2-spaces composed of stream-lines, then the stream-lines are geodesics.*

Let us now consider the case where V_2 is space-like. We shall apply (4.2), in which $t^i_{(3)}$, $t^i_{(4)}$ are unit vectors, normal to V_2 and to one another, but otherwise at our disposal, except for orientation. At each point of V_2 the velocity-vector λ^i may be resolved uniquely into a vector in V_2 and another vector normal to V_2: the latter will be a time-like vector, pointing into the future. Let us denote unit vectors in these directions by μ^i, $t^i_{(4)}$, respectively, and let $t^i_{(3)}$ be the unit vector perpendicular to $t^i_{(4)}$ and to V_2, with the proper orientation. This process defines $t^i_{(3)}$, $t^i_{(4)}$ uniquely (Fig. 1).

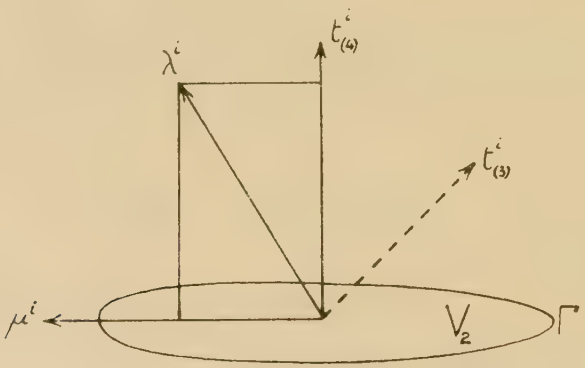

Fig. 1.

The diagram shows with full lines vectors lying in the tangent 3-space defined by λ^i and the tangent element to V_2, and by a broken line the vector $t^i_{(3)}$ normal to this 3-space.

† That is, reducible to a point by continuous deformation in space-time without leaving the 2-space of stream-lines.

We have then

$$\lambda^i = \alpha\mu^i + \beta t^i_{(4)} \quad (\alpha \geqslant 0,\ \beta > 0). \tag{4.5}$$

By virtue of the equations

$$\mu_i \mu^i = 1, \quad t_{(4)i} t^i_{(4)} = -1, \quad \mu_i t^i_{(4)} = 0, \quad \lambda_i \lambda^i = -1, \tag{4.6}$$

we deduce that

$$\alpha^2 - \beta^2 = -1, \quad \lambda_i \mu^i = \alpha. \tag{4.7}$$

Let us define the *obliquity* χ of λ^i to V_2 by

$$\sinh\chi = \lambda_i \mu^i = \alpha, \tag{4.8}$$

so that the obliquity vanishes if, and only if, λ^i is normal to V_2. Then

$$\beta = \cosh\chi, \tag{4.9}$$

and substitution from (4.5) in (4.2) gives, since $\epsilon_3 \epsilon_4 = -1$,

$$C = -2\int_{V_2} \omega^{*ij} t_{(3)i} (\lambda_j \operatorname{sech}\chi - \mu_j \tanh\chi)\, dS. \tag{4.10}$$

We may state the following result:

THEOREM V. *The kinematical circulation in a circuit* Γ *bounding a space-like 2-space* V_2 *is*

$$C = -2\int_{V_2} \omega^i t_{(3)i} \operatorname{sech}\chi\, dS + 2\int_{V_2} \omega^{*ij} t_{(3)i} \mu_j \tanh\chi\, dS, \tag{4.11}$$

where χ *is the obliquity of the stream-lines to* V_2, μ^i *a unit vector whose direction is that of the orthogonal projection of the velocity-vector on* V_2, *and* $t^i_{(3)}$ *a unit vector normal to* V_2 *and to the stream-line; the sense of* $t^i_{(3)}$ *is such that the orientation of the following tetrad in order is the same as that of the parametric lines of the coordinates*: (i) *the positive tangent to* Γ, (ii) *the outward normal to* Γ *in* V_2, (iii) $t^i_{(3)}$, (iv) λ^i, *the velocity-vector.*

If the motion is not irrotational, we cannot in general draw a 2-space normal to the stream-lines. Let us however draw any 3-space V_3 normal to the stream-line at a point A, and let us take in V_3 a 2-space V_2, containing A and bounded by a circuit Γ. Then, with the notation used above, we have at A

$$\chi = 0, \quad t^i_{(4)} = \lambda^i. \tag{4.12}$$

Hence we have this result as a deduction from (4.11):

THEOREM VI. *The principal part of the kinematical circulation in an infinitesimal circuit* Γ, *lying in a 2-space which cuts a stream-line normally at a point inside* Γ, *is*

$$C = -2\omega_{(n)}\,dS, \tag{4.13}$$

where dS *is the area enclosed by* Γ, *and*

$$\omega_{(n)} = \omega^i t_{(3)i}, \tag{4.14}$$

namely, the component of the kinematical vorticity-vector in the direction of the vector $t^i_{(3)}$, *which is perpendicular to the circuit and the stream-line and has a sense defined as in Theorem* V.

The above result generalizes the classical theorem of Stokes in hydrodynamics, the generalization holding, however, only for an infinitesimal circuit. The sense of description of the circuit is the opposite to that usually employed.

5. *Theorems concerning irrotational motion.*

Since, as has been shown, irrotationality implies the existence of 3-spaces normal to the congruence of stream-lines, we may state the following result:

THEOREM VII. *In an irrotational motion the kinematical flow is zero along any curve drawn in a 3-space normal to the congruence of stream-lines.*

Referring to (4.14), we note that, if $\omega_{(n)}$ vanishes for every unit vector $t^i_{(3)}$ normal to λ^i, then $\omega^i = 0$, because ω^i is always perpendicular to λ^i. Hence we may state this result:

THEOREM VIII. *If, for every infinitesimal circuit* Γ *normal to the stream-line at a point* A, *the limit of the ratio of the kinematical circulation to the area enclosed by* Γ *is zero, then the motion is irrotational at* A.

In an irrotational motion for which the stream-lines are geodesics we have

$$\omega_{ij}\lambda^j = 0, \quad \omega^{*ij}\lambda_j = 0. \tag{5.1}$$

By introducing temporarily a system of local normal coordinates for which all the components of λ^i vanish except λ^4, we see immediately that $\omega_{ij} = 0$.

The following result is given by (4.1):

THEOREM IX. *In an irrotational motion in which the stream-lines are geodesics, the kinematical circulation vanishes in every reducible† circuit. Conversely, if the kinematical circulation vanishes in every reducible† circuit, then the motion is irrotational and the stream-lines are geodesics.*

This is, of course, merely a well-known fact in the geometry of a normal congruence of geodesics, translated into hydrodynamical language.

PART III. *Fluid motion in general.*

6. *The energy-tensor and the equations of conservation.*

If T_{ij} is the symmetric energy-tensor of a continuous medium, the determinantal equation for X,

$$|T_{ij}+Xg_{ij}|=0, \tag{6.1}$$

has four roots. If $T_{ij}dx^i dx^j$ is a positive-definite or negative-definite form, these roots are real, in accordance with a well-known result. Or if it is possible to choose a system of coordinates (with x^4 time-like) such that at an assigned point we have

$$g_{14}=g_{24}=g_{34}=0, \quad T_{14}=T_{24}=T_{34}=0, \tag{6.2}$$

then the roots are real at that point. It does not appear possible to deduce from mechanical postulates inherent in the theory of relativity that the roots are always real‡; their reality will be accepted as a postulate in what follows.

On this assumption, the equations

$$T_{ij}\xi^j+Xg_{ij}\xi^j=0, \quad \xi_i\xi^i=\pm 1, \tag{6.3}$$

determine unit vectors in space-time. When the roots of (6.1) are distinct, four mutually orthogonal unit vectors are thus defined (except for sense).

† That is, reducible to a point by continuous deformation in space-time without leaving the fluid.

‡ I have elsewhere [*Trans. Roy. Soc. Canada*, III, 28 (1934), 127–171] developed the theory of the energy-tensor on a statistical basis, and given the physical interpretation of its components. In that paper the equations of conservation (6.5) were deduced from the law of conservation of momentum and energy without reference to the field equations. It is interesting to note that, if T_{ij} is the energy-tensor of a general electromagnetic field in vacuo, then the roots of (6.1) are real, but not distinct, being of the form $k, k, -k, -k$, *Principal null-directions defined in space-time by an electromagnetic field* [Univ. of Toronto Studies, Applied Mathematics Series 1 (1935)].

One of these is necessarily time-like, and we may take it pointing into the future. The *velocity-vector* λ^i (as already employed in Part II, without reference to the energy-tensor) is defined to be this vector, and the *proper energy-density*† ρ is defined to be the corresponding invariant X. The *stream-lines* consist of the congruence defined by the field λ^i. The three space-like vectors given by (6.3) define the *principal directions of stress*, and the corresponding values of X the *principal stresses*.

A *perfect fluid* is defined by the invariant condition that the three principal stresses are equal‡. Each will be denoted by $-p$, p being called the *pressure*. It can then be shown§ that the energy-tensor of a perfect fluid may be written in the commonly accepted form||

$$T_{ij} = (\rho+p)\lambda_i\lambda_j + pg_{ij}. \tag{6.4}$$

Regarding them as a basic hypothesis, a deduction from the field equations, or a deduction from a statistical theory based on the conservation of momentum and energy, as mentioned in a footnote, we shall accept the *equations of conservation*

$$T^{ij}_{\ \ ,j} = 0. \tag{6.5}$$

Substituting the contravariant form of (6.4), we have as *the equations of*

† As a heritage from Newtonian mechanics it is still customary to regard mass as the primary concept, and to define energy in terms of mass. This introduces an awkward factor c^2, which has no place in the general theory of relativity. It appears more satisfactory to regard energy as the primary concept, and mass as the derived concept. Energy-density and stress have the same physical dimensions.

‡ Approaching the question from the point of view of the Ricci principal directions and the field equations, Eisenhart (E., 207) has obtained conclusions equivalent to the above.

§ See *Trans. Royal Soc. Canada*, III, 28 (1934), 154, 163.

|| *Cf.* A. Einstein, *Ann. d. Phys.*, 49 (1916), 811, equation (58); R. C. Tolman, *Relativity, thermodynamics, and cosmology* (Oxford, 1934), 217, equation (85.7). Both of these equations require a change of sign for comparison with (6.4), because these authors use the signature $---+$. Einstein's equation differs formally from (6.4), having ρ instead of $\rho+p$, but a footnote to his paper shows that his ρ is equivalent to our $\rho+p$. Eddington, *Mathematical theory of relativity* (Cambridge, 1924), 122, would replace our $\rho+p$ by $\rho+4p$, ρ being the proper density, but this appears to be unsuitable, because for an electro-magnetic field we have $T^i_{\ ,i} = 0$ and thus, if the fluid consists of electromagnetic radiation, Eddington's definition would lead to the quite untenable conclusion that the energy-density is zero in such a case.

From a purely mathematical point of view it does not matter what physical names are given to the invariant coefficients in (6.4). But when we wish to investigate the consequences of particular relationships between these coefficients, it is important to know how these coefficients involve the pressure and density, in order that we may select from the infinitely many possible relationships those most interesting from a physical stand-point

motion of a perfect fluid

$$(\rho+p)(\lambda^i{}_{,j}\lambda^j+\lambda^i\lambda^j{}_{,j})+(\rho+p)_{,j}\lambda^i\lambda^j+p^{,i}=0. \tag{6.6}$$

These equations form the basis of our investigations. They may be written in the form

$$(\rho+p)\left(\frac{\delta\lambda^i}{\delta s}+\lambda^i\theta\right)+\lambda^i\frac{d}{ds}(\rho+p)+p^{,i}=0, \tag{6.7}$$

the differentiation being along a stream-line: θ is the expansion (3.9).

Multiplication of (6.7) by λ_i gives, by (3.1),

$$\frac{d\rho}{ds}+(\rho+p)\theta=0, \tag{6.8}$$

and substitution from this in (6.7) gives†

$$(\rho+p)\frac{\delta\lambda^i}{\delta s}+\lambda^i\frac{dp}{ds}+p^{,i}=0. \tag{6.9}$$

Hence we may state this result:

THEOREM X. *A perfect fluid moves in accordance with the five equations* (6.8), (6.9), *of which only four are independent.*

Thus we have only four equations for the determination of five quantities, namely, ρ, p and three independent components of the unit vector λ^i. The problem of fluid motion is therefore indeterminate, as in classical hydrodynamics, in the absence of a further equation—a pressure-density relation or other condition. This will be introduced in § 8; for the present we proceed without it.

7. *Relations between the stream-lines and the distributions of pressure and density.*

If we assume that

$$\rho\geqslant 0, \quad p\geqslant 0, \tag{7.1}$$

† The equations (6.8), (6.9) are equivalent to those given by Eisenhart, E., 209; for (6.8), see also Tolman, *op. cit.*, 221, and (for $p=0$) T. Levi-Civita, *Fondamenti di meccanica relativistica* (Bologna, 1928), 85.

it follows from (6.8) that *density decreases in an expanding fluid.* In the absence of further data, it is not legitimate to assume (as it would sometimes be convenient to do†) that *pressure* necessarily decreases in an expanding fluid. The deduction is, of course, legitimate if a pressure-density equation is assumed making $dp/d\rho$ positive.

Since $\delta\lambda^i/\delta s = 0$ are the conditions for a geodesic stream-line, we have the following results from (6.9):

THEOREM XI. *If all stream-lines are geodesics, then either the pressure is constant‡, or the stream-lines are normal to the 3-spaces $p = const.$ and the motion is therefore irrotational.*

THEOREM XII. *If a stream-line cuts the 3-spaces $p = const.$ orthogonally, then that stream-line is a geodesic.*

Substitution from the first of the Frenet formulae (2.4) in (6.9) gives

$$(\rho+p)\,k_{(1)}\,v^i_{(1)}+\lambda^i dp/ds+p^{,i}=0, \tag{7.2}$$

and multiplication by $v_{(1)i}$ then gives

$$(\rho+p)\,k_{(1)}+\partial p/\partial v_{(1)}=0, \tag{7.3}$$

where

$$\partial p/\partial v_{(1)}=p^{,i}\,v_{(1)i}=v^i_{(1)}\partial p/\partial x^i, \tag{7.4}$$

the rate of increase of p along the first normal to the stream-line. Hence we have these results:

THEOREM XIII. *The two-dimensional element containing the tangent and first normal to the stream-line contains the vector of pressure-gradient $(p^{,i})$. The stream line bends in the direction of decreasing pressure with a first curvature*

$$k_{(1)}=-\frac{\partial p/\partial v_{(1)}}{\rho+p}. \tag{7.5}$$

THEOREM XIV. *The second and third normals to the stream-line are tangent to the 3-space $p = const.$*

This last result may also be expressed by the equations

$$\partial p/\partial v_{(2)}=0, \quad \partial p/\partial v_{(3)}=0. \tag{7.6}$$

† *Cf.* Tolman, *op. cit.*, 397.
‡ *Cf.* E., 212.

The four equations (6.8), (7.5), (7.6) are equivalent to the original equations (6.6).

Let us now differentiate (7.2) along the stream-line, obtaining by (2.4)

$$(7.7)\quad (\rho+p)\,k_{(1)}(k_{(2)}\nu^i_{(2)}+k_{(1)}\lambda^i)+\nu^i_{(1)}\frac{d}{ds}\{k_{(1)}(\rho+p)\}$$
$$+k_{(1)}\nu^i_{(1)}dp/ds+\lambda^i d^2p/ds^2+p^{,ij}\lambda_j=0.$$

Multiplying in turn by λ_i, $\nu_{(1)i}$, $\nu_{(2)i}$, $\nu_{(3)i}$, we get

$$(7.8)\quad \begin{cases} -(\rho+p)\,k^2_{(1)}-d^2p/ds^2+p_{,ij}\lambda^i\lambda^j=0, \\ \dfrac{d}{ds}\{k_{(1)}(\rho+p)\}+k_{(1)}dp/ds+p_{,ij}\nu^i_{(1)}\lambda^j=0, \\ (\rho+p)\,k_{(1)}k_{(2)}+p_{,ij}\nu^i_{(2)}\lambda^j=0, \\ p_{,ij}\nu_{(3)}\lambda^j=0. \end{cases}$$

The last terms in these expressions suggest consideration of the infinitesimal *pressure-quadric*

$$(7.9)\qquad p_{,ij}dx^i dx^j=\text{const.},$$

which may be constructed by measuring off along the geodesic radii drawn from a point infinitesimal distances whose squares are inversely proportional to the values of d^2p/ds^2 calculated along these radii. Thus, from the last of (7.8) we conclude that *the tangent and third normal to a stream-line have directions conjugate to one another with respect to the pressure-quadric.*

Since a geodesic of a 3-space immersed in space-time is characterized by the condition that its first normal (relative to space-time) lies along the normal to the 3-space, the following deduction from (7.2) is immediate:

THEOREM XV. *If the pressure is constant along a stream-line, then that stream-line is a geodesic of the 3-space $p=$ const.*

The analogous result in classical hydrodynamics is easily established, but only in the absence of body-forces, a condition which makes the result inapplicable to some cases of physical interest. The relativistic result given above applies to a gravitating mass of fluid in steady rotation, because in it the pressure will obviously be constant along a stream-line.

The following are immediate deductions from (6.8):

THEOREM XVI. *If the density is constant along a stream-line, the expansion vanishes, and hence the 3-volume of the normal section of a thin tube of stream-lines enclosing the line in question remains constant.*

THEOREM XVII. *If the expansion vanishes throughout a fluid, the density is constant along each stream-line.*

8. *Conditions rendering the motion determinate.*

So far we have been dealing with an indeterminate problem, five independent variables being subjected to only four independent equations. We shall now consider cases in which the problem is made determinate by the provision of an extra equation.

Let us first consider an *incompressible* (but in general heterogeneous) fluid, which we shall define by the kinematical condition that its expansion θ is zero†. It follows from Theorem XVII that the density ρ is constant along each stream-line in an incompressible fluid. Thus we may say that *the motion of an incompressible (in general heterogeneous) fluid is to be found by solving for the five independent quantities contained in the set* ρ, p, λ^i *the five independent equations contained in the set*

$$(8.1) \qquad \begin{cases} (\rho+p)\lambda^i{}_{,j}\lambda^j+\lambda^i p_{,j}\lambda^j+p^{,i}=0, \\ \lambda^i{}_{,i}=0, \quad \rho_{,i}\lambda^i=0. \end{cases}$$

We are, of course, to bear in mind the identity $\lambda_i\lambda^i=-1$.

A *homogeneous incompressible fluid* is one in which the density ρ is constant, not only along each stream-line but throughout the fluid. Thus we may say that *the motion of a homogeneous incompressible fluid is to be found by solving for the four independent quantities contained in the set* p, λ^i *the four independent equations contained in the set*

$$(8.2) \qquad (\rho+p)\lambda^i{}_{,j}\lambda^j+\lambda^i p_{,j}\lambda^j+p^{,i}=0, \quad \lambda^i{}_{,i}=0,$$

ρ *being a constant.*

---

† There has been a difference of opinion regarding the proper definition of incompressibility. That employed here agrees with that used by Schwarzschild, *Sitzungsber. K. Preuss. Akad. d. Wiss.* (1916), 424, but Eddington (*op. cit.*, 122, 168–170) has objected strongly to this definition, preferring $T^i{}_i = \text{const.}$, equivalent to $\rho-3p = \text{const.}$: this, however, makes electromagnetic radiation incompressible, and has no connection with a simple kinematical law. The well-known solution of the field equations for the interior of the sun is based on Schwarzschild's definition: Eddington's definition has been applied to the same problem by N. R. Sen, *Monthly Notices Roy. Ast. Soc.*, 94 (1934), 550–564.

Although in the present paper we shall adopt the Schwarzschild definition of incompressibility, it must be pointed out that, on account of the results developed in § 21, an incompressible fluid cannot be regarded as the limiting case of a compressible fluid possessing a pressure-density equation with compressibility tending to zero.

A more general class of fluids, including the homogeneous incompressible fluid as a particular case, consists of those for which there exists a *pressure-density equation* of the form†

$$\rho = f(p). \tag{8.3}$$

Thus we may say that *the motion of a fluid possessing a pressure-density equation is to be found by solving for the four independent quantities contained in the set* ρ, p, λ^i *the four independent equations contained in the set*

$$\begin{cases} (\rho+p)\lambda^i{}_{,j}\lambda^j+\lambda^i p_{,j}\lambda^j+p^{,i}=0, \\ \rho_{,i}\lambda^i+(\rho+p)\lambda^i{}_{,i}=0. \end{cases} \tag{8.4}$$

As a particular case of such a fluid we may mention a *perfect gas*, for which (8.3) takes the form

$$\rho = ap^c, \tag{8.5}$$

a and c being constants. This formula includes the *isothermal* and *adiabatic* cases, the isothermal case corresponding to $c=1$. On substitution from (8.5), we have from (8.4) for an isothermal gas

$$(p^b\lambda^i)_{,j}\lambda^j+(p^b)^{,i}=0, \quad (p^{ab}\lambda^i)_{,i}=0 \quad [b=(a+1)^{-1}.] \tag{8.6}$$

Radiation satisfies (8.5) with $a=3$, $c=1$.

PART IV. *Motion of a fluid possessing a pressure-density equation.*

9. *The index-function and the variational principle.*

A relation of the form (8.3) is assumed throughout Part IV. Let us define the *index-function*‡ F by

$$F(p) = \exp\left(\int_{p_0}^{p}\frac{dp}{\rho+p}\right), \tag{9.1}$$

† Here, and in classical theory, we should employ an equation of state involving density, pressure and temperature, but classical hydrodynamics is chiefly concerned (for mathematical reasons) with limiting cases of incompressibility or isothermal or adiabatic motion for which the relation is of the form (8.3). We shall follow the same course.

‡ So called by analogy with the refractive index of a transparent medium, as developed below. A function ϕ introduced by Eisenhart (E., 216) is the same as $\log F$. The variational principle given below will also be found in Eisenhart's paper, developed by a method based on the geometry of paths.

where p_0 is an arbitrary constant; then

$$p_{,i}/(\rho+p) = F_{,i}/F, \tag{9.2}$$

and the equations of motion (8.4) may be written

$$\begin{cases} F\delta\lambda^i/\delta s+\lambda^i dF/ds+F^{,i}=0, \\ d\rho/ds+(\rho+p)\lambda^i_{,i}=0. \end{cases} \tag{9.3}$$

Let us define the *current-vector* Λ^i by

$$\Lambda^i = F\lambda^i; \tag{9.4}$$

then the first line of (9.3) may be written

$$\delta\Lambda^i/\delta s+F^{,i}=0, \tag{9.5}$$

and we may state this result:

THEOREM XVIII. *Along a stream-line the rate of change of the current vector is the negative of the gradient of the index-function.*

For a homogeneous incompressible fluid we have by (9.1) (with $p_0=0$)

$$F(p)=1+p/\rho, \quad \Lambda^i=(1+p/\rho)\lambda^i. \tag{9.6}$$

We may note that in most cases of physical interest the ratio p/ρ is small, so that in such cases Λ^i is approximately the same as λ^i: but such approximations are not used in the paper.

For an isothermal gas we shall choose $p_0=1$ in (9.1): then

$$F=p^b, \quad \Lambda^i=p^b\lambda^i, \tag{9.7}$$

where b is defined as in (8.6).

Let us now consider the variation of $\int Fds$, taken along any curve in space-time which has not a null-direction at any point. We shall regard p, and consequently F, as a known function of position in space-time. Since the indicator ϵ of the curve plays an important part, it is necessary to give the detail of what is otherwise a routine operation.

Let u be a parameter with a fixed range for the family of curves under consideration, and let

$$w=g_{ij}\dot{x}^i\dot{x}^j, \quad \dot{x}^i=dx^i/du. \tag{9.8}$$

Then

$$(9.9)\quad \delta\int_A^B F ds = \delta\int_{u_1}^{u_2} F(\epsilon w)^{\frac{1}{2}}\,du$$

$$= \left[F\frac{\partial}{\partial \dot{x}^i}(\epsilon w)^{\frac{1}{2}}\delta x^i\right]_{u_1}^{u_2} - \int_{u_1}^{u_2}\left\{\frac{d}{du}\left(F\frac{\partial}{\partial \dot{x}^i}(\epsilon w)^{\frac{1}{2}}\right) - \frac{\partial}{\partial x^i}\left(F(\epsilon w)^{\frac{1}{2}}\right)\right\}\delta x^i\,du.$$

Hence an extremal satisfies the equations

$$(9.10)\qquad \tfrac{1}{2}\frac{d}{ds}\left(F\epsilon\frac{\partial w}{\partial \dot{x}^i}\right) - \tfrac{1}{2}F\epsilon\frac{\partial w}{\partial x^i} - \frac{\partial F}{\partial x^i} = 0,$$

where we have put $u = s$, and consequently $\epsilon w = 1$. Now we know that

$$(9.11)\qquad \frac{d}{ds}\frac{\partial w}{\partial x^i} - \frac{\partial w}{\partial \dot{x}^i} = 2\frac{\delta t_i}{\delta s}, \qquad \frac{\partial w}{\partial \dot{x}^i} = 2t_i,$$

where t^i is the unit tangent vector, and thus the equations of an extremal are, in covariant form,

$$(9.12)\qquad F\frac{\delta t_i}{\delta s} + t_i\frac{dF}{ds} - \epsilon F_{,i} = 0.$$

Also, by (9.9), the variation from an extremal is

$$(9.13)\qquad \delta\int_A^B F\,ds = \epsilon\left[Ft_i\delta x^i\right]_A^B.$$

For a time-like extremal we have $\epsilon = -1$, and comparison of (9.12) with (9.3) gives us the following result:

THEOREM XIX. Variational principle†. *In a fluid possessing a pressure-density equation, the stream-lines satisfy the variational principle*

$$(9.14)\qquad \delta\int F\,ds = 0,$$

p being regarded as an assigned function of position in space-time, and F being the index-function (9.1).

Passage from a stream-line to an adjacent stream-line gives

$$(9.15)\qquad \delta\int_A^B F\,ds = -\left[\Lambda_i\delta x^i\right]_A^B.$$

The variational principle (9.14) is formally identical with Fermat's optical principle in a transparent medium of refractive index F. But the

† *Cf.* E., 216.

problem of determining the stream-lines is not the same as that of finding the rays in a given medium, even after allowing for the fact that we are dealing with a curved manifold of four dimensions with an indefinite line-element. In the optical problem the refractive index is supposed known, whereas in our problem F is not known: for its determination the last equation of (9.3) must be employed, jointly with the variational principle. Our problem is analogous to that of finding the rays and refractive index in a medium in which the refractive index depends on the distribution of the rays.

Nevertheless the variational principle is valuable, on account of the powerful methods associated with such a principle, originally developed by Hamilton† a century ago both in optics and dynamics. These methods are now familiar in the calculus of variations, but in the modification and development of Hamilton's method in dynamics by Jacobi and later writers much of the primitive conceptual simplicity has been lost. Thus, although Hamilton's principal function was the central idea on which his method in dynamics rested, Birkhoff‡ calls the total energy H of a system the Hamiltonian "principal function"; Hamilton's principal function was actually something quite different.

In § 14 we shall make direct use of Hamilton's optical method for the definition of the current-potential in an irrotational motion.

10. *Definitions of dynamical circulation and dynamical vorticity.*

The definitions given in § 3 were purely kinematical, the distributions of density and pressure not being involved. We shall now modify those definitions, defining "dynamical" instead of "kinematical" quantities in order that we may be able to use the variational principle to establish a body of theory having a close analogy to basic theory in classical hydrodynamics.

Attention is confined to fluids possessing a pressure-density equation (8.3), and hence an index-function F (9.1). We have already defined the current-vector Λ^i in (9.4).

The *dynamical flow* along any curve in space-time, joining points A and B, is defined as the integral

$$\int_A^B \Lambda_i dx^i, \tag{10.1}$$

† *Cf. The mathematical papers of Sir W. R. Hamilton*, 1 (Cambridge, 1931).

‡ "Dynamical systems", *American Math. Soc. Colloquium Publications*, 9 (1927), 52.

taken along the curve, and the *dynamical circulation* in any circuit is defined as

$$D = \oint \Lambda_i dx^i = \oint F\lambda_i dx^i, \tag{10.2}$$

the integral being taken round the circuit in an assigned sense.

The skew-symmetric *dynamical vorticity-tensor* is defined as

$$\Omega_{ij} = \tfrac{1}{2}(\Lambda_{j,i} - \Lambda_{i,j}), \tag{10.3}$$

the *conjugate dynamical vorticity-tensor* as

$$\Omega^{*ij} = \tfrac{1}{2}\eta^{ijkl}\Omega_{kl}, \tag{10.4}$$

and the *dynamical vorticity-vector* as

$$\Omega^i = \Omega^{*ij}\lambda_j, \tag{10.5}$$

λ_j being, of course, the velocity-vector.

We have then

$$\Omega_{ij} = \tfrac{1}{2}(F_{,i}\lambda_j - F_{,j}\lambda_i) + \tfrac{1}{2}F(\lambda_{j,i} - \lambda_{i,j}), \tag{10.6}$$

and

$$\Omega_{ij}\lambda^j = -\tfrac{1}{2}(F_{,i} + \lambda_i dF/ds + F\delta\lambda_i/\delta s), \tag{10.7}$$

the differentiation being along a stream-line. Hence, by (9.3), we have

$$\Omega_{ij}\lambda^j = 0. \tag{10.8}$$

Also

$$\begin{aligned} \Omega^i &= \tfrac{1}{2}\eta^{ijkl}\lambda_j\Omega_{kl} \\ &= \tfrac{1}{4}\eta^{ijkl}\lambda_j\{F_{,k}\lambda_l - F_{,l}\lambda_k + F(\lambda_{l,k} - \lambda_{k,l})\}, \end{aligned} \tag{10.9}$$

and on account of the skew-symmetry of the η-symbol this reduces to

$$\Omega^i = \tfrac{1}{4}F\eta^{ijkl}\lambda_j(\lambda_{l,k} - \lambda_{k,l}) = F\omega^i, \tag{10.10}$$

where ω^i is the kinematical vorticity-vector (3.8).

The dynamical and kinematical vorticity-vectors have therefore the same direction and we may without ambiguity define the directed *vortex-lines* as the curves in space-time which have the direction of either vorticity-vector. They are space-like curves, because

$$\Omega_i\lambda^i = \omega_i\lambda^i = 0.$$

We shall now show that, *if* μ^i, ν^i *are any two vectors perpendicular to* λ^i, *then*

$$\Omega^{*ij}\mu_i\nu_j = 0. \tag{10.11}$$

On account of the skew-symmetry of Ω^{*ij} it will obviously suffice to prove this result for the case where μ^i, ν^i are unit vectors forming an orthogonal triad with λ^i. Now at any point coordinates may be chosen so that there

$$\lambda^1 = \lambda^2 = \lambda^3 = \mu_4 = \nu_4 = 0. \tag{10.12}$$

Then by (10.8) we have

$$\Omega_{14} = \Omega_{24} = \Omega_{34} = 0, \tag{10.13}$$

and hence by (10.4)

$$\Omega^{*23} = \Omega^{*31} = \Omega^{*12} = 0, \tag{10.14}$$

so that (10.11) holds for this particular coordinate-system. But (10.11) is invariant, and therefore it is established for a general coordinate system.

11. *The permanence of dynamical circulation.*

Consider now the dynamical circulation D in a reducible circuit Γ drawn on a two-dimensional sheet S formed of stream-lines (Fig. 2).

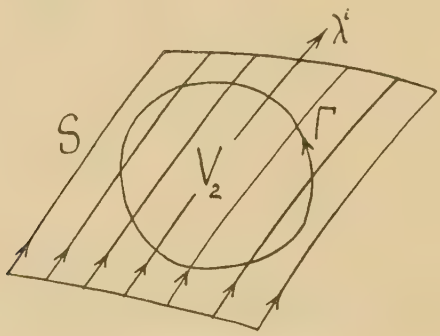

Fig. 2.

By (2.1) we have

$$D = \oint_\Gamma \Lambda_i dx^i = -2\int_{V_2} \Omega_{ij} t^i_{(1)} t^j_{(2)} dS, \tag{11.1}$$

where V_2 is the region of S enclosed by Γ and $t^i_{(1)}$, $t^i_{(2)}$ orthogonal unit vectors in S. We may choose $t^i_{(1)} = \lambda^i$: then by (10.8) the integrand vanishes, and we have the following result, which may also be obtained easily by

variation along the stream-lines without use of the generalized Green-Stokes theorem:

THEOREM XX. *The dynamical circulation vanishes in any reducible† circuit drawn on a two-dimensional sheet of stream-lines.*

Let us now take any circuit in space-time, and through its points draw a two-dimensional tube T of stream-lines. If Γ_1 and Γ_2 are two reconcilable circuits, each passing once round T, the preceding result may be applied to a reducible circuit on T formed of Γ_1, Γ_2 reversed, and the two edges of a cut along a stream-line joining Γ_1 and Γ_2 (Fig. 3). It follows immediately

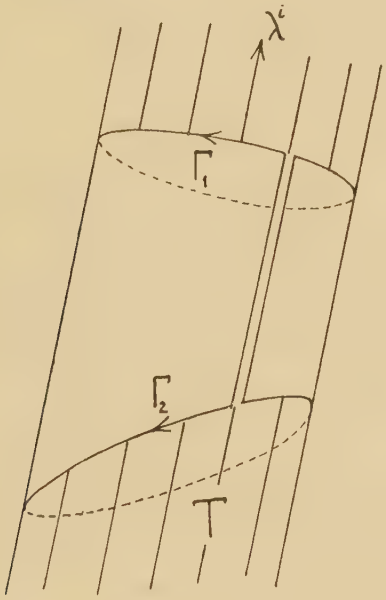

Fig. 3.

that the dynamical circulation in Γ_1 is equal‡ to that in Γ_2. This dynamical circulation is a fundamental invariant of the tube of stream-lines.

To evaluate this invariant, let us cut T by a space-like V_3, the curve of section being a circuit Γ. Let V_2 be any two-dimensional region in V_3, bounded by Γ. We have already calculated in (4.11) the kinematical circulation in a circuit bounding a space-like V_2: to get the dynamical

† That is, reducible to a point by continuous deformation in space-time without leaving the sheet of stream-lines.

‡ This is, of course, only a hydrodynamical expression of a classical result regarding integral invariants.

circulation we have simply to replace ω by Ω, so that

$$(11.2) \qquad D = -2\int_{V_2} \Omega^i t_{(3)i} \operatorname{sech}\chi\, dS + 2\int_{V_2} \Omega^{*ij} t_{(3)i}\mu_j \tanh\chi\, dS.$$

We may resolve the vector μ^i along λ^i and a unit vector ν^i which lies in the 2-space of μ^i and λ^i and is perpendicular to the latter: thus

$$(11.3) \qquad \mu^i = A\lambda^i + B\nu^i,$$

where

$$(11.4) \qquad A = -\mu^i\lambda_i = -\sinh\chi$$

by (4.8). When we substitute from (11.3) in (11.2), the term in B disappears, since by (10.11)

$$(11.5) \qquad \Omega^{*ij} t_{(3)i}\nu_j = 0.$$

Hence, using (10.5), we have this result, analogous to a fundamental theorem in classical hydrodynamics†:

THEOREM XXI. Permanence of dynamical circulation. *The dynamical circulation D is the same in all reconcilable circuits passing once round a two-dimensional tube of stream-lines, and is given by*

$$(11.6) \qquad D = -2\int_{V_2} \Omega^i t_{(3)i} \cosh\chi\, dS,$$

where V_2 is any space-like two-dimensional cross-section of the tube, Ω^i the dynamical vorticity-vector, $t^i_{(3)}$ a unit vector perpendicular to the stream-line and to V_2, and χ the obliquity of the stream-line to V_2, as defined in (4.8). *The sense of $t^i_{(3)}$ is such that the following directions in order have the orientation of the parametric lines of the coordinates*: (i) *the tangent to the circuit bounding V_2, in the assigned positive sense,* (ii) *the outward normal to the circuit in V_2,* (iii) $t^i_{(3)}$, (iv) λ^i, *the velocity-vector.*

12. *The strength of a vortex-tube.*

Consider a space-like two-dimensional sheet S formed of vortex-lines, and a reducible circuit Γ drawn on it (Fig. 4). The formula (11.6) may be applied directly to give the dynamical circulation in Γ. But $t^i_{(3)}$ is, by

† H. Lamb, *Hydrodynamics* (Cambridge, 1932), 36.

definition, perpendicular to S, and hence

$$\Omega^i t_{(3)i} = 0, \tag{12.1}$$

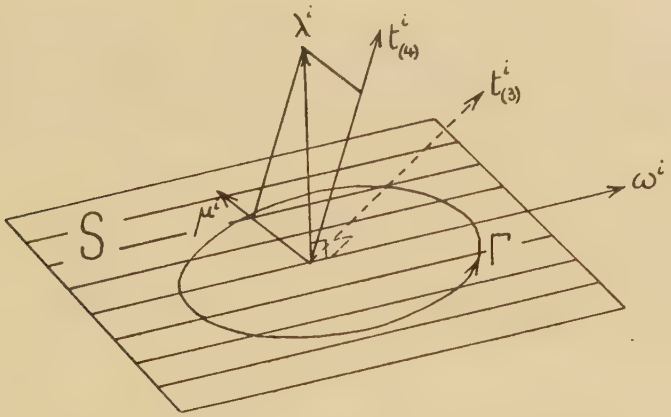

Fig. 4.

since Ω^i lies in S. Thus we have this result:

THEOREM XXII. *The dynamical circulation vanishes in every reducible† circuit drawn on a space-like two-dimensional sheet of vortex-lines.*

Let us now take a two-dimensional tube T of vortex-lines, drawn through the points of a space-like circuit, so that T is a space-like 2-space. It follows immediately from the preceding result that if Γ_1 and Γ_2 are two reconcilable circuits, each passing once round T (Fig. 5), then the dynamical circulation in Γ_1 is the same as that in Γ_2. Following classical terminology, we shall call this the *strength* of the vortex-tube.

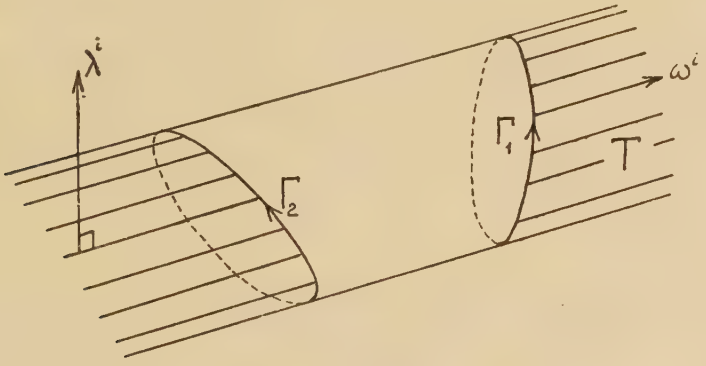

Fig. 5.

† That is, reducible to a point by continuous deformation in space-time without leaving the sheet of vortex-lines.

To find the strength of a space-like vortex-tube, we take a space-like two-dimensional cross-section, and apply the method leading to (11.6). Hence we have this result:

THEOREM XXIII. Theorem of Helmholtz†. *The dynamical circulation D is the same in all reconcilable circuits passing once round a space-like two-dimensional tube of vortex-lines. It is called the strength of the tube and is given by*

$$D = -2\int_{V_2} \Omega^i t_{(3)i} \cosh\chi \, dS, \tag{12.2}$$

where V_2 is a space-like two-dimensional cross-section of the tube, and the notation is as in Theorem XXI.

13. *The permanence of irrotational motion.*

Irrotational motion was defined kinematically in § 3, the condition of irrotationality being the vanishing of the kinematical vorticity-vector ω^i. Although we have, in § 10, introduced dynamical definitions, it is not necessary to modify the definition of irrotational motion, on account of the simple relation (10.10) between the kinematical and dynamical vorticity-vectors. We have in fact for an irrotational motion

$$\omega^i = \Omega^i = 0. \tag{13.1}$$

Let there be a motion in which the stream-lines are normal to a portion V_3' of a 3-space. Let us measure off along the stream-lines segments for which the values of $\int F\,ds$, starting from V_3', have a common value. The extremities of these segments form a 3-space V_3, and it follows from (9.15) that the stream-lines cut V_3 orthogonally, a well-known property of extremals. Since the existence of a singly-infinite set of 3-spaces normal to the stream-lines establishes the irrotational character of the motion, we may state this result:

THEOREM XXIV. Permanence of irrotational motion. *If throughout a four-dimensional region R_4 of space-time the motion is irrotational (or if the stream-lines are normal to a 3-space in R_4), the motion was and will be irro-*

† Lamb, *op. cit.*, 202. It is to be noted that the classical result is purely kinematical, whereas our present result has been established by virtue of the equations of motion.

tational for all the past and future history given by drawing the stream-lines through the points of R_4.

This result carries into the general theory of relativity a basic theorem of classical hydrodynamics†. As a particular physical application, the following is worthy of note. Let there be a sphere of incompressible fluid, remote from all other matter and without rotation, as in the solution of Schwarzschild for the solar field. This state is, of course, irrotational. The result just established tells us that if, in its subsequent history, this fluid comes under the gravitational influence of other bodies, any motion produced must be irrotational. Spin cannot be generated in it by a gravitational field. This applies, of course, also to any fluid possessing a pressure-density equation (8.3).

14. *The current-potential.*

We proceed now to define for an irrotational motion of a fluid satisfying (8.3) a current-potential analogous to the velocity-potential of classical hydrodynamics.

Let V_3' be any one of the 3-spaces normal to the stream-lines in an irrotational motion, and let x^i be the coordinates of any point B. Let A be the point where the stream-line through B cuts V_3'. We define the *current-potential* ϕ at B as

$$\phi = \int_A^B F\,ds, \tag{14.1}$$

the integral being taken along the stream-line, and ϕ being considered as a function of the coordinates of B.

Giving an arbitrary infinitesimal displacement δx^i to B (with a consequent displacement of A on the same V_3'), we have by (9.15)

$$\delta\phi = -\Lambda_i \delta x^i, \tag{14.2}$$

and hence *the covariant components of the current-vector are related to the current-potential by the equations*

$$F\lambda_i = \Lambda_i = -\partial\phi/\partial x^i. \tag{14.3}$$

† Lamb, *op. cit.*, 37.

When the section of the fluid by a space-like 3-space is multiply connected, ϕ may be multiple-valued. In such a case the point A, used in (14.1), is not unique. The stream-line through B cuts the normal 3-space V_3' in an infinite sequence of points A_1, A_2, ... Fig. 6 shows the geometry

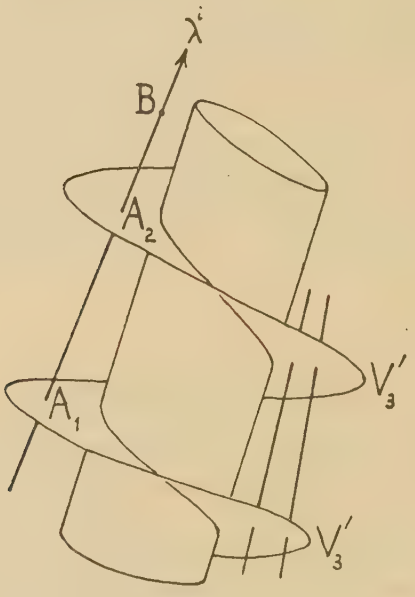

Fig. 6.

of the situation schematically, space-time being regarded for diagrammatic purposes as deprived of one spatial dimension. If we consider for simplicity a doubly-connected region (like the history of the interior of a torus) the values of $\int_{A_n}^{A_{n+1}} F ds$ taken between two consecutive points of the sequence A_1, A_2, ... are independent of n. The integral corresponds to the circulation in an irreducible circuit in classical hydrodynamics, but it is not a circulation, because the path of integration is not closed. We may call it the *fundamental flow*. If we denote it by Φ, then the current-potential ϕ will be multiple-valued, with an additive indeterminacy $m\Phi$, where m is an integer. In the case of connectivity of order N, there will be $N-1$ fundamental flows Φ_1, Φ_2, ... and an additive indeterminacy

$$m_1\Phi_1+m_2\Phi_2+\ldots$$

in ϕ, where m_1, m_2, ... are integers. As in classical hydrodynamics, the fluid may be made simply-connected by the introduction of barriers, across

which ϕ is discontinuous. These barriers may conveniently be composed of stream-lines.

If we proceed along any curve in the fluid, from a point P to a point Q, taking any determination of ϕ consistent with continuity, it is evident from (14.2) that

$$\phi_Q - \phi_P = -\int_P^Q \Lambda_i dx^i. \tag{14.4}$$

Hence we have the following result:

THEOREM XXV. *In an irrotational motion the dynamical circulation vanishes for every reducible† circuit.*

This also follows by (2.1) from the fact that

$$\Omega_{ij} = \tfrac{1}{2}(\Lambda_{j,i} - \Lambda_{i,j}) = -\tfrac{1}{2}(\phi_{,ji} - \phi_{,ij}) = 0. \tag{14.5}$$

Much of the basic theory of classical hydrodynamics follows from the fact that, in the irrotational motion of a homogeneous incompressible fluid, the velocity-potential satisfies Laplace's equation, which, incidentally, is a purely kinematical deduction. Let us investigate the partial differential equation satisfied by the current-potential in the present theory.

Let us define the Hamiltonian operator H by

$$H\phi = -g^{ij}\phi_{,i}\phi_{,j}, \tag{14.6}$$

the minus sign being inserted to make $H\phi$ positive, $\phi_{,i}$ being a time-like vector. We have then from (14.3)

$$H\phi = F^2. \tag{14.7}$$

With this equation we are to associate the last of (9.3), namely,

$$d\rho/ds + (\rho + p)\lambda^i{}_{,i} = 0. \tag{14.8}$$

By (14.3) we have

$$\lambda^i = -F^{-1}\phi^{,i}, \quad \lambda^i{}_{,i} = F^{-2}F_{,i}\phi^{,i} - F^{-1}\Delta\phi, \tag{14.9}$$

where Δ is the generalized Laplacian operator, so that

$$\Delta\phi = g^{ij}\phi_{,ij}. \tag{14.10}$$

† That is, reducible to a point by continuous deformation in space-time without leaving the fluid.

Also by (9.1)

$$\frac{\rho_{,i}}{\rho+p} = \frac{d\rho}{dp}\frac{F_{,i}}{F}, \tag{14.11}$$

and so (14.8) may be written

$$\Delta\phi + (d\rho/dp - 1)\, F^{-1} F_{,i}\phi^{,i} = 0. \tag{14.12}$$

Let us define, as a function of p or of F,

$$G = \int_{p_0}^{p} \left(\frac{d\rho}{dp} - 1\right) \frac{dp}{\rho+p}, \tag{14.13}$$

so that by (9.1)

$$(d\rho/dp - 1)\, F^{-1} = dG/dF. \tag{14.14}$$

Substituting this value in (14.12), and substituting for $F_{,i}$ from (14.7), we obtain the following result, in which all radicals have their positive values, since by its definition F is positive:

THEOREM XXVI. *The current-potential ϕ satisfies the partial differential equation obtained by eliminating F from the equations*

$$(H\phi)^{\frac{1}{2}} = F, \quad \Delta\phi + [(H\phi)^{\frac{1}{2}}]_{,i}\phi^{,i} dG/dF = 0, \tag{14.15}$$

where H is the Hamiltonian operator (14.6), *Δ the generalized Laplacian operator* (14.10), *and G is defined as a function of F by the equations*

$$\rho = f(p), \qquad F = \exp\left(\int_{p_0}^{p} \frac{dp}{\rho+p}\right), \qquad G = \int_{p_0}^{p} \left(\frac{d\rho}{dp} - 1\right) \frac{dp}{\rho+p}, \tag{14.16}$$

the function f being supposed assigned.

If a function ϕ has been found to satisfy the partial differential equation, the distributions of pressure and density are given by the first of (14.15), and the velocity-vector λ^i by $\lambda_i = -F^{-1}\phi_{,i}$. It is easy to verify directly that the mot on so obtained satisfies the equations of motion (6.6).

Let us find the partial differential equation satisfied by ϕ for some special cases.

For a homogeneous incompressible fluid we have by (9.6)

$$\rho = \text{const.}, \quad F = 1 + p/\rho, \quad G = -\log F, \quad dG/dF = -F^{-1}, \tag{14.17}$$

and thus ϕ satisfies

$$\Delta\phi = \tfrac{1}{2}(\log H\phi)_{,i}\phi^{,i}. \tag{14.18}$$

This equation, which replaces the Laplacian equation of classical hydrodynamics, may be obtained more directly by the relation $\lambda^i{}_{,i} = 0$ of (8.2): by (14.3) and (14.7) we obtain

$$\left[\frac{\phi^{,i}}{(H\phi)^3}\right]_{,i} = 0, \tag{14.19}$$

which reduces to (14.18). In the form (14.19) it is obvious that the equation is invariant under a transformation $\phi = \psi(\phi')$, where ψ is an arbitrary function.

For an isothermal gas we have by (8.5) and (9.7)

$$\begin{cases} \rho = ap, \quad F = p^b, \quad G = (a-1)\log F, \quad dG/dF = (a-1)/F, \\ \qquad\qquad\qquad\qquad b = (a+1)^{-1}, \end{cases} \tag{14.20}$$

and thus ϕ satisfies

$$\Delta\phi + \tfrac{1}{2}(a-1)(\log H\phi)_{,i}\phi^{,i} = 0. \tag{14.21}$$

We may apply this result to radiation by putting $a = 3$.

It is interesting to note that, by (14.15), ϕ satisfies the wave-equation $\Delta\phi = 0$ if $dG/dF = 0$: this condition is satisfied if, and only if, the pressure-density equation is of the form

$$\rho = p + C, \tag{14.22}$$

where C is a constant. The pressure is then, by the first of (14.15), given by

$$p = -\tfrac{1}{2}C + C'H\phi, \tag{14.23}$$

where C' is a constant.

15. *Uniqueness theorem for a homogeneous incompressible fluid.*

In classical hydrodynamics we have a uniqueness theorem, which states that if a homogeneous incompressible fluid occupies a simply-connected region bounded by a surface on which the normal velocity is assigned, there cannot exist two different irrotational motions in the fluid.

It is easy to establish the same result in general relativity in the following manner.

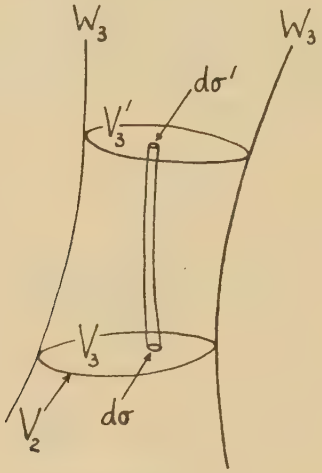

Fig. 7.

Let there be an irrotational motion of a homogeneous incompressible fluid, specified by the current-potential ϕ. Let V_3 be a simply-connected portion of one of the 3-spaces normal to the stream-lines, bounded by a closed 2-space V_2 (Fig. 7.) The stream-lines through the points of V_2 form a 3-dimensional tube W_3 in space-time, which is the history of the boundary of the portion of the fluid in question. If in W_3 we take another congruence of time-like curves, other than the stream-lines, we may regard the lines of this congruence as the histories of material particles forming an envelope enclosing the fluid, and W_3 as the history of this envelope. The condition that a time-like 3-space should be a possible envelope of a given motion with current-potential ϕ is obviously $\partial\phi/\partial n = 0$, ∂n being an element of the normal to the 3-space: or, alternatively, the condition is that the 3-space should be formed of stream-lines.

Now suppose that there exists another irrotational motion with current-potential ϕ', consistent with the same boundary W_3, and let V_3' be the region cut off from one of the normal 3-spaces of this motion by the tube W_3.

Consider now the stream-lines of the ϕ-motion drawn through the points of V_3. All these lines cut V_3', since no line can escape across W_3. If we draw a thin tube of stream-lines through an element of V_3 of 3-volume $d\sigma$, this tube will cut V_3' in a 3-volume which we shall denote by $d\sigma'$. *Since the fluid is, by hypothesis, incompressible,* the 3-volume of the normal section

of the tube remains constant ($d\sigma$). If we compare at V_3' the 3-volumes of the normal section ($d\sigma$) and the section by V_3' ($d\sigma'$), we have†

$$d\sigma = d\sigma' |\lambda_i \nu^i|, \tag{15.1}$$

where λ^i is the unit tangent vector to the tube and ν^i the unit vector normal to V_3'. But

$$|\lambda_i \nu^i| = \cosh \chi, \tag{15.2}$$

where χ is the obliquity of the stream-line of the ϕ-motion to V_3', in the sense of (4.9). Hence, integrating over V_3', we have

$$\sigma = \int_{V_3'} \cosh \chi \, d\sigma', \tag{15.3}$$

where σ is the 3-volume of V_3. Hence

$$\sigma \geqslant \sigma', \tag{15.4}$$

where σ' is the 3-volume of V_3'; the sign of equality holds only if $\chi = 0$ all over V_3', *i.e.* if all the stream-lines of the ϕ-motion cut V_3' normally.

If we now take thin tubes of the stream-lines of the ϕ'-motion, and apply the same process, we prove that

$$\sigma' \geqslant \sigma. \tag{15.5}$$

Therefore $\sigma' = \sigma$ and $\chi = 0$, showing that the stream-lines of the ϕ-motion are normal to all the normal 3-spaces of the ϕ'-motion; in fact, *the two motions have the same stream-lines and normal surfaces.*

Hence we may state the following result:

THEOREM XXVII. Uniqueness theorem. *In a space-time of given metric there exists at most one irrotational motion of a homogeneous incompressible fluid consistent with a given history of the boundary, the region contained within the boundary being simply-connected*‡.

† *Cf.* J. L. Synge, *Trans. Roy. Soc. Canada*, III, 28 (1934), 166.

‡ If the region was multiply-connected, we should have to augment the actual boundary W_3 by barriers. The proof would then break down, because, in general, stream-lines would cross the barriers in at least one of the two motions and we could not deduce simultaneously (15.4) and (15.5).

The interest of this theorem is actually less than might appear at first sight, on account of the neglect of the field equations. For the theorem is only proved for a space-time of given metric, whereas two different irrotational motions, if they exist, give different energy-tensors and hence different metrics. From a purely geometrical point of view the theorem is one concerning normal congruences of zero expansion in a given Riemannian space, and as such may be of interest. Moreover, although we cannot regard it as of physical significance as far as exact theory is concerned, it may be applied in an approximate way. Thus, we may have to deal with a fluid motion in which the gravitational effect of the fluid itself is insignificant in comparison with that of other matter: then the metric of space-time may be regarded as defined by the other matter, and the uniqueness theorem may be applied to the fluid.

16. *Co-moving coordinates in irrotational motion.*

In this section, Greek suffixes will have the range 1, 2, 3, with the summation convention for that range.

Co-moving coordinates† may be employed for any fluid motion. Taking any singly infinite family of space-like 3-spaces, we assign a system of coordinates x^α on one of them, V_3', and choose for x^4 a parameter increasing monotonically as we pass towards the future, x^4 being constant on each of the 3-spaces. To any point B in space-time we assign the x^4 belonging to the 3-space passing through B and the x^α belonging to the point A where the stream-line through B cuts V_3'. Thus the stream-lines are parametric lines of x^4 and we have throughout space-time

$$\lambda^\alpha = 0, \quad g_{44}(\lambda^4)^2 = -1. \tag{16.1}$$

In the case of an irrotational motion of a fluid possessing a pressure-density equation, it seems natural to take for the family of space-like 3-spaces the 3-spaces normal to the stream-lines, and to take

$$x^4 = \int F\,ds, \tag{16.2}$$

the integral being calculated from an assigned normal 3-space. We have then throughout the fluid, by (14.1),

$$\phi = x^4, \tag{16.3}$$

† R. C. Tolman, *op. cit.*, 302. These coordinates were also used by Eisenhart (E., 210), but without anticipation of the results given below.

and so

$$\phi_{,\alpha} = 0, \quad \phi_{,4} = 1. \tag{16.4}$$

Also, by the conditions of orthogonality,

$$g_{\alpha 4} = 0, \quad g^{\alpha 4} = 0, \tag{16.5}$$

and by (16.2)

$$dx^4 = F\,ds = F(-g_{44})^{\frac{1}{2}}\,dx^4, \tag{16.6}$$

so that

$$g_{44} = -F^{-2}, \quad g^{44} = -F^2. \tag{16.7}$$

The fundamental form of space-time then reads

$$g_{\alpha\beta}\,dx^\alpha\,dx^\beta - F^{-2}(dx^4)^2. \tag{16.8}$$

The components of the velocity-vector are

$$\lambda^\alpha = 0, \quad \lambda^4 = F; \quad \lambda_\alpha = 0, \quad \lambda_4 = -F^{-1}, \tag{16.9}$$

and those of the current-vector are

$$\Lambda^\alpha = 0, \quad \Lambda^4 = F^2; \quad \Lambda_\alpha = 0, \quad \Lambda_4 = -1. \tag{16.10}$$

By (14.6) we have

$$H\phi = -g^{44}\phi_{,4}\phi_{,4} = F^2, \tag{16.11}$$

so that the first of (14.15) is automatically satisfied. We have also by (14.10)

$$\Delta\phi = g^{\alpha\beta}\phi_{,\alpha\beta} + g^{44}\phi_{,44}. \tag{16.12}$$

Now

$$\phi_{,\alpha\beta} = -\tfrac{1}{2}F^2\,\partial g_{\alpha\beta}/\partial x^4, \quad \phi_{,44} = F^{-1}\partial F/\partial x^4, \tag{16.13}$$

and so

$$\Delta\phi = -\tfrac{1}{2}F^2 g^{\alpha\beta}\partial g_{\alpha\beta}/\partial x^4 - F\partial F/\partial x^4. \tag{16.14}$$

Substitution in the second equation of (14.15) yields the following result, which may also be obtained directly from (14.8) on evaluating $\lambda^i{}_{;i}$

by means of (16.8) and (16.9):

THEOREM XXVIII. *For co-moving coordinates in irrotational motion, in which x^4 is the dynamical flow along the stream-lines, the fundamental form of space-time is* (16.8) *and the equations of motion reduce to the single equation*

$$g^{\alpha\beta}\frac{\partial g_{\alpha\beta}}{\partial x^4}+2\left(\frac{1}{F}+\frac{dG}{dF}\right)\frac{\partial F}{\partial x^4}=0, \tag{16.15}$$

where F and G are as in (14.16).

The equation (16.15) may also be written

$$g^{\alpha\beta}\frac{\partial g_{\alpha\beta}}{\partial x^4}+\frac{2}{\rho+p}\frac{\partial \rho}{\partial x^4}=0. \tag{16.16}$$

For a homogeneous incompressible fluid the equation reduces to

$$g^{\alpha\beta}\partial g_{\alpha\beta}/\partial x^4=0, \tag{16.17}$$

and for an isothermal gas, as in (8.5), to

$$g^{\alpha\beta}\frac{\partial g^{\alpha\beta}}{\partial x^4}+2ab\frac{\partial}{\partial x^4}(\log p)=0 \quad [\rho=ap,\ b=(a+1)^{-1}]. \tag{16.18}$$

PART V. *Fluid motion in flat space-time.*

17. *Preliminary remarks.*

There is no mathematical incongruity in studying the motion of a fluid in flat space-time; but from a physical point of view such a theory cannot be exact, because the presence of matter produces curvature in space-time. We may, however, validate our theory from the physical point of view by regarding it as a first approximation, applicable to a limiting case where the density and pressure tend to zero.

All preceding results were developed against the background of an arbitrary Riemannian space-time: we now specialize that background by the condition of flatness, introducing (as we then can) coordinates x^i such that the components of the fundamental tensor are

$$g_{11}=g_{22}=g_{33}=1, \quad g_{44}=-1, \quad g_{ij}=0 \quad (i\neq j). \tag{17.1}$$

18. *Hydrostatics.*

By *hydrostatics* we shall understand the theory of those fluid motions for which the stream-lines are all parallel to a single vector in our flat space-time. A system of coordinates, satisfying (17.1), then exists such that

$$\lambda_\alpha = 0, \quad \lambda^4 = 1; \quad \lambda_\alpha = 0, \quad \lambda_4 = -1, \tag{18.1}$$

Greek suffixes having the range 1, 2, 3. The general equations of motion (6.6) then give

$$p_{,\alpha} = 0, \quad \rho_{,4} = 0. \tag{18.2}$$

Hence we have the following result (*cf.* Theorems XI and XVII):

THEOREM XXIX. *In hydrostatic equilibrium the pressure is constant on each 3-flat normal to the (straight) stream-lines, and the density is constant along each stream-line.*

Let us now consider hydrostatics in a fluid possessing a pressure-density equation (8.3). By (18.2) we have

$$p_{,\alpha} = 0, \quad \frac{d\rho}{dp} p_{,4} = 0, \tag{18.3}$$

and hence the following result:

THEOREM XXX. *If a fluid with a pressure-density equation* (8.3) *is in hydrostatic equilibrium, the pressure (and hence the density) is constant throughout if the fluid is compressible: but if it is incompressible, the pressure is constant over each 3-flat normal to the (straight) stream-lines, but may vary as we pass from one of these 3-flats to another.*

This distinction between the compressible and incompressible cases corresponds to the fact that in classical hydrodynamics a body of fluid remains in equilibrium under the application of a pressure, constant in space but variable in time, applied over its surface, if and only if it is incompressible.

19. *Steady motion and Bernoulli's theorem.*

By a *steady motion* we shall understand one in which, for some system of coordinates in our flat space-time satisfying (17.1), the values of λ^i, ρ, and p are independent of x^4.

When the conditions for steady character are introduced, the equations of motion (6.8), (6.9) give

$$(19.1)\qquad \begin{cases} d\rho/ds+(\rho+p)\theta=0, \\ (\rho+p)\,d\lambda^\alpha/ds+\lambda^\alpha\,dp/ds+p^{,\alpha}=0, \\ (\rho+p)\,d\lambda^4/ds+\lambda^4\,dp/ds=0, \end{cases}$$

Greek suffixes having, as usual, the range 1, 2, 3, and tensorial derivatives being replaced by ordinary derivatives. For the expansion θ we have, by (3.9),

$$(19.2)\qquad \theta=\lambda^\alpha{}_{,\alpha}=\partial\lambda^\alpha/\partial x^\alpha.$$

The last of (19.1) gives, along a stream-line,

$$(19.3)\qquad \lambda^4=A\exp\left(-\int\frac{dp}{\rho+p}\right),$$

where A is a constant depending on the particular stream-line.

Let us now consider a fluid possessing a pressure-density equation (8.3). For such (19.3) gives

$$(19.4)\qquad \lambda^4=AF^{-1},$$

where F is the index-function (9.1). Now

$$(19.5)\qquad \lambda^\alpha\lambda^\alpha-\lambda^4\lambda^4=-1,$$

and if we define the *relative velocity-vector* as the 3-vector λ^α and its magnitude q by

$$(19.6)\qquad q=(\lambda^\alpha\lambda^\alpha)^{\frac{1}{2}},$$

we have

$$(19.7)\qquad q^2=-1+(\lambda^4)^2=-1+A^2F^{-2},$$

i.e.

$$(19.8)\qquad F^2q^2+F^2=\text{const.},$$

along each stream-line.

This result also follows from the first three equations of (9.5), which now read

$$(19.9)\qquad d\Lambda^\alpha/ds+F^{,\alpha}=0.$$

Since, from the steady character of the motion,

$$dF/ds = F_{,i}\lambda^i = F_{,\alpha}\lambda^\alpha = F^{,\alpha}\lambda^\alpha, \tag{19.10}$$

multiplication of (19.9) by Λ^α and integration give along a stream-line the relation, equivalent to (19.8),

$$Q^2 + F^2 = \text{const.}, \tag{19.11}$$

where Q is the magnitude of the *relative current-vector* Λ^α, so that

$$Q = (\Lambda^\alpha \Lambda^\alpha)^{\frac{1}{2}} = Fq. \tag{19.12}$$

Hence we have this result:

THEOREM XXXI. Bernoulli's theorem. *In flat space-time the steady motion of a fluid possessing a pressure-density equation* (8.3) *satisfies the condition* (19.11) *expressing the constancy along a stream-line of the sum of the squares of the magnitude of the relative current-vector and the index-function.*

The connection with the classical theorem of Bernoulli† is easily seen, if we consider a fluid for which the ratio p/ρ is small and the velocity small compared with that of light. For such a fluid we have approximately by (9.1)

$$F = \exp\left(\int_{p_0}^{p} \frac{dp}{\rho}\right) = 1 + \int_{p_0}^{p} \frac{dp}{\rho}, \tag{19.13}$$

and hence (19.8) or (19.11) gives approximately

$$\tfrac{1}{2}q^2 + \int_{p_0}^{p} dp/\rho = \text{const.}, \tag{19.14}$$

the classical formula, q being now the ratio of Newtonian velocity to the velocity of light.

20. *Steady irrotational motion.*

Let us now consider in flat space-time, with coordinates satisfying (17.1), a steady irrotational motion of a fluid possessing a pressure-density equation (8.3). By (14.3) we have

$$\Lambda_\alpha = -\partial\phi/\partial x^\alpha, \quad \Lambda_4 = -\partial\phi/\partial x^4, \tag{20.1}$$

† Lamb, *op. cit.*, 21.

where ϕ is the current-potential. By the last of these equations we have (since Λ_4 is by hypothesis independent of x^4)

$$\phi = -\Lambda_4 x^4 + \bar{\phi}(x^1, x^2, x^3), \tag{20.2}$$

where $\bar{\phi}$ is an arbitrary function, and by the first three

$$\Lambda_\alpha = x^4 \partial\Lambda_4/\partial x^\alpha - \partial\bar{\phi}/\partial x^\alpha. \tag{20.3}$$

But by hypothesis Λ_α is independent of x^4, and so

$$\partial\Lambda_4/\partial x^\alpha = 0, \quad \Lambda_4 = \text{const.}, \quad \Lambda^4 = \text{const.}, \tag{20.4}$$

since Λ_4 is independent of x^4. Now we have identically

$$\Lambda^\alpha \Lambda^\alpha - \Lambda^4 \Lambda^4 = -F^2, \tag{20.5}$$

and hence, in the notation of (19.11),

$$Q^2 + F^2 = \text{const.}, \tag{20.6}$$

but whereas in (19.11) the expression was constant only along each streamline, now it is constant throughout the fluid. Hence we have the following result, which, like Theorem XXXI, extends to relativity a familiar equation of classical hydrodynamics†.

THEOREM XXXII. *In a steady irrotational motion in flat space-time of a fluid possessing a pressure-density equation, the sum of the squares of the magnitude of the relative current-vector and the index-function is constant throughout the fluid.*

In the case of a homogeneous incompressible fluid this fact is expressed accurately by the equation

$$(1+p/\rho)^2 q^2 + (1+p/\rho)^2 = \text{const.}, \tag{20.7}$$

where $q^2 = \lambda^\alpha \lambda^\alpha$, the square of the magnitude of the relative velocity-vector.

21. *The velocity of propagation of small disturbances.*

Let us suppose that throughout a fluid possessing a pressure-density equation there exists a hydrostatic regime, as described in § 18. We shall suppose the fluid to be compressible, so that the index-function F is constant throughout the fluid : let us denote this constant value by F_0.

† Lamb., *op. cit.*, 21.

We have then, by (18.1),

$$\Lambda_\alpha = 0, \quad \Lambda_4 = -F_0, \tag{21.1}$$

and consequently by (20.1) the current-potential is

$$\phi_0 = F_0 x^4. \tag{21.2}$$

Now let a small irrotational disturbance be given to the fluid. Let the index-function of the disturbed motion be

$$F = F_0 + \bar{F}, \tag{21.3}$$

where $\bar{F}$ is a small function of the space-time coordinates, and let the current-potential be

$$\phi = \phi_0 + \psi = F_0 x^4 + \psi, \tag{21.4}$$

where ψ is small. In developing the theory of the small disturbance we shall neglect second and higher powers of $\bar{F}$, ψ, and their derivatives.

With the notation of (14.6) we have

$$H\phi = -\phi_{,\alpha}\phi_{,\alpha} + \phi_{,4}\phi_{,4} = -\psi_{,\alpha}\psi_{,\alpha} + (F_0 + \psi_{,4})(F_0 + \psi_{,4}), \tag{21.5}$$

and so, to the first order,

$$H\phi = F_0^2 + 2F_0\psi_{,4}, \quad (H\phi)^{\frac{1}{2}} = F_0 + \psi_{,4}. \tag{21.6}$$

Retaining only terms of the first order, (21.3) and (14.15) give

$$\psi_{,4} = \bar{F}, \tag{21.7}$$

and

$$\Delta\psi + (dG/dF)_0 \psi_{,4i}\phi^{,i} = 0, \tag{21.8}$$

the subscript zero indicating evaluation for $F = F_0$. To the same degree of approximation this gives

$$\Delta\psi - (F dG/dF)_0 \psi_{,44} = 0, \tag{21.9}$$

or

$$\psi_{,\alpha\alpha} - (1 + F dG/dF)_0 \psi_{,44} = 0. \tag{21.10}$$

But this is a wave-equation, and we may state the following result:

THEOREM XXXIII. *In flat space-time a small irrotational disturbance is propagated with a velocity* [*cf.* (14.14)]

$$v = (1 + F\,dG/dF)^{-\frac{1}{2}} = (dp/d\rho)^{\frac{1}{2}}, \tag{21.11}$$

the velocity of light in vacuo being unity.

This result is formally identical with the formula for the velocity of sound in classical hydrodynamics†. *We are led to the unexpected result that as the fluid tends to incompressibility* ($dp/d\rho \to \infty$), *the velocity of propagation does not tend to the fundamental velocity, that of light, but tends to infinity.* This result is apparently in flagrant opposition to a fundamental concept of relativity, namely, that it is impossible to send a signal with a velocity greater than that of light. It would appear proper to draw the conclusion that there is a lower limit to the compressibility of compressible fluids possessing a pressure-density equation, and that in fact for such fluids

$$dp/d\rho \leqslant 1. \tag{21.12}$$

No difficulty, however, arises in the case of absolutely incompressible fluids, because for them a wave-motion of the type discussed will not exist.

We may note that when the perfect fluid consists of radiation, the velocity of propagation of a disturbance is $1/\sqrt{3}$. This would also be the velocity of propagation in a fluid incompressible in the sense of Eddington.

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† Lamb, *op. cit.*, 477.

FUNCTIONS WITH DOMINANT SINGULARITIES OF THE GENERALIZED ALGEBRAIC-LOGARITHMIC TYPE (II): ON THE ORDER OF THE HADAMARD PRODUCT

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1. *Introduction.* We are concerned with relations between the singularities of the function-elements represented by the power series

$$(1) \qquad f(z) = \sum_0^\infty a_n z^n, \quad g(z) = \sum_0^\infty b_n z^n, \quad h(z) = \sum_0^\infty a_n b_n z^n,$$

each of finite non-zero radius of convergence. Hadamard† has obtained the following result connecting $f(z)$, $g(z)$, and $h(z)$.

THEOREM A. *If α and β denote the vertices of the star domains, with respect to the origin, of $f(z)$ and $g(z)$ respectively, then $h(z)$ is regular in the star domain whose vertices are the points $\alpha\beta$.*

If the series in (1) are the developments, within the respective circles of convergence, of three functions each of which is uniform over the entire plane, then Theorem A concerns three uniform functions‡. In any other case it is a relation between given branches of each of three functions, the term "function" being used in the widest sense. It is this case which arises in the present paper, and it is not our intention to embark on a discussion of the relationships existing between three complete functions. The difficulties of such a study have been pointed out by Faber and Ostrowski§. There is, however, one matter which is relevant to the present discussion. If the two given functions have each only singularities of algebraic-logarithmic type, then they satisfy a linear differential

† 2, Ch. VII; 3; 4, § 88; 7; 9, Ch. III; 10, Ch. VI.
‡ 3, 239; 6, 285–286.
§ 6, 11.

equation of Fuchsian type, and the same is true of the Hadamard product function†. Thus the points $z=0$ and $z=\infty$ may be branch points of the product function even when $z=0$ is not singular for the particular branches $f(z)$, $g(z)$, and $h(z)$‡. .

In the problem under discussion the question whether the point $\alpha\beta$ is *effectively* singular for $h(z)$ is critical. The example§

$$f(z)=\Sigma z^{2n}, \quad g(z)=\Sigma z^{2n+1}, \quad h(z)=0$$

shows that some, or even all, of the points $\alpha\beta$ need not be singular for $h(z)$. Pólya‖ has obtained sufficient conditions under which the point $\alpha\beta$ is singular for $h(z)$. They are set out in

THEOREM B. *Suppose that each of $f(z)$ and $g(z)$ has only a countable number of singular points, at α_μ and β_ν respectively, and that these are all isolated and lie at the vertices of the respective star domains, and that $f(z)$ and $g(z)$ are regular at all other boundary points of these domains. Then all the points on the boundary of the star domain of $h(z)$ have the form $\gamma=\alpha_\mu\beta_\nu$, and γ is certainly a singular point if only one pair of indices μ, ν exist such that $\gamma=\alpha_\mu\beta_\nu$.*

In the case of polar singularities Borel¶ has obtained a remarkably precise result. This has been extended by Faber¶ in the form given below.

THEOREM C. *If the singularities of $f(z)$ and $g(z)$ each form an isolated†† set of points, and if $f(z)$ and $g(z)$ have, at the points α and β respectively, poles of orders p and q, and if $\alpha\beta\neq\alpha'\beta'$, where α' and β' are any other singular or external‡‡ points of $f(z)$ and $g(z)$ respectively, then $h(z)$ has at the point $\alpha\beta$ a pole of order $p+q-1$.*

Under the hypotheses of Theorem B we extend Theorem C to the case of the generalized algebraic-logarithmic singularity (§5). This type of singularity has been studied in some detail by Jungen§§ who also proved the following theorem‖‖.

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† See Theorem D *infra*.
‡ **4**, 348; **6**, 291.
§ **2**, 294; **3**, 244; **4**, 348.
‖ **13**, §93, Theorem IX.
¶ **3**, 242; **4**, 348–349; **6**, 294; **8**, Theorem 7; **9**, 327; **10**, 19.
†† **14**, §84.
‡‡ **4**, 348; **6**, 288–289.
§§ **8**. See also **15**.
‖‖ **8**, 297–306. In particular, Theorem E.

THEOREM D. *If each of the series $f(z)$ and $g(z)$ is an integral of a linear homogeneous differential equation of Fuchsian type then the same is true of $h(z)$.*

In other words, if $f(z)$ and $g(z)$ possess only singularities of algebraic-logarithmic type, then the same is true of $h(z)$. In particular, Pólya† has obtained the more precise result given below.

THEOREM E. *If $f(z)$ has a singularity of algebraic-logarithmic type at the point α as its only singularity on the circle of convergence, and if $g(z)$ has, at the points β_ν on its circle of convergence, only isolated singular points, then the points $\alpha\beta_\nu$ are effectively singular for $h(z)$.*

In this connection it should be noted that an isolated singular point may be either critical or non-critical‡.

2. *Statement of the problem.* These results introduce a problem of considerable importance. Suppose, for convenience, that all three series possess unit radius of convergence and let the orders of $f(z)$, $g(z)$ on $|z|=1$ be ω and ω' respectively. The problem may be formulated as follows: *under what conditions is the order of $h(z)$ equal to $\omega+\omega'-1$?*

Theorems A, B, D, and E provide the first step towards a solution since they are concerned with the location of the singularities of $h(z)$. Theorem C provides a solution in the simple case in which the dominant singularities§ of $f(z)$ and $g(z)$ on $|z|=1$ are poles. We extend this to the case in which the dominant singularities of $f(z)$ and $g(z)$ on $|z|=1$ are each of algebraic-logarithmic type§. The results obtained are the more precise in that a relation is established, not only between the orders, but also between the weights§ of the functions concerned. In this connection the following theorem, which is an immediate consequence of the definition of the weight of a function, is of some use.

THEOREM F. *If the weights of $f(z)$ and $g(z)$ on $|z|=1$ are respectively $[\sigma_1, k_1]$, $[\sigma_2, k_2]$, then the weight of $h(z)$ on $|z|=1$ cannot exceed*

$$[\sigma_1+\sigma_2-1, k_1+k_2].$$

In particular, if the orders of $f(z)$ and $g(z)$ on $|z|=1$ are respectively ω and ω' then that of $h(z)$ cannot exceed $\omega+\omega'-1$.

† 13, §92, Theorem VIII, part III.
‡ 13, 735–736.
§ 15, §4.

In the case in which the dominant singularities of each of $f(z)$ and $g(z)$ on $|z| = 1$ are of algebraic-logarithmic type† the relations become more precise. For in this case positive constants C_1 and C_2 exist†, such that, for every $n > n_0$,

$$|a_n| < C_1 n^{\sigma_1 - 1}(\log n)^{k_1}, \quad |b_n| < C_2 n^{\sigma_2 - 1}(\log n)^{k_2},$$

and therefore

$$|a_n b_n| < C_1 C_2 n^{\sigma_1 + \sigma_2 - 2}(\log n)^{k_1 + k_2}$$

for every $n > n_0$.

Theorems B and D show that, if $f(z)$ and $g(z)$ have all their singularities of algebraic-logarithmic type, then the singularities of $h(z)$ are all of this type, and they are to be found at every point $\alpha_\mu \beta_\nu$ which is not duplicated. Singularities may or may not be found at the points $\alpha_\mu \beta_\nu$ which are duplicated, but if such a point is singular for $h(z)$, then the singularity is of algebraic-logarithmic type. Theorem E deals with the case in which the singularities on the circle of convergence are of algebraic-logarithmic type.

In § 3 the characteristic properties of the Hadamard product of two algebraic-logarithmic singularities are considered in full detail, leading to the generalization of Theorem C in § 5. The case in which $f(z)$ and $g(z)$ have each only algebraic-logarithmic singularities on the circle of convergence is dealt with in § 6, while the more general case in which $f(z)$ and $g(z)$ have each dominant singularities of algebraic-logarithmic type on the circle of convergence is treated in § 7. The outstanding result, true in all cases, may be expressed in the following form: *if the weights of* $f(z)$ *and* $g(z)$ *on* $|z| = 1$ *are respectively* $[\sigma_1, k_1]$, $[\sigma_2, k_2]$, *then that of* $h(z)$ *is* $[\sigma_1 + \sigma_2 - 1, k_1 + k_2]$, *provided that at least one* $\alpha_\mu \beta_\nu$ *is not duplicated.* The proofs of Theorems 3–7 depend essentially on a process of dissection of the Taylor series appropriate to the occasion. A discussion on this subject is given in § 4.

The last part of the paper is concerned with the coefficient problem of the three series, that is, the problem of obtaining information concerning the singularities of $h(z)$ from the sequences of coefficients (a_n), (b_n) of $f(z)$ and $g(z)$ respectively. In the general theory‡ the distribution of zero coefficients in (a_n) and (b_n) leads to a knowledge of the distribution of zero coefficients in the sequence $(a_n b_n)$, and this, in turn, gives information about the nature and position§ of the singularities of $h(z)$. In this paper we are concerned with the determination of the weights of the dominant singu-

† **15**, Theroem D.
‡ **13**, § 68.
§ **12**, § 2.

larities of $h(z)$, and for this purpose it is necessary to know how the dominant coefficients in the sequence (a_n) combine with those of the sequence (b_n) to form dominant coefficients of the sequence $(a_n b_n)$.

If each of $f(z)$ and $g(z)$ has $|z|=1$ as its circle of convergence, then $h(z)$ may have no singularities on this circle or only singularities of less than the statutory weight. Arithmetical conditions on the sequences (a_n) and (b_n) are found which determine these cases. It is then shown that, *if $h(z)$ is of maximum weight on $|z|=1$, it cannot have more than $l_1 l_2$ dominant singularities nor less than $\min(l_1, l_2)$ dominant singularities on $|z|=1$, where l_1 and l_2 are the numbers of dominant singularities of $f(z)$ and $g(z)$ on this circle.*

3. *The product of two algebraic-logarithmic singularities.* Suppose first that each of the series for $f(z)$ and $g(z)$ in (1) represents the expansion about the origin of a given branch of a single algebraic-logarithmic singularity with a single dominant element, at the points α and β respectively, with the respective weights† $[\sigma_1, k_1]$, $[\sigma_2, k_2]$. The type† of the first element is then (s_1, k_1) unless s_1-1 is a negative integer, when it is (s_1, k_1+1), and that of the second is (s_2, k_2), unless s_2-1 is a negative integer, in which case the type is (s_2, k_2+1). The constants are connected by the relations $R(s_1)=\sigma_1$, $R(s_2)=\sigma_2$. Jungen‡ has proved that the coefficients in the series for $f(z)$ and $g(z)$ take the form

$$(2)\qquad \begin{cases} a_n = \dfrac{A n^{s_1-1}(\log n)^{k_1}}{\alpha^n}\{1+O(1/\log n)\}, \\ b_n = \dfrac{B n^{s_2-1}(\log n)^{k_2}}{\beta^n}\{1+O(1/\log n)\}, \end{cases}$$

where A and B are non-zero constants.

It follows from Theorem E that $h(z)=\Sigma a_n b_n z^n$ represents the expansion about the origin of a certain branch of an algebraic-logarithmic singularity at the point $\alpha\beta$. From (2)

$$(3)\qquad a_n b_n = \frac{AB n^{s_1+s_2-2}}{(\alpha\beta)^n}(\log n)^{k_1+k_2}\{1+O(1/\log n)\}$$

and thus the singularity is of type $(s_1+s_2-1,\ k_1+k_2)$ if s_1+s_2-2 is not a negative integer and of type $(s_1+s_2-1,\ k_1+k_2+1)$ if s_1+s_2-2 is

† 8, 274.
‡ 8, 273, 275.

a negative integer. In either case the weight is

$$[\sigma_1+\sigma_2-1, \quad k_1+k_2].$$

Hence we have

THEOREM 1. *If each of $f(z)$ and $g(z)$ has only one singularity, and that of algebraic-logarithmic type with a single dominant element, situated at the points α and β and of weights $[\sigma_1, k_1]$, $[\sigma_2, k_2]$ respectively, then $h(z)$ possesses only one singularity, and this is also of the same kind, situated at the point $\alpha\beta$ and of weight $[\sigma_1+\sigma_2-1, \ k_1+k_2]$.*

This theorem includes, in particular, the case of the product of two algebraic singularities, each with a single dominant element†, when, of course, $k_1=0$, $k_2=0$. A special case of this arises when s_1+s_2-1 is a positive integer, the dominant element then being polar. If s_1 and s_2 are not themselves positive integers, this differs from the case of Theorem C in that there are elements of smaller weight which are non-polar, so that the complete product function is not uniform.

We next consider the case in which the singularity of $f(z)$ at the point α contains r_1 elements of greatest weight, each equal to $[\sigma_1, k_1]$, but of *different types*, while the singularity of $g(z)$ at the point β contains r_2 elements of greatest weight, each equal to $[\sigma_2, k_2]$, also of different types. Jungen‡ has proved that the coefficients in the two series take the form

$$(4) \qquad \begin{cases} a_n = \dfrac{n^{\sigma_1-1}(\log n)^{k_1}}{\alpha^n}\left\{\sum_{\mu=1}^{r_1} A_\mu e^{i\alpha_\mu \log n}+O(1/\log n)\right\}, \\ b_n = \dfrac{n^{\sigma_2-1}(\log n)^{k_2}}{\beta^n}\left\{\sum_{\nu=1}^{r_2} B_\nu e^{i\beta_\nu \log n}+O(1/\log n)\right\}, \end{cases}$$

where the A_μ, B_ν are non-zero constants, the α_μ are real and different and the β_ν are real and different.

It follows from Theorem E that $h(z)=\Sigma a_n b_n z^n$ represents the expansion about the origin of a certain branch of a function with an algebraic-logarithmic singularity at the point $\alpha\beta$. From (4)

$$(5) \qquad a_n b_n = \frac{n^{\sigma_1+\sigma_2-2}(\log n)^{k_1+k_2}}{(\alpha\beta)^n}\left\{\sum_{\mu,\nu=1}^{r_1,r_2} A_\mu B_\nu e^{i(\alpha_\mu+\beta_\nu)\log n}+O(1/\log n)\right\}.$$

This singularity, of weight $[\sigma_1+\sigma_2-1, \ k_1+k_2]$, is of general type, since all the $\alpha_\mu+\beta_\nu$ cannot be the same without all the α_μ being equal and all the

† See **3**, 243; **8**, 298, for a discussion of this case.

‡ **8**, 281.

β_ν being equal. The number of elements of greatest weight in the singularity is at least equal to the number of $\alpha_\mu+\beta_\nu$ which are not duplicated and cannot exceed the number of $\alpha_\mu+\beta_\nu$ which are different. Evidently the number of different elements cannot exceed $r_1 r_2$; we proceed to find a lower bound for the number of non-duplicated elements in the product singularity.

Consider the set of r_1-1 relations

$$(6)\qquad \begin{cases} \alpha_1+\beta_1 = \alpha_2+\beta_2 = \ldots = \alpha_{r_2}+\beta_{r_2}, \\ \alpha_2+\beta_1 = \alpha_3+\beta_2 = \ldots = \alpha_{r_2+1}+\beta_{r_2}, \\ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ \alpha_{r_1-1}+\beta_1 = \alpha_{r_1}+\beta_2 = \ldots = \alpha_{r_1+r_2-2}+\beta_{r_2}, \end{cases}$$

each connecting r_2 of the $\alpha_\mu+\beta_\nu$, in which it is supposed that $r_1 \geqslant r_2$, and that each of the $r_2(r_1-1)$ quantities $\alpha_\mu+\beta_\nu$ occurs only once. The $\alpha_\mu+\beta_\nu$ form r_1-1 sets of r_2 equal numbers; this is the greatest possible of equalities, for the addition of one further relation is sufficient to ensure that every α_μ is the same and every β_ν is the same.

When the relations (6) hold, the number of $\alpha_\mu+\beta_\nu$ which are not duplicated is r_2 and the number which are different is r_1+r_2-1. Hence the number of different elements in the singularity typified by (5) is at least equal to min (r_1, r_2) and at most equal to $r_1 r_2$.

That the smaller value can occur is shown by the example

$$a_n = n^{\sigma_1-1}(\log n)^{k_1}\{e^{i\theta\log n}-e^{2i\theta\log n}+e^{3i\theta\log n}+O(1/\log n)\},$$

$$b_n = n^{\sigma_2-1}(\log n)^{k_2}\{e^{i\theta\log n}+e^{2i\theta\log n}+O(1/\log n)\},$$

for which

$$a_n b_n = n^{\sigma_1+\sigma_2-2}(\log n)^{k_1+k_2}\{e^{2i\theta\log n}+e^{5i\theta\log n}+O(1/\log n)\}.$$

On the other hand, if one of the singularities has only a single dominant element, so that $r_2 = 1$, then the product singularity has

$$r_1 = r_1+r_2-1 = r_1 r_2$$

dominant elements.

It may be noted that the coefficients $A_\mu B_\nu$ are derived from those of the component singularities as in the simpler case of (3). On writing

$$(7)\qquad s_1 = \sigma_1+ia_\mu, \quad s_2 = \sigma_2+i\beta_\nu,$$

we see that the corresponding element of (5) is of type (s_1+s_2-1, k_1+k_2), *in general*.

In particular, if s_1+s_2-2 is a negative integer, then $s_1+s_2=\sigma_1+\sigma_2$ and $\alpha_\mu+\beta_\nu=0$, from (7). Thus at most one element of the product singularity can be of the exceptional type $(s_1+s_2-1,\ k_1+k_2+1)$.

In the same way, if s_1+s_2-1 is a positive integer and $k_1=0$, $k_2=0$, then the product singularity has a single polar element among its dominant terms provided that the corresponding $\alpha_\mu+\beta_\nu=0$.

The results obtained above may be stated as follows:

THEOREM 2. *If each of $f(z)$ and $g(z)$ has only one singularity, and that of algebraic-logarithmic type, situated at the points α and β and of weights $[\sigma_1, k_1]$, $[\sigma_2, k_2]$ respectively, at least one of the two being of general type, then $h(z)$ possesses only one singularity, also of general type, and it is situated at the point $\alpha\beta$ and is of weight $[\sigma_1+\sigma_2-1,\ k_1+k_2]$.*

It may be noted that results (2) and (4) were proved by Jungen† for the more general case in which the functions in question have each on their circles of convergence only singularities of algebraic-logarithmic type, of which the singularities at α and β are of greater weight than the others on their circles. This indicates the possibility of the further investigations of §§ 5 and 6.

4. *The problem of dissection.* In order to make use of the results of § 3 for the purpose of considering the Hadamard product of two functions, each of which has at least one algebraic-logarithmic singularity (among other singularities), it is necessary to separate the Taylor series of each of the functions into two parts, one of which represents a given branch of the selected algebraic-logarithmic singularity and the other the remainder of the function.

There are two points of view from which the general problem of dissection may be considered. One is due to Faber‡ and depends on an extension of Weierstrass' results on the representation of a function with prescribed singularities at a set of isolated points§. Faber has obtained an expression for the function which is regular at every point except the set of singular points and their limiting points§ and uniform except on the appropriate cuts arising from those singularities which are critical points. Convergence is assured by means of the removal of increasing sets of terms from the expansions associated with the separate singularities. It follows

† *Loc. cit.*

‡ **5.**

§ For an account of the properties of such sets from this point of view see **14**, § 84.

that, in the neighbourhood of a regular point, the function can be expressed as the sum of an infinite number of Taylor series, each associated with a singularity.

The other point of view is to be found in the work of Aronszajn and Pólya, among others†. Here the function is given and also the nature and position of its singularities. The separation is effected by means of contours which divide the singular points into sets as required. The corresponding representation of the dissected function takes the form of a set of contour integrals. This method is very powerful, and can be used not only for isolated singularities but also for the more general types classified by Pólya‡. The method has, however, the disadvantage that, in separating off any singularity, other than a pole or isolated essential singularity, the effort to secure uniformity introduces an artificial singularity whose negative occurs in the expression for the remainder of the function.

The method of Faber is simpler in dealing with non-critical points, but the separation of a critical singularity introduces such serious difficulties that Faber specifically excluded them from his treatment of the Hadamard multiplication of isolated singular points§.

In the generalization of Theorem C with the replacement of poles by algebraic-logarithmic singularities, we therefore follow Pólya‖ in using the second method of separation and in imposing the restriction that the set of points at which the function is singular has no finite limiting point.

Let the singularities of the function be at the points a_ν, where

$$|a_1| \leqslant |a_2| \leqslant |a_3| \leqslant \ldots \leqslant |a_\nu| \leqslant \ldots .$$

Since there is no finite limiting point, a circle $|z| = a$, say, can be drawn which encloses the points $a_1, a_2, \ldots, a_q$. From these points describe outward drawn radii to meet $|z| = a$ at the points $a_1', a_2', \ldots, a_q'$. From each point a_ν' describe a loop containing a_ν but not the origin. Each loop can be deformed into an arbitrarily small circle around a_ν and a return traverse of the segment $a_\nu a_\nu'$, so that none of the deformed loops intersect. Thus

$$f(z) = \frac{1}{2\pi i}\int_{C'} \frac{f(w)\,dw}{w-z} + \sum_{\nu=1}^{q} \frac{1}{2\pi i}\int_{C_\nu} \frac{f(w)\,dw}{w-z}, \tag{8}$$

$$= f^*(z) + \sum_{\nu=1}^{q} f_\nu(z),$$

† See 1 and 13 for further references.
‡ 13, §69.
§ 6, p. 296.
‖ 13, §93.

where C' consists of the sum of the open arcs $a_\nu' a'_{\nu+1}$, traversed positively, and C_ν consists of the deformed loop containing the point a_ν, traversed negatively. Evidently the function $f^*(z)$ is regular at all points inside the circle $|z| = a$, while $f_\nu(z)$ is regular and uniform at every point exterior to the segment $a_\nu a_\nu'$ and has a singularity at the point a_ν. It is regular, but not necessarily uniform, at inner points of $a_\nu a_\nu'$ and a_ν' may be a singular point. In the case of a pole this is not so, but in the case of a critical point it certainly is, and in the case of an algebraic-logarithmic singularity $f_\nu(z)$ has a singularity of the same kind† at the point a_ν'.

We shall make use of this method of dissection in §§ 5–7; we now give two simple examples‡ of its use.

If $f(z)$ is the sum of a function $\phi(z)$ having all its singularities outside the circle $|z| = 2$ and the function $\log(1-z)$, then, evaluating as in (8), we have

$$f^*(z) = \phi(z) + \log(z-2), \quad f_1(z) = \log\{(z-1)/(z-2)\}.$$

If $f(z)$ is the sum of $\phi(z)$ and $\sqrt{(z-1)}$, then from (8)

$$f_1(z) = \frac{2}{\pi i} + \frac{\sqrt{(z-1)}}{\pi i} \log\left\{\frac{\sqrt{(z-1)}-1}{\sqrt{(z-1)}+1}\right\}, \quad f^*(z) = \phi(z) + \sqrt{(z-1)} - f_1(z).$$

It is easily seen that the function $f_1(z)$ has the singularity $\sqrt{(z-1)}$ at $z = 1$ and a logarithmic singularity at $z = 2$.

5. *The generalization of the Borel-Faber theorem.* We now consider the case in which each of $f(z)$ and $g(z)$ has its singularities at a set of isolated points with no finite limiting point. The singularities of $f(z)$ at the set of points (a_μ) and of $g(z)$ at the set of points (β_ν) may be of any of the types considered by Pólya§ so long as one at least of each set is an algebraic-logarithmic singularity.

Let $f(z)$ have at the point a_u an algebraic-logarithmic singularity of weight $[\sigma_1, k_1]$ and let $g(z)$ have at the point β_v an algebraic-logarithmic singularity of weight $[\sigma_2, k_2]$. The dissection of $f(z)$ is carried out as in § 4 with a_u as an interior point of $|z| = a$. Similarly in the case of $g(z)$ a circle $|z| = b$ is taken not passing through any singular point of $g(z)$ but enclosing the point β_v. In accordance with the notation of § 4 we may write

$$(9) \qquad f(z) = f^*(z) + \sum_{\mu=1}^{q_1} f_\mu(z), \quad g(z) = g^*(z) + \sum_{\nu=1}^{q_2} g_\nu(z),$$

† The integral of an algebraic-logarithmic function is a function of the same kind.

‡ *Cf.* **1**, 55–57.

§ **13**, § 68.

where α_μ, $(\mu = 1, 2, \ldots, q_1)$, is the set of singular points contained within $|z| = a$ and β_ν, $(\nu = 1, 2, \ldots, q_2)$, is the set of singular points contained within $|z| = b$; and the associated singularities on $|z| = a$ and $|z| = b$, are at the points α_μ', $(\mu = 1, 2, \ldots, q_1)$, and β_ν', $(\nu = 1, 2, \ldots, q_2)$, respectively.

In the notation of Pólya† the Hadamard product $h(z)$ of the series $f(z)$ and $g(z)$ may be written as follows

$$h(z) = \sum_{\mu=1}^{q_1} \sum_{\nu=1}^{q_2} [f_\mu g_\nu] + \sum_{\mu=1}^{q_1} [f_\mu g^*] + \sum_{\nu=1}^{q_2} [f^* g_\nu] + [f^* g^*]. \tag{10}$$

From Theorem A it follows that none of the singularities belonging to the functions in the last three groups can lie within the circle $|z| = c$, where $c \leqslant a|\beta_1|$, $b|\alpha_1|$, and that the only possible singularity of $[f_\mu g_\nu]$ in this circle is at the point $\alpha_\mu\beta_\nu$, since none of the points $\alpha_\mu'\beta_\nu$, $\alpha_\mu\beta_\nu'$, $\alpha_\mu'\beta_\nu'$ lies inside the circle $|z| = c$. Thus the only singularities of $h(z)$ within the circle $|z| = c$ are at the points $\alpha_\mu\beta_\nu$, $(\mu = 1, 2, \ldots, q_1;\ \nu = 1, 2, \ldots, q_2)$. Some of these points may lie outside the circle, but if a and b are so chosen as to satisfy the inequalities $|\alpha_u\beta_v| < c \leqslant a|\beta_1|$, $b|\alpha_1|$, then certainly the point $\alpha_u\beta_v$ will lie within the circle $|z| = c$.

Now the only singularities of $f_u(z)$ and $g_v(z)$ are at the points α_u, α_u' and β_v, β_v' respectively, and these are all of algebraic-logarithmic type. Hence, by Theorem D, the function $[f_u . g_v]$ has all its singularities of this type. Moreover, by Theorem E, the function $[f_u . g_v]$ certainly has a singularity at the point $\alpha_u\beta_v$, and is regular and uniform except on the ray running from this point outwards through the point $\alpha_u'\beta_v'$ to infinity. On this ray $[f_u . g_v]$ is not uniform but is regular save possibly at the points $\alpha_u'\beta_v$, $\alpha_u\beta_v'$, $\alpha_u'\beta_v'$.

It is evident that, if the point $\alpha_u\beta_v$ is not duplicated, then every function-element on the right-hand side of (10) is regular at this point except $[f_u . g_v]$ which has an algebraic-logarithmic singularity thereat. On writing

$$\Sigma a_n z^n = \Sigma a_n' z^n + \Sigma a_n'' z^n, \quad \Sigma b_n z^n = \Sigma b_n' z^n + \Sigma b_n'' z^n,$$

where $\Sigma a_n' z^n$, $\Sigma b_n' z^n$ represent the expansions about the origin of specified branches of $f_u(z)$, $g_v(z)$ respectively, we see that

$$h(z) = \Sigma a_n' b_n' z^n + \Sigma(a_n' b_n'' + a_n'' b_n' + a_n'' b_n'') z^n,$$

where $\Sigma a_n' b_n' z^n$ has an algebraic-logarithmic singularity at the point $\alpha_u\beta_v$ and $\Sigma(a_n' b_n'' + a_n'' b_n' + a_n'' b_n'') z^n$ is regular there.

† 13, § 94.

Consider the series $\Sigma a_n' z^n$, $\Sigma b_n' z^n$. It has already been pointed out (following Theorem 2) that results (2) and (4) hold also in the case of these series. Thus the coefficients $a_n' b_n'$ satisfy relations (3) or (5), as the case may be, and so the singularity of $h(z)$ at the point $\alpha_u \beta_v$ is of weight $[\sigma_1+\sigma_2-1, k_1+k_2]$. Hence we have

THEOREM 3. *Suppose that each of $f(z)$ and $g(z)$ has only a countable number of singular points, at (α_μ) and (β_ν) respectively, and that these are all isolated and lie at the vertices of the respective star domains. If $f(z)$ and $g(z)$ have, at the points α_u and β_v respectively, algebraic-logarithmic singularities of weights $[\sigma_1, k_1]$, $[\sigma_2, k_2]$, and if $\alpha_u \beta_v \neq \alpha_\mu \beta_\nu$ (except when $\mu = u$ and $\nu = v$), then $h(z)$ has at the point $\alpha_u \beta_v$ an algebraic-logarithmic singularity of weight $[\sigma_1+\sigma_2-1, k_1+k_2]$.*

The characteristic details of the product singularity may be obtained as in §3. In particular, if the singularities of $f(z)$ and $g(z)$ at the points α_u and β_v are each poles we obtain a modified form of Theorem C.

6. *Algebraic-logarithmic singularities on the circle of convergence.* We next consider the case in which each of the functions $f(z)$ and $g(z)$ has on its circle of convergence only singularities of algebraic-logarithmic type. For convenience we shall suppose the radius of convergence of $f(z)$, $g(z)$, and $h(z)$ to be unity except where otherwise stated.

Since the singularities of $f(z)$ on $|z| = 1$ are all isolated, a circle $|z| = a$, with $a > 1$, exists which separates the singular points on $|z| = 1$ from those external to this circle. In the same way a circle $|z| = b$, with $b > 1$, can be used for the dissection of $g(z)$. Applying the methods of §§4 and 5, we obtain a special case of Theorem 3, owing to the fact that the singularities of $f(z)$ at the points α_μ, $(\mu = 1, 2, \ldots, q_1)$, all lie on $|z| = 1$, and the singularities of $g(z)$ at the points β_ν, $(\nu = 1, 2, \ldots, q_2)$, all lie on $|z| = 1$. It follows from Theorem 3 that all the singularities of $h(z)$ at the points $\alpha_\mu \beta_\nu$, $(\mu = 1, 2, \ldots, q_1;\ \nu = 1, 2, \ldots, q_2)$ lie on $|z| = 1$ and are of algebraic-logarithmic type. Hence we have

THEOREM 4. *If $f(z)$ and $g(z)$ have on their circles of convergence only algebraic-logarithmic singularities, at the points α_μ and β_ν respectively, then $h(z)$ only has algebraic-logarithmic singularities. These are to be found at all the points $\alpha_\mu \beta_\nu$ which are not duplicated. They may or may not exist at other points $\alpha_\mu \beta_\nu$.*

This result differs from that of Theorem B in that no restriction is placed on the number or nature of the singularities which $f(z)$ and $g(z)$ may possess outside the circle of convergence. It is, in fact, a more general form of Theorem E.

The weight of each singularity and its character, in detail, may be determined as in § 3. In particular the following result holds:

THEOREM 5. *If l_1 of the q_1 singularities at the points α_μ^* are dominant and of weight $[\sigma_1, k_1]$, and l_2 of the q_2 singularities of $g(z)$ at the points β_ν^* are dominant and of weight $[\sigma_2, k_2]$, then $h(z)$ has on $|z| = 1$ exactly $l_1 l_2$ dominant singularities of weight $[\sigma_1+\sigma_2-1, k_1+k_2]$, provided that the points $\alpha_\mu^* \beta_\nu^*$ are all different. If this is not so then there are at least as many dominant singularities as there are $\alpha_\mu^* \beta_\nu^*$ which are different.*

7. *Dominant singularities of algebraic-logarithmic type.* Finally, suppose that the functions, of which $f(z)$ and $g(z)$ represent particular branches, have on $|z| = 1$ dominant algebraic-logarithmic singularities of the respective weights $[\sigma_1, k_1]$, $[\sigma_2, k_2]$ at the points α_μ^*, $(\mu = 1, 2, \ldots, l_1)$ and β_ν^*, $(\nu = 1, 2, \ldots, l_2)$, respectively. By definition† $f(z)$ is the sum of two functions, one of weight $[\sigma_1, k_1]$, possessing only algebraic-logarithmic singularities on $|z| = 1$, and the other of weight $[\sigma_1', k_1']$, less than $[\sigma_1, k_1]$, on $|z| = 1$. Similarly $g(z)$ is the sum of two functions, one of weight $[\sigma_2, k_2]$, possessing only algebraic-logarithmic singularities on $|z| = 1$, and the other of weight $[\sigma_2', k_2']$, less than $[\sigma_2, k_2]$, on $|z| = 1$. We therefore write

$$f(z) = \Sigma a_n' z^n + \Sigma a_n'' z^n, \quad g(z) = \Sigma b_n' z^n + \Sigma b_n'' z^n, \tag{11}$$

where $\Sigma a_n' z^n$, $\Sigma b_n' z^n$ are the expansions about the origin of two functions having only algebraic-logarithmic singularities on $|z| = 1$ of the respective weights $[\sigma_1, k_1]$, $[\sigma_2, k_2]$, while the weights of $\Sigma a_n'' z^n$, $\Sigma b_n'' z^n$ on $|z| = 1$ are $[\sigma_1', k_1']$, $[\sigma_2' k_2']$ respectively. Now $h(z)$ may be expressed as the sum of the four series

$$h(z) = \Sigma a_n' b_n' z^n + \Sigma a_n' b_n'' z^n + \Sigma a_n'' b_n' z^n + \Sigma a_n'' b_n'' z^n,$$

and Theorem F shows that the weights of $\Sigma a_n' b_n'' z^n, \Sigma a_n'' b_n' z^n, \Sigma a_n'' b_n'' z^n$ on $|z| = 1$ cannot exceed

$$[\sigma_1+\sigma_2'-1, k_1+k_2'], \quad [\sigma_1'+\sigma_2-1, k_1'+k_2], \quad [\sigma_1'+\sigma_2'-1, k_1'+k_2'], \tag{12}$$

respectively, while Theorem 5 shows that $\Sigma a_n' b_n' z^n$ is exactly of weight $[\sigma_1+\sigma_2-1, k_1+k_2]$ on $|z| = 1$, provided that at least one $\alpha_\mu^* \beta_\nu^*$ is not duplicated. It follows from (12) that $h(z)$ is of weight $[\sigma_1+\sigma_2-1, k_1+k_2]$ on $|z| = 1$ and possesses $l \leqslant l_1 l_2$ dominant singularities of algebraic-logarithmic type on this circle, and that $l = l_1 l_2$ if all the $\alpha_\mu^* \beta_\nu^*$ are

† **15**, 211.

different. The characteristic details of each dominant singularity may be obtained as in § 3. Hence we have

THEOREM 6. *If $f(z)$ and $g(z)$ have on $|z| = 1$ dominant singularities of algebraic-logarithmic type at the points α_μ^*, $(\mu = 1, 2, \ldots, l_1)$, and β_ν^*, $(\nu = 1, 2, \ldots, l_2)$, of the respective weights $[\sigma_1, k_1]$, $[\sigma_2, k_2]$, then $h(z)$ has $l \leqslant l_1 l_2$ dominant algebraic-logarithmic singularities on $|z| = 1$ of weight $[\sigma_1+\sigma_2-1, k_1+k_2]$, provided that at least one $\alpha_\mu^* \beta_\nu^*$ is not duplicated.*

If either of the series $\Sigma a_n'' z^n$, $\Sigma b_n'' z^n$ in (11) has dominant algebraic-logarithmic singularities on $|z| = 1$, then a further separation can be effected. Since the number of algebraic-logarithmic singularities of $\Sigma a_n'' z^n$, $\Sigma b_n'' z^n$ on $|z| = 1$ is finite†, it follows that after a finite number of separations the reduced functions will no longer have dominant algebraic-logarithmic singularities on $|z| = 1$. For convenience we suppose $[\sigma_1', k_1']$, $[\sigma_2', k_2']$ to be the weights of the reduced functions on $|z| = 1$. The series representing the displaced singularities are added to $\Sigma a_n' z^n$, $\Sigma b_n' z^n$ respectively, and the dissection process exhibited in (9) is applied to the resultant series so that we can now remove the algebraic-logarithmic singularities of weights not exceeding $[\sigma_1', k_1']$, $[\sigma_2', k_2']$ on $|z| = 1$ respectively, and add them to the corresponding reduced functions. In this way we obtain the dissection

$$f(z) = \Sigma a_n^* z^n + \Sigma a_n^{**} z^n, \quad g(z) = \Sigma b_n^* z^n + \Sigma b_n^{**} z^n,$$

where $\Sigma a_n^* z^n$, $\Sigma b_n^* z^n$ have on $|z| = 1$ only algebraic-logarithmic singularities, of weights exceeding $[\sigma_1', k_1']$, $[\sigma_2', k_2']$ respectively, while the series $\Sigma a_n^{**} z^n$, $\Sigma b_n^{**} z^n$ have the weights $[\sigma_1', k_1']$, $[\sigma_2', k_2']$ on $|z| = 1$ respectively. The former are called the *dominant*‡ functions and the latter the *residual*‡ functions of $f(z)$ and $g(z)$ respectively. Let the dominant functions have, as in Theorem 6, singularities of weight $[\sigma_1, k_1]$ at the points α_μ^*, $(\mu = 1, 2, \ldots, l_1)$, and singularities of weight $[\sigma_2, k_2]$ at the points β_ν^*, $(\nu = 1, 2, \ldots, l_2)$, on $|z| = 1$ respectively, and let q_1', q_2' be the total number of their respective singularities, all of algebraic-logarithmic type, on $|z| = 1$. As with the previous dissection, $h(z)$ may be written in the form

$$h(z) = \Sigma a_n^* b_n^* z^n + \Sigma a_n^* b_n^{**} z^n + \Sigma a_n^{**} b_n^* z^n + \Sigma a_n^{**} b_n^{**} z^n.$$

Now $\Sigma a_n^* b_n^* z^n$ has, by Theorem 5, only singularities of algebraic-logarithmic type on $|z| = 1$, and, provided that at least one $\alpha_\mu^* \beta_\nu^*$ is not

† **8**, 267–268.

‡ **15**, Theorem H.

duplicated, its weight on $|z|=1$ is $[\sigma_1+\sigma_2-1,\ k_1+k_2]$, while if all the $\alpha_\mu{}^*\beta_\nu{}^*$ are different there are $l=l_1l_2$ dominant singularities.

Theorem F shows that the respective weights of $\Sigma a_n{}^*b_n{}^{**}z^n$, $\Sigma a_n{}^{**}b_n{}^*z^n$, $\Sigma a_n{}^{**}b_n{}^{**}z^n$ on $|z|=1$ cannot exceed

$$(13)\quad [\sigma_1+\sigma_2'-1,\ k_1+k_2'],\quad [\sigma_1'+\sigma_2-1,\ k_1'+k_2],\quad [\sigma_1'+\sigma_2'-1,\ k_1'+k_2'],$$

in which σ_1', σ_2', k_1', k_2' may have a significance different from that in (11) It follows from (13) that $h(z)$ is of weight $[\sigma_1+\sigma_2-1,\ k_1+k_2]$ on $|z|=1$ and possesses $l\leqslant l_1l_2$ dominant singularities of algebraic-logarithmic type on this circle, provided that at least one $\alpha_\mu{}^*\beta_\nu{}^*$ is not duplicated, and that the weight of the residual function cannot exceed the greater of $[\sigma_1+\sigma_2'-1,\ k_1+k_2']$, $[\sigma_1'+\sigma_2-1,\ k_1'+k_2]$. Evidently the series $\Sigma a_n{}^*b_n{}^*z^n$ may contain singularities of smaller weight than this (as follows from Theorem 4), so that $\Sigma a_n{}^*b_n{}^*z^n$ is not necessarily the dominant function of $h(z)$, although it certainly contains the dominant function, provided that at least one $\alpha_\mu{}^*\beta_\nu{}^*$ is not duplicated. Thus the number of singularities q' of the dominant function satisfies the relation $q'\leqslant q_1'q_2'$. Hence we have

THEOREM 7. *If the dominant functions of $f(z)$ and $g(z)$ are of weights $[\sigma_1,\ k_1]$, $[\sigma_2,\ k_2]$ respectively on $|z|=1$, and if their residual functions are of weights $[\sigma_1',\ k_1']$, $[\sigma_2',\ k_2']$ respectively, then $h(z)$ is of weight*

$$[\sigma_1+\sigma_2-1,\ k_1+k_2]$$

and its residual function is of weight not exceeding the greater of

$$[\sigma_1+\sigma_2'-1,\ k_1+k_2'],\ [\sigma_1'+\sigma_2-1,\ k_1'+k_2].$$

The fact that the residual function may have the circle $|z|=1$ as a natural boundary does not affect the result except that, in this case, the dominant algebraic-logarithmic singularities are only the *dominant elements* of the singularities concerned.

8. *The coefficient problem.* Pólya† has shown that the question whether, associated with a singular point α of $f(z)$ and a singular point β of $g(z)$, there exists a singular point $\alpha\beta$ of the product series $h(z)$ depends on the coefficient densities‡ of the sequences (a_n) and (b_n). The corresponding problem concerning the relations between the weights of the three singularities involves the question of the density and distribution of the *dominant* coefficients in the sequences (a_n) and (b_n).

† 13, § 68.

‡ 12, 556–562.

The case arising in Theorem 1, in which each of $f(z)$ and $g(z)$ represents the expansion of a given branch of a single algebraic-logarithmic singularity, with a single dominant element, is particularly simple. Formulae (2) and (3) show that every coefficient is a dominant one in the case of all three functions, and this remains true when the three functions have each a single dominant singularity of the above type on the circle of convergence. When this singularity contains a number of dominant elements, as in Theorem 2, the behaviour of the dominant coefficients is still fairly regular. The information available with regard to the distribution of the dominant coefficients in the more general case of a function having on its circle of convergence a number of dominant singularities of algebraic-logarithmic type is almost complete†, and we make use of it to state corresponding results concerning the series $f(z)$, $g(z)$ and $h(z)$.

Suppose that $f(z)$ and $g(z)$ have on $|z| = 1$ dominant singularities of algebraic-logarithmic type in number l_1 and l_2 and of the respective weights $[\sigma_1, k_1]$, $[\sigma_2, k_2]$. Then, as in Theorem 6, the series $h(z)$ has l dominant singularities of algebraic-logarithmic type of weight $[\sigma, k]$, where

$$l \leqslant l_1 l_2, \quad \sigma = \sigma_1 + \sigma_2 - 1, \quad k = k_1 + k_2,$$

in which it is assumed that $l \geqslant 1$, while, if r_1, r_2, and r, in the case of $f(z)$, $g(z)$, and $h(z)$ respectively, are the smallest numbers of dominant elements in a dominant singularity, then, as in § 3,

$$\min(r_1, r_2) \leqslant r \leqslant r_1 r_2.$$

Results concerning the relations between the dominant coefficients in the sequences (a_n), (b_n), and $(a_n b_n)$ are set out in the five lemmas below.

The first concerns the size of the dominant coefficients.

LEMMA 1. *Positive constants C_1, C_2, and C exist such that*

$$\overline{\lim_{n\to\infty}} |a_n| / n^{\sigma_1 - 1} (\log n)^{k_1} = C_1, \quad \overline{\lim_{n\to\infty}} |b_n| / n^{\sigma_2 - 1} (\log n)^{k_2} = C_2,$$

and

$$\overline{\lim_{n\to\infty}} |a_n b_n| / n^{\sigma - 1} (\log n)^{k} = C.$$

The second concerns the densities of the sequences of dominant coefficients.

LEMMA 2. *Given positive numbers h_1, h_2, and h, each less than 1, it is possible to select positive constants C_1', C_2', and C' such that*

$$|a_n| > C_1' n^{\sigma_1 - 1} (\log n)^{k_1}, \quad |b_n| > C_2' n^{\sigma_2 - 1} (\log n)^{k_2},$$

† 15, § 4.

and

$$|a_n b_n| > C' n^{\sigma-1} (\log n)^k$$

hold for sequences of n of the respective lower densities h_1/l_1, h_2/l_2, *and* h/l.

The third concerns the distributions of the dominant coefficients.

LEMMA 3. *For every positive* η_1, η_2, *and* η, *however small, there exist infinite sequences of n such that the inequalities*

$$\sum_{\nu=1}^{l_1} |a_{n-\nu-r}| \geqslant (C_1'' - \eta_1) n^{\sigma_1-1} (\log n)^{k_1},$$

$$\sum_{\nu=1}^{l_2} |b_{n-\nu-r}| \geqslant (C_2'' - \eta_2) n^{\sigma_2-1} (\log n)^{k_2},$$

and

$$\sum_{\nu=1}^{l} |a_{n-\nu-r} b_{n-\nu-r}| \geqslant (C'' - \eta) n^{\sigma-1} (\log n)^k,$$

hold respectively for $r = 0, \pm 1, \pm 2, \ldots, \pm q$, *where* q *is* $o(n/\log n)$ *and and* C_1'', C_2'', C'' *are certain positive constants.*

The fourth lemma concerns the density of small coefficients.

LEMMA 4. *If* $\epsilon_1(n)$, $\epsilon_2(n)$, *and* $\epsilon(n)$ *are positive decreasing functions tending to zero with* $1/n$, *the inequalities*

$$|a_n| > \epsilon_1(n) n^{\sigma_1-1} (\log n)^{k_1}, \quad |b_n| > \epsilon_2(n) n^{\sigma_2-1} (\log n)^{k_2},$$

and

$$|a_n b_n| > \epsilon(n) n^{\sigma_1-1} (\log n)^k$$

hold for every n save possibly a sequence of density zero.

The fifth lemma concerns the distribution of small coefficients.

LEMMA 5. *If the residual functions of* $f(z)$, $g(z)$, *and* $h(z)$ *are of weights less than* $[\sigma_1 - r_1 + 1, k_1]$, $[\sigma_2 - r_2 + 1, k_2]$ *and* $[\sigma - r + 1, k]$ *respectively, then there exist pairs of positive numbers* (A_1, B_1), (A_2, B_2) *and* (A, B) *such that, for* $n > n_0$,

$$A_1 n^{\sigma_1-r_1} (\log n)^{k_1} \leqslant |a_{n-1}| + |a_{n-2}| + \ldots + |a_{n-m_1}| \leqslant B_1 n^{\sigma_1-1} (\log n)^{k_1},$$

$$A_2 n^{\sigma_2-r_2} (\log n)^{k_2} \leqslant |b_{n-1}| + |b_{n-2}| + \ldots + |b_{n-m_2}| \leqslant B_2 n^{\sigma_2-1} (\log n)^{k_2},$$

$$A n^{\sigma-r} (\log n)^k \leqslant |a_{n-1} b_{n-1}| + |a_{n-2} b_{n-2}| + \ldots + |a_{n-m} b_{n-m}| \leqslant B n^{\sigma-1} (\log n)^k,$$

where $m_1 \leqslant q_1' r_1$, $m_2 \leqslant q_2' r_2$ *and* $m \leqslant q' r$.

As in Theorem 7, the weights of the residual functions of $f(z)$ $g(z)$ and $h(z)$ are respectively $[\sigma_1', k_1']$, $[\sigma_2', k_2']$ *and* $[\sigma, ' k']$, and the numbers of singu-

larities possessed by the dominant functions are q_1', q_2', and q' respectively, where $q' \leqslant q_1' q_2'$.

In particular, when $f(z)$, $g(z)$, and $h(z)$ have only singularities of algebraic-logarithmic type on $|z| = 1$, in number q_1, q_2, and q respectively, then $q_1' = q_1$, $q_2' = q_2$ and $q' = q$.

Now the conditions of Lemma 5 demand that

$$[\sigma_1', k_1'] < [\sigma_1 - r_1 + 1, k_1], \quad [\sigma_2', k_2'] < [\sigma_2 - r_2 + 1, k_2] \tag{14}$$

and

$$[\sigma', k'] < [\sigma - r + 1, k]. \tag{15}$$

From Theorem 7 the last relation can be written

$$\max\{[\sigma_1 + \sigma_2', k_1 + k_2'], \quad [\sigma_1' + \sigma_2, k_1' + k_2]\} < [\sigma_1 + \sigma_2 - r + 1, k_1 + k_2]$$

and therefore

$$[\sigma_1', k_1'] < [\sigma_1 - r + 1, k_1], \quad [\sigma_2', k_2'] < [\sigma_2 - r + 1, k_2].$$

Thus condition (15) is sufficient to ensure the truth of Lemma 5 when $r \geqslant \max.(r_1, r_2)$ and (14) is sufficient when $r = r_2$. Examples of each case occur in § 4.

9. *Reduction of the coefficient problem.* The interest in the results of § 8 lies not so much in the inequalities satisfied by the coefficients of the series $h(z)$ as in *the way in which the dominant members of the sequence* (a_n) *combine with the dominant members of the sequence* (b_n) *to form dominant members of the sequence* $(a_n b_n)$. In this connection one condition must be satisfied, namely that $h(z)$ shall possess at least one algebraic-logarithmic singularity of weight $[\sigma_1 + \sigma_2 - 1, k_1 + k_2]$, and this condition is certainly satisfied if at least one $\alpha_\mu \beta_\nu$ is not duplicated. In view of the necessity for this condition we first explore the possibilities in the general case.

Since the radius of convergence of $f(z)$ and $g(z)$ is unity, it follows that

$$\overline{\lim_{n\to\infty}} \frac{1}{n} \log|a_n| = 0, \qquad \overline{\lim_{n\to\infty}} \frac{1}{n} \log|b_n| = 0.$$

If the sequence $(a_n b_n)$ is such that $\overline{\lim_{n\to\infty}} \frac{1}{n} \log|a_n b_n| < 0$, then $h(z)$ has no singularity on $|z| = 1$.

Suppose, however, that the radius of convergence of $h(z)$ is 1. The weight $[\sigma, k]$ of $\psi(z) = \Sigma c_n z^n$ is an arithmetical function of the sequence of

coefficients (c_n) and is defined by the following relations†

$$(16) \qquad \sigma = 1 + \overline{\lim_{n\to\infty}} \frac{\log|c_n|}{\log n}, \qquad k = \overline{\lim_{n\to\infty}} \frac{\log|c_n| - (\sigma-1)\log n}{\log(\log n)}.$$

Thus if the weights $[\sigma_1, k_1]$, $[\sigma_2, k_2]$ *and* $[\sigma, k]$ of the respective functions $f(z)$, $g(z)$, and $h(z)$ are such that either $\sigma < \sigma_1+\sigma_2-1$ or $\sigma = \sigma_1+\sigma_2-1$ and $k < k_1+k_2$, this condition can be expressed by means of inequalities connecting the coefficients of the sequences (a_n), (b_n), and $(a_n b_n)$, as in (16). In this case the dominant singularities of $h(z)$ on $|z| = 1$ are not of the greatest possible weight $[\sigma_1+\sigma_2-1, k_1+k_2]$. We illustrate these two cases by means of an example.

Let $\psi(z) = \Sigma c_n z^n$ have on $|z| = 1$ a single dominant singularity of algebraic-logarithmic type at the point $z = 1$. The Hadamard product of $\psi(z)$ and $1/(1-z^m)$ is a function $\Sigma c_{mn} z^{mn}$ which has exactly m dominant singularities of algebraic-logarithmic type, situated at the points $e^{2\pi i\nu/m}$, $(\nu = 1, 2, \ldots, m)$, each of the same weight $[\sigma, k]$ as those of $\psi(z)$. Each of the set of functions

$$(17) \qquad \Sigma c_{mn} z^{mn}, \quad \Sigma c_{mn+1} z^{mn+1}, \ \ldots, \quad \Sigma c_{mn+m-1} z^{mn+m-1}$$

possesses this property and the Hadamard product of any pair of them is the constant zero. Now replace the set (17) by

$$\Sigma c_{mn} z^{mn} + \Sigma c_n' z^n, \quad \Sigma c_{mn+1} z^{mn+1} + \Sigma c_n' z^n, \ \ldots, \ \Sigma c_{mn+m-1} z^{mn+m-1} + \Sigma c_n' z^n,$$

where $\Sigma c_n' z^n$ has a radius of convergence greater than 1. The Hadamard product of any pair of the new set of functions has no singularity on $|z| = 1$. If, on the other hand, the weight of $\Sigma c_n' z^n$ on $|z| = 1$ is $[\sigma', k']$, less than $[\sigma, k]$, then the weight of the product function, which is at most $[\sigma+\sigma'-1, k+k']$, is less than the greatest possible weight $[2\sigma-1, 2k]$.

The coefficient problem is statistical in nature, and the solution in the general case is best exhibited by considering three cases of special importance: that in which the dominant singularities in the product function either vanish or are reduced in weight, in which case all the $\alpha_\mu \beta_\nu$ derived from the dominant singularities of the component functions are duplicated; that in which none of the $\alpha_\mu \beta_\nu$ are duplicated; and finally that in which the minimum number of $\alpha_\mu \beta_\nu$ survive without the product function being reduced in weight.

We have already discussed the first case, and it will be assumed from now onwards that $h(z)$ possesses at least one dominant algebraic-logarithmic

† 15, § 4.

singularity of the best possible weight $[\sigma_1+\sigma_2-1,\ k_1+k_2]$. The arithmetical conditions which govern the second case are stated in Lemmas 1–3 of § 8 with $l = l_1 l_2$. The first lemma merely states in arithmetical form the condition that the sequence $(a_n b_n)$ is of weight $[\sigma_1+\sigma_2-1,\ k_1+k_2]$. From Lemma 2, since we can take h_1, h_2, and h as near to 1 as we like, it follows that *the sequence of dominant coefficients in* (a_n) *of lower density at least* $1/l_1$ *combines with the sequence of dominant coefficients in* (b_n) *of lower density at least* $1/l_2$ *to form a sequence of dominant coefficients in* $(a_n b_n)$ *of lower density at least* $1/l_1 l_2$. This is approximately the result which would arise from the theory of probability on the assumption that neither of the sequences (a_n), (b_n) is privileged. Evidently this latter condition is fulfilled if none of the $\alpha_\mu \beta_\nu$ is duplicated. In this case Lemma 3 shows that *over groups of coefficients of increasing extent, at least* 1 *in every* l_1, l_2 *or* $l_1 l_2$, *as the case may be, is dominant.*

In general, provided that $h(z)$ possesses $l(\geqslant 1)$ dominant algebraic-logarithmic singularities of weight $[\sigma_1+\sigma_2-1,\ k_1+k_2]$, these two results hold with l in place of $l_1 l_2$. It would appear then that the cancellation of singularities (at least in so far as their dominant elements are concerned), increases the density of the dominant coefficients in $(a_n b_n)$, provided that at least one singularity of the best possible weight, $[\sigma_1+\sigma_2-1,\ k_1+k_2]$, survives.

This leads to the third case in which $h(z)$ possesses a minimum number of surviving dominant singularities. The following example provides an illustration. Consider the function $\Sigma c_{mn} z^{mn}$ introduced earlier. It has at the points $e^{2\pi i\nu/m}$ dominant algebraic-logarithmic singularities of weight $[\sigma,\ k]$. The Hadamard product of this function with itself, $\Sigma c_{mn}^2 z^{mn}$ also has its only dominant singularities at the points $e^{2\pi i\nu/m}$. They are of algebraic-logarithmic type and of weight $[2\sigma-1,\ 2k]$. In fact the function has the minimum number of singularities of this weight. *In general* $h(z)$ *cannot possess less than min.* $(l_1,\ l_2)$ *singularities of weight* $[\sigma_1+\sigma_2-1,\ k_1+k_2]$ *on* $|z| = 1$ *unless it has none at all.* This is readily shown by the method of § 3 following (6).

That the minimum number of singularities can arise is shown by the following example. The Hadamard product of the functions

$$\frac{1}{1-ze^{i\theta}} - \frac{1}{1-z} + \frac{1}{1-ze^{-i\theta}}, \quad \frac{1}{1-z} + \frac{1}{1-ze^{i\theta}}$$

is the function

$$\frac{1}{1-ze^{2i\theta}} + \frac{1}{1-ze^{-i\theta}}.$$

The set of functions

$$1/(1-z^m), \quad z/(1-z^m), \quad \ldots, \quad z^{m-1}/(1-z^m), \tag{18}$$

in which m is a positive integer, can be used to extract further information about the distribution of the dominant coefficients of a given function $\psi(z) = \Sigma c_n z^n$ which has on $|z| = 1$ dominant singularities of algebraic-logarithmic type at the points $\alpha_1, \alpha_2, \ldots, \alpha_l$, each of weight $[\sigma, k]$. The Hadamard product of $\psi(z)$ with each member of (18) in turn gives rise to the series of functions (17). Each member of this series has dominant algebraic-logarithmic singularities at some of the points $\alpha_\mu e^{2\pi i\nu/m}$. If none of these ml points is duplicated, we can use Lemmas 1–3 to discuss in greater detail the distribution of the dominant coefficients in the sequence (c_n). These results are set out below.

LEMMA 1*. *Positive constants C_r exist such that*

$$\overline{\lim_{n\to\infty}} \frac{|c_{mn+r}|}{n^{\sigma-1}(\log n)^k} = C_r \quad (r = 1, 2, \ldots, m).$$

LEMMA 2*. *Given positive numbers h_r, $(r = 1, 2, \ldots, m)$, each less than 1, it is possible to select positive constants C_r' such that the relations*

$$|c_{mn+r}| > C_r' n^{\sigma-1}(\log n)^k \quad (r = 1, 2, \ldots, m)$$

hold for sequences of n of the respective lower densities h_r/lm.

LEMMA 3*. *For every positive η_r, however small, there exist infinite sequences of n such that the inequalities*

$$\sum_{\nu=1}^{l} |c_{mn-\nu-t-r}| \geqslant (C_r'' - \eta_r) n^{\sigma-1}(\log n)^k \quad (\nu = 1, 2, \ldots, m),$$

hold respectively for $t = 0, \pm 1, \pm 2, \ldots, \pm q$, where q is $o(n/\log n)$ and the C_r'' are certain positive constants.

These results are valid only for integers m such that none of the lm quantities $\alpha_\mu e^{2\pi i\nu/m}$, $(\mu = 1, 2, \ldots, l; \quad \nu = 1, 2, \ldots, m)$, is duplicated. Thus every value of m for which this restriction is operative gives information concerning the distribution of dominant coefficients in the subsequences (c_{mn+r}), $(r = 1, 2, \ldots, m)$, of the set of coefficients (c_n) of the given function. On the other hand values of m for which these results do not hold indicate the existence of special relations among the affixes α_μ of the dominant singularities of the function on $|z| = 1$.

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A CLASS OF SELF-RECIPROCAL FUNCTIONS CONNECTED WITH SUMMATION FORMULAE

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1. *Introduction.* It has been observed that, for $p \leqslant 3$, the function

$$t^{-\frac{1}{2}-\frac{1}{4}p}\{\bar{P}_p(t)-1\},$$

where
$$\bar{P}_p(t) = \sum_{n\leqslant t}{}' r_p(n) - \frac{(\pi t)^{\frac{1}{2}p}}{\Gamma(1+\frac{1}{2}p)},$$

and $r_p(n)$ is the number of ways of expressing n as the sum of the squares of p integers, is self-reciprocal* with respect to transformations having the kernel $\pi J_{1+\frac{1}{2}p}(2\pi x^{\frac{1}{2}})$; *i.e.*

$$\pi\int_0^\infty t^{-\frac{1}{2}-\frac{1}{4}p}\{\bar{P}_p(t)-1\}J_{1+\frac{1}{2}p}(2\pi x^{\frac{1}{2}}t^{\frac{1}{2}})\,dt = x^{-\frac{1}{2}-\frac{1}{4}p}\{\bar{P}_p(x)-1\}. \qquad (1.1)$$

In the case $p = 2$ this equation is equivalent to the formula

$$\sum_{n\leqslant x}{}' r(n) = \pi x + x^{\frac{1}{2}}\sum_{n=1}^{\infty} r(n)\,n^{-\frac{1}{2}}J_1(2\pi n^{\frac{1}{2}}x^{\frac{1}{2}}).$$

This is a particular case of the formula†

$$\sum_{a\leqslant n\leqslant b}{}' r(n)f(n) = \pi\int_a^b f(t)\,dt + \pi\sum_{n=1}^{\infty} r(n)\int_a^b f(t)\,J_0(2\pi n^{\frac{1}{2}}t^{\frac{1}{2}})\,dt, \qquad (1.2)$$

which involves the kernel $\pi J_0(2\pi x^{\frac{1}{2}})$.

* G. H. Hardy, *Messenger of Math.*, 57 (1927), 113–120 (120); G. H. Hardy and E. C. Titchmarsh, *Quart. Journal of Math.*, 1 (1930), 196–231 (213).

† E. Landau, *Vorlesungen über Zahlentheorie*, 2 (1927), 274.

Ferrar* has proved that, associated with every Dirichlet's series

$$\psi(s) = \sum_{n=1}^{\infty} a_n n^{-s} \tag{1.3}$$

of a certain class, and the function $a(s)$ defined by

$$\psi(s) = a(s)\psi(1-s), \tag{1.4}$$

there is a summation formula of the type

$$\sum'_{\alpha \leqslant n \leqslant \beta} a_n f(n) = \int_\alpha^\beta f(t)\, dR_0(t) + \sum_{n=1}^{\infty} a_n \int_\alpha^\beta f(t)\, K(nt)\, dt, \tag{1.5}$$

where $K(x)$ is a Fourier kernel.

Only a preliminary investigation without detailed conditions on $f(x)$ has so far been published†. However, we can identify the appropriate kernel $K(x)$ by considering the series

$$\sum_{1 \leqslant n \leqslant x} (x-n)^\alpha a_n.$$

In this paper we show that the formula (1.1) can be generalized in the same way. We find that, with suitable conditions,

$$\int_0^\infty \left\{ \sum'_{1 \leqslant n \leqslant t} a_n - R_0(t) \right\} t^{-1} K_1(xt)\, dt = \left\{ \sum'_{1 \leqslant n \leqslant x} a_n - R_0(x) \right\} x^{-1}, \tag{1.6}$$

where

$$K_1(x) = \frac{1}{x} \int_0^x K(t)\, dt - K(x) \tag{1.7}$$

and $K_1(x)$ is a Fourier kernel.

In order to cover examples such as $a_n = r_3(n)$ it is necessary to consider Dirichlet's series of a wider class and functions $a(s)$ defined by

$$\psi(s) = a(s)\psi(\beta - s) \tag{1.8}$$

in place of (1.4). Our results are accordingly extended to cover such Dirichlet's series.

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* W. L. Ferrar, *Compositio Mathematica*, 1 (1935), 344–360.

† A second paper by Ferrar appeared, *Compositio Mathematica*, 4 (1937), 394–405, after the present paper had been accepted for publication.

2. *Formalities.* Suppose that there exists a sequence $\{a_n\}$ of real numbers, and a positive β such that $\psi(s)$, defined by (1.3), has an abscissa of absolute convergence not greater than β, and satisfies the functional equation (1.8).

Let us write (1.8) in the form

$$A(s) = \frac{\psi\{\frac{1}{2}(\beta-1)+s\}}{\psi\{\frac{1}{2}(\beta+1)-s\}}. \tag{2.1}$$

From Perron's formula*, we have

$$\sum_{n\leqslant x}{}' a_n n^{-\frac{1}{2}(\beta-1)} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \psi\{\tfrac{1}{2}(\beta-1)+s\}\, x^s s^{-1}\, ds,$$

where the dash in the sum on the left indicates that the last term is to be halved if x is an integer, and the integral is

$$\lim_{T\to\infty}\int_{c-iT}^{c+iT}.$$

If we now move the line of integration to the left of the origin, we have

$$\sum_{n\leqslant x}{}' a_n n^{-\frac{1}{2}(\beta-1)} = R_0(x)x^{-\frac{1}{2}(\beta-1)} + \psi\{\tfrac{1}{2}(\beta-1)\} + \frac{1}{2\pi i} \times \int_{-b-i\infty}^{-b+i\infty} A(s)\,\psi\{\tfrac{1}{2}(\beta+1)-s\}\, x^s\, s^{-1}\, ds,$$

where $R_0(x)$ is the sum of the residues of

$$\psi\{\tfrac{1}{2}(\beta-1)+s\}\, x^{\frac{1}{2}(\beta-1)+s}\, s^{-1}$$

at the poles of $\psi\{\frac{1}{2}(\beta-1)+s\}$ in the strip $-b<\sigma<c$.

If we substitute the series (1.3) for $\psi\{\frac{1}{2}(\beta+1)-s\}$ and integrate formally term by term, we have

$$\sum_{n\leqslant x}{}' a_n n^{-\frac{1}{2}(\beta-1)} = R_0(x)\, x^{-\frac{1}{2}(\beta-1)} + \psi\{\tfrac{1}{2}(\beta-1)\} + \sum_{n=1}^{\infty} a_n n^{-\frac{1}{2}(\beta+1)}\, \frac{1}{2\pi i}\int_{-b-i\infty}^{-b+i\infty} A(s)\,(nx)^s\, s^{-1}\, ds. \tag{2.2}$$

If we now take the Stieltjes integral of a function $f(x)$ with respect to either side of (2.2), and formally invert the order of summation and

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* E. C. Titchmarsh, *The theory of functions* (1932), 300.

integration, we have

$$\sum_{p<n<q}' a_n n^{\frac{1}{2}(\beta-1)} f(n) = -\int_p^q f(x)\, d\{x^{\frac{1}{2}(\beta-1)} R_0(x)\}$$
$$+\frac{1}{2\pi i}\sum_{n=1}^{\infty} a_n n^{-\frac{1}{2}(\beta+1)} \int_p^q f(x)\, d\left\{\int_{-b-i\infty}^{-b+i\infty} A(s)(nx)^s s^{-1}\, ds\right\}$$
$$= \int_p^q f(x)\, d\{x^{-\frac{1}{2}(\beta-1)} R_0(x)\}$$
$$+\frac{1}{2\pi i}\sum_{n=1}^{\infty} a_n n^{-\frac{1}{2}(\beta-1)} \int_p^q f(x) \left\{\int_{-b-i\infty}^{-b+i\infty} A(s)(nx)^{s-1}\, ds\right\} dx.$$

That is

$$\sum_{p\leqslant n\leqslant q}' a_n n^{-\frac{1}{2}(\beta-1)} f(n) = \int_p^q f(x)\, d\{x^{-\frac{1}{2}(\beta-1)} R_0(x)\} + \sum_{n=1}^{\infty} a_n n^{-\frac{1}{2}(\beta-1)} g(n), \quad (2.3)$$

where
$$g(x) = \int_p^q f(y)\, K(xy)\, dy, \quad (2.31)$$

and
$$K(x) = \frac{1}{2\pi i}\int_{-b-i\infty}^{-b+i\infty} A(s)\, x^{s-1}\, dx = \frac{1}{2\pi i}\int_{1+b-i\infty}^{1+b+i\infty} A(1-s)\, x^{-s}\, ds. \quad (2.32)$$

From (2.1) we have

$$A(s)\, A(1-s) = 1, \quad (2.33)$$

and hence $K(x)$ is a Fourier kernel*.

3. *The series* $\sum_{n\leqslant x} (x-n)^{\alpha} a_n$. The relevant theorems for the summation formula are:—

THEOREM 1. *If*

(i) *there is a sequence* $\{a_n\}$ *of real numbers, and a non-negative* β *such that* $\psi(s)$, *defined by* (1.3), *has an abscissa of absolute convergence not greater than* β,

(ii) *there is a positive* b *such that* $\psi(s)$ *can be analytically continued in the strip* $-b \leqslant \sigma \leqslant \beta$, *and the only singularities of* $\psi(s)$ *are poles, finite in number, none of which lies in* $-b \leqslant \sigma \leqslant 0$;

(iii) $|\psi(s)| = O(|t|^K)$ *for some* K, *and* $|t| > t_0$, $\sigma \geqslant -b$;

* G. H. Hardy and E. C. Titchmarsh, *Proc. London Math. Soc.* (2), 35 (1932), 116–155; G. N. Watson, *Proc. London Math. Soc.* (2), 35 (1932), 156–199.

(iv) $A(s)$, *defined by* (2.1), *satisfies*

$$|A(-b-\tfrac{1}{2}\beta+\tfrac{1}{2}+it)| = O(|t|^B)$$

for some B, *and* $|t| > t_1$, *then*

$$\frac{1}{\Gamma(\alpha)} \sum_{1\leqslant n\leqslant x} a_n (x-n)^{\alpha-1} = \frac{x^{\alpha-1}\psi(0)}{\Gamma(\alpha)} + R_{\alpha-1}(x) + \sum_{n=1}^{\infty} a_n n^{1-\alpha-\beta} A_\alpha(nx), \tag{3.1}$$

where $R_{\alpha-1}(x)$ *is the sum of the residues of*

$$\frac{\psi(s)\,\Gamma(s)\,x^{s+\alpha-1}}{\Gamma(s+\alpha)}$$

at the poles of $\psi(s)$ *in the strip* $0 \leqslant \sigma \leqslant \beta$,

$$A_\alpha(y) = \frac{y^{\alpha+\frac{1}{2}\beta-\frac{1}{2}}}{2\pi i} \int_{-b-\frac{1}{2}(\beta-1)-i\infty}^{-b-\frac{1}{2}(\beta-1)+i\infty} \frac{A(s)\,\Gamma\{\frac{1}{2}(\beta-1)+s\}}{\Gamma\{\alpha+\frac{1}{2}(\beta-1)+s\}}\, y^{s-1}\,ds, \tag{3.2}$$

and $\alpha > \max(B+1,\ K+1)$.

THEOREM 2. *If, in addition to the assumptions* (i)–(iv),

(v) $$|A(\sigma+it)| = O(|t|^B)$$

for $-b-\frac{1}{2}(\beta-1) \leqslant \sigma \leqslant \frac{1}{2}$ *and* $|t| > t_1$,

(vi) $$\frac{A(s)\,\Gamma\{\frac{1}{2}(\beta-1)+s\}}{\Gamma\{\alpha+\frac{1}{2}(\beta-1)+s\}}$$

has no poles in the strip

$$-b-\tfrac{1}{2}(\beta-1) \leqslant \sigma \leqslant \tfrac{1}{2},$$

then

$$\chi(y) = \operatorname*{l.i.m.}_{T\to\infty} \frac{1}{2\pi} \int_{-T}^{T} \frac{A(\frac{1}{2}-it)}{\frac{1}{2}-it}\, y^{\frac{1}{2}-it}\,dt \tag{3.3}$$

exists, and is a Fourier kernel in Watson's sense. Further,* $A_{\alpha+1}(x)$ *is the* α*-th fractional integral of*

$$x^{\frac{1}{2}(\beta-1)}\chi(x) - \tfrac{1}{2}(\beta-1)\int_0^x \chi(t)\,t^{\frac{1}{2}(\beta-3)}\,dt.$$

These theorems are simple extensions of those of Ferrar for the case $\beta = 1$, and are proved in the same way.

* G. N. Watson, *loc. cit.* above.

In the case above the connection between $A_{a+1}(x)$ and $\chi(x)$ may also be expressed by regarding $A_{a+1}(x)$ as the $(a+1)$-th fractional integral of $\chi'(x)x^{\frac{1}{2}(\beta-1)}$. However, if we do not assume the differentiability of $\chi(x)$, the relation can only be expressed as in Theorem 2.

4. *The self-reciprocal function.* Let

$$F(x) = x^{-\frac{1}{4}(\beta+1)}\left\{\sum_{1\leqslant n\leqslant x}{}' a_n - R_0(x)\right\}, \tag{4.1}$$

and let $\phi(s)$ be the Mellin transform of $F(x)$; that is to say, let

$$\phi(s) = \int_0^\infty F(x)\,x^{s-1}\,dx. \tag{4.2}$$

In most of the cases which arise in practice we know that there exist inequalities of the forms*

$$\sum_{1\leqslant n\leqslant x}{}' a_n - R_0(x) = O(x^\eta) \quad \text{as} \quad x\to\infty, \tag{4.3}$$

$$R_0(x) = O(x^\theta) \quad \text{as} \quad x\to 0, \tag{4.31}$$

where
$$\theta > \tfrac{1}{2}\beta+\epsilon > \tfrac{1}{2}\beta-\epsilon > \eta. \tag{4.32}$$

With these conditions (4.2) converges when

$$\tfrac{1}{2}(\beta+1)-\theta < \sigma < \tfrac{1}{2}(\beta+1)-\eta.$$

Also $R_0(x)$, from its definition, must be the sum of a finite number of terms of the form $ax^m(\log x)^n$, where $m > -b+\frac{1}{2}(\beta-1)$; or, if we choose $b < \frac{1}{2}(\beta+1)$, then $m > -1$; and n is a positive integer or zero. Let us write

$$R_0(x) = \Sigma\, ax^m(\log x)^n.$$

Then

$$\begin{aligned}\phi(s) &= \int_0^\infty \left\{\sum_{1\leqslant n\leqslant x}{}' a_n - R_0(x)\right\} x^{s-\frac{1}{2}\beta-\frac{3}{2}}\,dx \\ &= -\Sigma\, a\int_0^1 (\log x)^n\, x^{m+s-\frac{1}{2}\beta-\frac{3}{2}}\,dx + \int_1^\infty \left\{\sum_{1\leqslant n\leqslant x}{}' a_n - R_0(x)\right\} x^{s-\frac{1}{2}\beta-\frac{3}{2}}\,dx \\ &= -\Sigma\,\frac{a(-)^n n!}{(m+s-\frac{1}{2}\beta-\frac{1}{2})^{n+1}} + \int_1^\infty \left\{\sum_{1\leqslant n\leqslant x}{}' a_n - R_0(x)\right\} x^{s-\frac{1}{2}\beta-\frac{3}{2}}\,dx.\end{aligned}$$

* In other cases, such as $a_n = r_4(n)$, these inequalities either have not been proved, or do not hold, and our result is not necessarily true.

This obviously gives the analytic continuation of $\phi(s)$ into the region $\sigma \leqslant \frac{1}{2}(\beta+1)-\theta$, and we have there

$$\phi(s) = -\Sigma \frac{a(-)^n n!}{(m+s-\frac{1}{2}\beta-\frac{1}{2})^{n+1}} + \int_1^\infty \left\{ \sum_{1\leqslant n\leqslant x}' a_n \right\} x^{s-\frac{1}{2}\beta-\frac{3}{2}}\, dx - \int_1^\infty R_0(x)\, x^{s-\frac{1}{2}\beta-\frac{3}{2}}\, dx.$$

Evaluating this last integral as before, we find that it cancels with the first term, and we have

$$\begin{aligned}\phi(s) &= \sum_{\nu=1}^{\infty} (a_1+\ldots+a_\nu) \int_\nu^{\nu+1} x^{s-\frac{1}{2}\beta-\frac{3}{2}}\, dx \\ &= \sum_{\nu=1}^{\infty} \frac{(a_1+\ldots+a_\nu)}{s-\frac{1}{2}(\beta+1)} \left\{(\nu+1)^{s-\frac{1}{2}(\beta+1)} - \nu^{s-\frac{1}{2}(\beta+1)}\right\} \\ &= \sum_{\nu=1}^{\infty} \frac{a_\nu\, \nu^{s-\frac{1}{2}(\beta+1)}}{\{\frac{1}{2}(\beta+1)-s\}} = \frac{\psi\{\frac{1}{2}(\beta+1)-s\}}{\{\frac{1}{2}(\beta+1)-s\}}. \end{aligned} \tag{4.4}$$

Now let us put

$$\chi_1(x) = \text{l.i.m.}_{T\to\infty} \frac{1}{2\pi} \int_{-T}^{T} \frac{A(\frac{1}{2}-it)}{\frac{1}{2}-it} \frac{\frac{1}{2}\beta+it}{\frac{1}{2}\beta-it} x^{\frac{1}{2}-it}\, dt. \tag{4.5}$$

The integral exists, since

$$\left| \frac{A(\frac{1}{2}-it)}{\frac{1}{2}-it} \frac{\frac{1}{2}\beta+it}{\frac{1}{2}\beta-it} \right| = \left| \frac{\frac{1}{2}\beta+it}{(\frac{1}{2}-it)(\frac{1}{2}\beta-it)} \right|$$

and consequently belongs to $L^2(-\infty, \infty)$. Also it follows that

$$\frac{\chi_1(xy)}{y} = \text{l.i.m.}_{T\to\infty} \frac{1}{2\pi i} \int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT} \frac{A(1-s)}{1-s} \frac{\frac{1}{2}(\beta-1)+s}{\frac{1}{2}(\beta+1)-s} x^{1-s} y^{-s}\, ds.$$

Hence

$$\frac{\chi_1(xy)}{y}, \quad \frac{A(1-s)}{1-s} \frac{\frac{1}{2}(\beta-1)+s}{\frac{1}{2}(\beta+1)-s} x^{1-s}$$

are a pair of Mellin transforms of L^2.

Further, by (4.1), (4.3), (4.31), and (4.32),

$$\begin{aligned} F(y) &= O(y^{-\frac{1}{2}+\epsilon}) \quad \text{as} \quad y\to 0, \\ &= O(y^{-\frac{1}{2}-\epsilon}) \quad \text{as} \quad y\to\infty, \end{aligned}$$

and hence $F(y)$ belongs to $L^2(0, \infty)$. Hence

$$F(y), \qquad \frac{\psi\{\frac{1}{2}(\beta+1)-s\}}{\{\frac{1}{2}(\beta+1)-s\}},$$

$$\left.\begin{matrix} 1 & (0<y<x) \\ 0 & (y \geqslant x) \end{matrix}\right\}, \qquad \frac{x^s}{s},$$

are pairs of Mellin transforms of L^2, and, by Parseval's theorem for Mellin transforms and (2.1),

$$\int_0^\infty F(y)\,\frac{\chi_1(xy)}{y}\,dy = \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{\psi\{\frac{1}{2}(\beta-1)+s\}}{\{\frac{1}{2}(\beta-1)+s\}}\,\frac{A(1-s)}{1-s}\,\frac{\frac{1}{2}(\beta-1)+s}{\frac{1}{2}(\beta+1)-s}\,x^{1-s}\,ds$$

$$= \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{\psi\{\frac{1}{2}(\beta+1)-s\}}{\{\frac{1}{2}(\beta+1)-s\}}\,\frac{x^{1-s}}{1-s}\,ds = \int_0^x F(y)\,dy.$$

5. *The relation between $\chi(x)$ and $\chi_1(x)$.* If $\beta > 0$,

$$\beta x^{-\frac{1}{2}(\beta-1)}\int_0^x \chi(u)\,u^{\frac{1}{2}(\beta-3)}\,du - \chi(x)$$

$$= \beta x^{-\frac{1}{2}(\beta-1)} \lim_{T\to\infty} \frac{1}{2\pi}\int_0^x u^{\frac{1}{2}(\beta-3)}\,du \int_{-T}^{T} \frac{A(\frac{1}{2}-it)}{\frac{1}{2}-it}\,u^{\frac{1}{2}-it}\,dt - \chi(x)$$

$$= \beta x^{-\frac{1}{2}(\beta-1)} \lim_{T\to\infty} \frac{1}{2\pi}\int_{-T}^{T} \frac{A(\frac{1}{2}-it)}{\frac{1}{2}-it}\,\frac{x^{\frac{1}{2}\beta-it}}{\frac{1}{2}\beta-it}\,dt - \chi(x)$$

$$= \lim_{T\to\infty} \frac{\beta}{2\pi}\int_{-T}^{T} \frac{A(\frac{1}{2}-it)}{\frac{1}{2}-it}\,\frac{x^{\frac{1}{2}-it}}{\frac{1}{2}\beta-it}\,dt - \text{l.i.m.}_{T\to\infty}\,\frac{1}{2\pi}\int_{-T}^{T} \frac{A(\frac{1}{2}-it)}{\frac{1}{2}-it}\,x^{\frac{1}{2}-it}\,dt$$

$$= \text{l.i.m.}_{T\to\infty}\,\frac{1}{2\pi}\int_{-T}^{T} \frac{A(\frac{1}{2}-it)}{\frac{1}{2}-it}\,\frac{\frac{1}{2}\beta+it}{\frac{1}{2}\beta-it}\,x^{\frac{1}{2}-it}\,dt = \chi_1(x).$$

If $\beta = 0$,

$$\chi_1(x) = -\text{l.i.m.}_{T\to\infty}\,\frac{1}{2\pi}\int_{-T}^{T} \frac{A(\frac{1}{2}-it)}{\frac{1}{2}-it}\,x^{\frac{1}{2}-it}\,dt = -\chi(x).$$

Hence, for all $\beta \geqslant 0$,

$$\chi_1(x) = \beta x^{-\frac{1}{2}(\beta+1)}\int_0^x \chi(u)\,u^{\frac{1}{2}(\beta-3)}\,du - \chi(x).$$

Hence we have

THEOREM 3. *If, in addition to the assumptions* (i)–(vi),

(vii) *the inequalities* (4.3), (4.31), *and* (4.32) *are satisfied,*

then
$$\int_0^x F(y)\,dy = \int_0^\infty F(y)\,\frac{\chi_1(xy)}{y}\,dy,$$

where
$$F(x) = x^{-\frac{1}{2}(\beta+1)}\left\{\sum_{1\leqslant n\leqslant x}{}' a_n - R_0(x)\right\}$$

and
$$\chi_1(x) = \underset{T\to\infty}{\text{l.i.m.}}\,\frac{1}{2\pi}\int_{-T}^{T}\frac{A(\frac{1}{2}-it)}{\frac{1}{2}-it}\,\frac{\frac{1}{2}\beta+it}{\frac{1}{2}\beta-it}\,x^{\frac{1}{2}-it}\,dt.$$

Further
$$\chi_1(x) = \beta x^{-\frac{1}{2}(\beta+1)}\int_0^x \chi(u)\,u^{\frac{1}{2}(\beta-3)}\,du - \chi(x). \tag{5.1}$$

6. *Examples.* In most of the cases arising in practice $\chi(x)$ and $\chi_1(x)$ are everywhere differentiable, with differential coefficients $K(x)$ and $K_1(x)$. We can easily deduce a relation between $K(x)$ and $K_1(x)$ from (5.1). We find

$$K_1(x) = \beta x^{-\frac{1}{2}(\beta+1)}\int_0^x t^{\frac{1}{2}(\beta-1)}K(t)\,dt - K(x). \tag{6.1}$$

Some examples are:

a_n	$K(x)$	$F(x)$	$K_1(x)$
1	$2\cos 2\pi x$	$\dfrac{[x]-x}{x}$	$2\pi x^{\frac{1}{2}}J_{\frac{3}{2}}(2\pi x)$
0 (n even) $(-1)^{\frac{1}{2}(n-1)}$ (n odd)	$\sin\frac{1}{2}\pi x$	$\dfrac{\sum'_{1\leqslant 2n+1\leqslant x}(-1)^n}{x}$	$\frac{1}{2}\pi x^{\frac{1}{2}}J_{-\frac{3}{2}}(\frac{1}{2}\pi x)$
$r_p(n)$ $(p\leqslant 3)$	$\pi J_{\frac{1}{4}p-1}(2\pi x^{\frac{1}{2}})$	$\dfrac{\sum'_{1\leqslant n\leqslant x} r_p(n) - (\pi x)^{\frac{1}{2}p}/\Gamma(1+\frac{1}{2}p)}{x^{\frac{1}{4}p+\frac{1}{2}}}$	$\pi J_{\frac{1}{4}p+1}(2\pi x^{\frac{1}{2}})$
$d(n)$	$-2\pi Y_0(4\pi x^{\frac{1}{2}})$ $+4K_0(4\pi x^{\frac{1}{2}})$	$\dfrac{\sum'_{1\leqslant n\leqslant x} d(n) - x(\log x+2\gamma-1)}{x}$	$-2\pi Y_2(4\pi x^{\frac{1}{2}})$ $-4K_2(4\pi x^{\frac{1}{2}})$
$\sigma_k(n)$ $(\lvert k\rvert<\frac{1}{2})$	$2\pi G_k(4\pi x^{\frac{1}{2}})$	$\dfrac{\sum'_{1\leqslant n\leqslant x}\sigma_k(n)-\psi_k(x)}{x^{\frac{1}{2}k+1}}$	$2\pi H_{k+2}(4\pi x^{\frac{1}{2}})$

In the last example* $\sigma_k(n)$ is the sum of the k-th powers of the divisors of n, and

$$G_k(x) = -\sin \tfrac{1}{2}\pi k\, J_k(x) - \cos \tfrac{1}{2}\pi k \left\{ Y_k(x) - \frac{2}{\pi} K_k(x) \right\}$$

$$H_k(x) = \sin \tfrac{1}{2}\pi k\, J_k(x) + \cos \tfrac{1}{2}\pi k \left\{ Y_k(x) + \frac{2}{\pi} K_k(x) \right\},$$

$$\psi_k(x) = x\,\zeta(1-k) + x^{1+k}\,\zeta(1+k)/(1+k)$$

when $k \neq 0$, and

$$\psi_0(x) = x(\log x + 2\gamma - 1).$$

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* See A. Oppenheim, *Proc. London Math. Soc.* (2), 26 (1927), 295–350. For the required inequalities see J. R. Wilton, *Proc. London Math. Soc.* (2), 36 (1933), 391–426 (418).

INEQUALITIES CONNECTED WITH BOUNDED p-TH POWER VARIATION IN THE WIENER SENSE AND WITH INTEGRATED LIPSCHITZ CONDITIONS

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1. *Introduction.* This note belongs to the same order of ideas as my work on non-absolute Riemann-Stieltjes integration, term by term integration, Fourier series, etc. [**11**–**14**] and my joint researches with E. R. Love [**9**, **10**]. (*Cf.* also N. Wiener [**8**], G. H. Hardy and J. E. Littlewood [**2**–**6**].)

The main question treated here, which arose out of a proof of a theorem on Fourier series (*cf.* below § 2), concerns the Wiener class, and the integrated Lipschitz class, of a "Stieltjes Faltung" $\int f(x, t)\,dg(t)$, particularly one of the more special form $\int f(x+t)\,dg(t) = S(x)$. I owe to Prof. J. E. Littlewood an extremely suggestive proof of the inequality

$$V_\lambda(S) \leqslant K V_p(f) V_q(g) \quad \text{for} \quad \frac{1}{\lambda} < \frac{1}{p} + \frac{1}{q} - 1, \tag{1.1}$$

subject to certain inessential restrictions. There are analogous results in terms of integrated Lipschitz conditions, the Stieltjes Faltung being interpreted where necessary in a weak sense which determines it for almost all x.

Finally, I discuss some inequalities for multiple integrals connected with (1.1).

2. *Origin of the inequality* (1.1). If f and g are periodic and we have $f \prec W_p$, $g \prec W_q$ in the period 2π, where $1/p+1/q > 1$, we may write

$$\int_0^{2\pi} s_n(f)\,dg \to \int_0^{2\pi} \tfrac{1}{2}\{f(x+0)+f(x-0)\}\,dg, \tag{2.1}$$

where $s_n(f)$, the sum to n terms of the Fourier series of f, tends to

$$\tfrac{1}{2}\{f(x+0)+f(x-0)\}.$$

The relation (2.1), which we have written in the form of term by term integration, is, in fact, identical with the Parseval equation

$$(2.2) \qquad \pi \sum_{n=1}^{\infty} n(a_n b_n' - a_n' b_n) = \int_0^{2\pi} \tfrac{1}{2}\{f(x+0)+f(x-0)\}\, dg,$$

established at the end of [11] in a rather indirect manner.

Now (2.1) can be made to follow directly from general theorems on term by term integration [11], a slight device {*cf.* the proof of Lemma (iv) [11], p. 279} disposing of the minor complication arising out of possible discontinuities of f, *provided that for a sufficiently small* $\epsilon > 0$ [*i.e.* where $1/(p+\epsilon)+1/q > 1$] *the numbers* $V_{p+\epsilon}[s_n(f)]$ *can be shown to be bounded in* n. The functions $s_n(f)$ are, however, precisely of the Stieltjes Faltung form

$$(2.3) \qquad s_n(f) = \int_0^{2\pi} f(x+t)\, dg_n(t),$$

where, for the functions $g_n(t)$ [indefinite integrals of the Fourier kernels $\sin(n+\frac{1}{2})t/(\pi \sin \frac{1}{2}t)$], we have $V_{1+\delta}(g_n) < K_\delta$ (independent of n) for every $\delta > 0$. With the help of the inequality (1.1) we thus find, when $\delta < \epsilon/p$, which is the same as $1/(p+\epsilon) < 1/p+1/(1+\delta)-1$, the relations

$$V_{p+\epsilon}[s_n(f)] \leqslant K V_p(f) V_{1+\delta}(g_n) \leqslant K V_p(f),$$

where the K's may depend, of course, on ϵ, but not on n. It was the fact that the inequality (1.1) leads to such a direct proof of (2.2) that suggested investigating the questions to be discussed here. The first proof of (1.1), with a few inessential restrictions, was the one communicated to me by Prof. Littlewood and is reproduced in the next section. The precise theorem established by Littlewood is as follows:

(2.4) *Suppose that* $p > 1$, $q > 1$, $0 < 1/\lambda < 1/p+1/q-1$. *Then there exists a finite constant* $K = K(p, q, \lambda)$ *such that, for any two functions* $f(x)$ *and* $g(x)$ *with continuous derivatives and period* 1, *the upper bounds* A, B, C *of the sums*

$$(2.5) \qquad \sum_{i=1}^{N} |f(x_i)-f(x_{i-1})|^p, \quad \sum_{i=1}^{N} |g(x_i)-g(x_{i-1})|^q, \quad \text{and} \quad \sum_{i=1}^{N} S|(x_i)-S(x_{i-1})|^\lambda,$$

for $0 = x_0 \leqslant x_1 \leqslant \ldots \leqslant x_N = 1$, *satisfy the inequality*

$$C^{1/\lambda} \leqslant K A^{1/p} B^{1/q}, \tag{2.6}$$

the function $S(x)$ *denoting the Stieltjes Faltung*

$$\iint_T f'(x+u)\, g'(t)\, du\, dt, \tag{2.7}$$

where T *is the triangle* $0 \leqslant u < t \leqslant 1$.

3. *Littlewood's proof of* (2.4). We may suppose that $A = B = 1$, by trivial transformations. Writing $B(x', x'')$ for the upper bound of the sum

$$\sum_{i=1}^{N} |g(x_i) - g(x_{i-1})|^q,$$

where $x' = x_0 \leqslant x_1 \leqslant \ldots \leqslant x_N = x''$, we observe that our hypotheses imply that $B(x', x'')$ is continuous in x' and x'' and non-negative, and that, if $x' \leqslant x'' \leqslant x'''$,

$$B(x', x'') + B(x'', x''') \leqslant B(x', x'''). \tag{3.1}$$

This being so, there must evidently exist a point t such that

$$B(0, t) \leqslant \tfrac{1}{2} \quad \text{and} \quad B(t, 1) \leqslant \tfrac{1}{2}.$$

For similar reasons, denoting by E_0 the set of the two points 0 and 1, we can define by induction a finite set E_ν of $2^\nu + 1$ points such that, if α and β are consecutive points of $E_{\nu-1}$ in increasing order, the common part of E_ν with the closed interval $[\alpha, \beta]$ consists of the points α, β together with one further point γ for which $\alpha < \gamma < \beta$ and

$$B(\alpha, \gamma) \leqslant \tfrac{1}{2} B(\alpha, \beta) \quad \text{and} \quad B(\gamma, \beta) \leqslant \tfrac{1}{2} B(\alpha, \beta). \tag{3.2}$$

We shall write the points of E_ν in increasing order

$$t_0^\nu, \quad t_1^\nu, \quad \ldots, \quad t_{2^\nu}^\nu.$$

On account of (3.2) we evidently have, for $i = 1, 2, \ldots, 2^\nu$,

$$B(t_{i-1}^\nu, t_i^\nu) \leqslant 2^{-\nu},$$

and *a fortiori*

$$|g(t_i^\nu) - g(t_{i-1}^\nu)| \leqslant 2^{-\nu/q}. \tag{3.3}$$

We observe further that the points t_i^ν are everywhere dense in the open set in which $g'(t) \neq 0$. This being so, we denote by R_i^ν the rectangle in the

(t, u) plane given by

$$t_{2i}^{\nu+1} \leqslant u < t_{2i+1}^{\nu+1} < t \leqslant t_{2i+2}^{\nu+1}. \tag{3.4}$$

If (u, t) is any point of the triangle T and if $g'(t) \neq 0$, there is one and only one rectangle R_i^ν containing (u, t). For, since $u < t$ and $g'(t) \neq 0$, there exists a point $t_k^{\nu+1}$ with minimum index ν such that $u < t_k^{\nu+1} < t$. This implies that k is odd, $k = 2i+1$ say, and that (3.4) must hold; thus (u, t) belongs to R_i^ν. Conversely, if (u, t) belongs to R_i^ν, we see that $t_{2i+1}^{\nu+1}$ is the point $t_k^{\nu+1}$ with minimum index ν such that $u < t_k^{\nu+1} < t$, and hence that R_i^ν is unique.

Thus, f' and g' being continuous, we have

$$S(x) = \sum_{\nu=0}^{\infty} \sum_{i=1}^{2^\nu} \iint_{R_i^\nu} f'(x+u)\, g'(t)\, du\, dt = \sum_\nu c_\nu(x), \tag{3.5}$$

where

$$c_\nu(x) = \sum_{i=1}^{2^\nu} \{f(x+t_{2i+1}^{\nu+1}) - f(x+t_{2i}^{\nu+1})\}\{g(t_{2i+2}^{\nu+1}) - g(t_{2i+1}^{\nu+1})\}.$$

This being so, we have now to form the upper bound $C^{1/\lambda}$ of the expression

$$\left(\sum_{l=1}^{N} |S(x_l) - S(x_{l-1})|^\lambda\right)^{1/\lambda}$$

for $0 = x_0 \leqslant x_1 \leqslant \ldots \leqslant x_N = 1$. By the converse of Hölder's inequality, this upper bound is also that of the expression

$$\sum_{l=1}^{N} \{S(x_l) - S(x_{l-1})\} d_l \tag{3.6}$$

for $0 = x_0 \leqslant x_1 \leqslant \ldots \leqslant x_N = 1$ and $\Sigma |d_l|^{\lambda'} = 1$, the index λ' (conjugate to λ) being finite since the conditions $p > 1$, $q > 1$, $1/p + 1/q - 1 > 1/\lambda > 0$ imply $0 < 1/\lambda' < 1$. On account of (3.5), the expression (3.6) may be written in the form ΣC_ν, where, for fixed ν,

$$C_\nu = \sum_{l=1}^{N} \{c_\nu(x_l) - c_\nu(x_{l-1})\} d_l = \sum_{l=1}^{N} \sum_{i=1}^{2^\nu} a_{i,l} d_l g_i, \tag{3.7}$$

$$a_{i,l} = f(x_l + t_{2i+1}^{\nu+1}) - f(x_l + t_{2i}^{\nu+1}) - f(x_{l-1} + t_{2i+1}^{\nu+1}) + f(x_{l-1} + t_{2i}^{\nu+1}),$$

$$g_i = g(t_{2i+2}^{\nu+1}) - g(t_{2i+1}^{\nu+1});$$

these expressions, by our hypotheses, satisfy inequalities of the type

$$\sum_i |a_{i,l}|^p \leqslant K(p), \quad \sum_l |a_{i,l}|^p \leqslant K(p), \quad \sum_l |d_l|^{\lambda'} \leqslant 1, \tag{3.8}$$

and, by (3.3),

$$\sum_i |g_i|^{q_1} \leqslant 2^{-\epsilon\nu q_1}, \tag{3.9}$$

where q_1 is chosen to satisfy $1/\lambda = 1/p + 1/q_1 - 1$, and where

$$\epsilon = \frac{1}{q} - \frac{1}{q_1} = \frac{1}{p} + \frac{1}{q} - 1 - \frac{1}{\lambda} > 0.$$

Now the conditions on the exponents λ, p, q_1 imply that

$$q_1 > 1, \quad \lambda' > 1, \quad \frac{p}{q_1'} + \frac{p}{\lambda} = 1 \quad \text{and} \quad \frac{1}{q_1'} + \frac{1}{\lambda} < 1$$

(the last since $p > 1$). We may therefore apply to (3.7) the inequality 274, p. 198 of [**7**] (which extends a corresponding theorem of W. H. Young) and we derive from (3.8) and (3.9) that

$$|C_\nu| \leqslant K(p)\, 2^{-\epsilon\nu}.$$

This evidently gives $|\Sigma C_\nu| < K(p, \epsilon) = K(p, q, \lambda)$, since $\epsilon > 0$. It follows that the upper bound of (3.6) is at most $K(p, q, \lambda)$ and (2.4) is established.

With hardly any change in the argument, we can establish a more general result, which corresponds to Theorem (6.1) below, subject to continuity of $\partial f(x, y)/\partial y$ and $g'(y)$. Actually, both in (2.4) and in this extension, the restrictions concerning existence and continuity of derivates are unnecessary, and their removal offers only secondary difficulties which, however, render the proof a good deal more elaborate. We shall establish the theorems without these restrictions, when we derive them from results concerning integrated Lipschitz conditions (*cf.* below, § 6).

A more difficult question concerns the range of the exponent λ. We have not been able to ascertain whether (1.1) still holds for $1/\lambda = 1/p + 1/q - 1$.

4. *A criterion for bounded p-th power variation.* We know (*cf.* [**11**], p. 259 (8.4) and p. 260, that for any periodic function $f \prec W_p$ in the interval (0, 1)

$$\left\{\int_0^1 |f(x+h) - f(x)|^p\, dx\right\}^{1/p} \leqslant h^{1/p}\, V_p(f),$$

and *a fortiori*

$$\int_0^1 |f(x+h) - f(x)|\, dx \leqslant h^{1/p}\, V_p(f).$$

Moreover, since we have $V_p\big(f[\phi(t)]\big) \leqslant V_p(f)$ whenever $\phi(t)$ is a monotonic function in the interval $(0, 1)$ and $\phi(0) = 0$, $\phi(1) = 1$, we see that

$$\int_0^1 |f[\phi(t+h)]-f[\phi(t)]|\,dt \leqslant h^{1/p}\,V_p(f). \tag{4.1}$$

We shall prove a partial converse of this inequality.

(4.2) *Suppose that, for some a* $p > 1$, *a function* $f(x)$ *of period* 1 *satisfies the condition*

$$\int_0^1 |f[\phi(t+h)]-f[\phi(t)]|\,dt \leqslant Ah^{1/p}$$

for every monotonic function $\phi(t)$ *in* $(0, 1)$ *such that* $\phi(0) = 0$, $\phi(1) = 1$, *the constant* A *being independent of* ϕ. *Then*

$$V_{p+\epsilon}(f) \leqslant K\,.\,A,$$

where

$$K = K(p, \epsilon) < \infty \quad \text{for} \quad \epsilon > 0.$$

The rather indirect proof that we shall give, leaves open the question whether the ϵ can be dispensed with; and we have no *Gegenbeispiel* to exclude this possibility (which is intimately connected with that of the truth of (1.1) in the limiting case $1/\lambda = 1/p+1/q-1$).

We begin with the following simple lemma:

(4.3) *Let* g *be a continuous function of* W_q *in* $(0, 1)$ *and let* $\gamma(x)$ *be the upper bound of the sums*

$$\sum_{i=1}^{N} |g(x_i)-g(x_{i-1})|^q \tag{4.4}$$

for $0 = x_0 \leqslant x_1 \leqslant \ldots \leqslant x_N = x$. *Then* $\gamma(x)$ *is continuous and monotonic and we have, for* $\beta > \alpha$,

$$|g(\beta)-g(\alpha)|^q \leqslant \gamma(\beta)-\gamma(\alpha). \tag{4.5}$$

We need only establish the continuity of $\gamma(x)$, the rest being obvious. Let ϵ be any positive number and let s_n denote the sum of the first n terms of (4.4). By definition of $\gamma(x)$, there exists a sum (4.4) exceeding the value $\gamma(x)-\epsilon$, and we then have also $s_n \geqslant \gamma(x_n)-\epsilon$, and therefore, for $x_{n-1} \leqslant t < x_n$,

$$\gamma(t) \geqslant s_{n-1}+|g(t)-g(x_{n-1})|^q \geqslant \gamma(x_n)-\epsilon-|g(x_n)-g(x_{n-1})|^q+|g(t)-g(x_{n-1}|^q.$$

The inequality $\gamma(x-0) \geqslant \gamma(x)-\epsilon$ follows by choosing $n = N$ and making $t \to x-0$, while the inequality $\gamma(t) \geqslant \gamma(t+0)-\epsilon$ follows by making $x \to t+0$ and therefore $x_n \to t+0$. Since ϵ is arbitrary, the increasing function $\gamma(x)$ is thus continuous.

We now pass on to the proof of Theorem (4.2).

Let g be any function of period 1 with continuous derivative g' and let $B(g)$ denote the convex functional $V_q(g)$, where q is conjugate to $p+\epsilon$. For any $\eta > 0$, the function $\{\gamma(t)+\eta . t\}/\{B(g)+\eta\}$ [where $\gamma(t)$ is defined as in Lemma (4.3)] is continuous and increases strictly from 0 to 1 with t. Denoting by $\phi(t)$ the inverse function continued outside (0, 1) by writing $\phi(t+1) = 1+\phi(t)$, we find

$$|g[\phi(t+h)]-g[\phi(t)]| \leqslant \{B(g)+\eta\}h^{1/q}, \tag{4.7}$$

and hence

$$\left|\int_0^1 \{f[\phi(t+h)]-f[\phi(t)]\}\{g[\phi(t)]-g[\phi(t-h)]\}\,dt\right| \leqslant A\Big(B(g)+\eta\Big)h^{1+\epsilon_1},$$

where $\epsilon_1 > 0$ depends on ϵ, p. Hence, by results established in connection with binary derivation of a Lebesgue Faltung (*cf.* [**14**], § 4), we find that

$$\left|\int_0^1 f[\phi(t)\,dg[\phi(t)]\right| \leqslant K(\epsilon, p)\,A\{B(g)+\eta\}$$

and therefore, changing the variable and making $\eta \to 0$, that

$$\left|\int_0^1 f(x)\,g'(x)\,dx\right| \leqslant K(\epsilon, p)\,A\,B(g).$$

This shows that for functions g of period 1 and continuous derivative, the linear functional $L(g) = \int_0^1 fg'\,dx$ is majorized by the convex functional $Q(g) = K(\epsilon, p)\,A\,B(g)$. By the Minkowski-Banach continuation principle ([**1**], 27–29), we can continue $L(g) \leqslant Q(g)$ to all continuous g of W_q, and we must have (*cf.* [**9**], 248), in the sub-class $W_q{}^*$,

$$L(g) = -\int_0^1 g\,df_1,$$

where $f_1 \prec W_{q'}$, and, moreover, where

$$V_{q'}(f_1) \leqslant 8K(\epsilon, p)\,A.$$

It is clear that $f_1(1) = f_1(0)$, since $L(g)$ vanishes when g is a constant. We may therefore write, when g is periodic,

$$L(g) = \int_0^1 f_1(x)\,dg(x),$$

and hence, for all periodic g with continuous g',

$$\int_0^1 \{f_1(x)-f(x)\}g'(x)\,dx = 0,$$

from which it follows that f_1-f is constant almost everywhere.

For the same reason, given any monotonic $\phi(t)$ which increases from 0 to 1 with t, the function $f[\phi(t)]$ is equivalent to a function $f_1(t)$ such that

$$V_{q'}(f_1) \leqslant 8KA.$$

In particular, choosing $\phi(t)$ to have constant values x_i throughout corresponding intervals, we derive for $0 = x_0 \leqslant x_1 \leqslant \ldots \leqslant x_N = 1$,

$$\Big\{\sum_{i=1}^{N} |f(x_i)-f(x_{i-1})|^{q'}\Big\}^{1/q'} \leqslant 8KA,$$

and (4.2) follows at once, since $q' = p+\epsilon$.

5. *Some results concerning the integrated Lipschitz class of a generalized Stieltjes Faltung.* Let $f(x, y)$ and $g(y)$ have period 1 in y and let

$$F(x, y) = \int_0^1 f(x, y+t)g(t)\,dt, \qquad S_\nu(x) = 2^\nu\{F(x, 0)-F(x, 2^{-\nu})\}.$$

From what has been shown in [**14**] §§ 2–4, it follows that the binary derivative of $-F$ with respect to y,

$$(5.1)\quad \lim_{\nu\to\infty} S_\nu(x) = -\tfrac{1}{2}\sum_{\nu=1}^{\infty} 2^\nu \int_0^1 \{f(x, t+2^{-\nu})-f(x, t)\}\{g(t)-g(t-2^{-\nu})\}\,dt,$$

is a generalization of the Riemann-Stieltjes integral

$$\int_0^1 f(x, t)\,dg(t).$$

More generally, we shall describe as a *generalized Stieltjes Faltung* $S(x)$ a function such that

$$\lim_{\nu\to\infty} \int_0^1 |S_\nu(x)-S(x)|\,dx = 0.$$

We now prove the following result.

(5.2) THEOREM. *Let $f(x, y)$ and $g(y)$ have period 1 in y and let*

$$\int_0^1\int_0^1 |f(x+h, y)-f(x, y)|\,dx\,dy \leqslant A_1 h^{1/r},$$

$$\int_0^1\int_0^1 |f(x, y+h)-f(x, y)|\,dx\,dy \leqslant A_2 h^{1/p},$$

$$|g(y+h)-g(y)| \leqslant Bh^{1/q},$$

where $r>0$, $p>0$, $q>0$, $1/p+1/q-1=1/\lambda_0>0$. *Then*

$$\int_0^1 |S(x+h)-S(x)|\,dx \leqslant Ch^{1/\mu_0},$$

where $\mu_0=r\lambda_0/p$ *and*

$$C=K(p, q, r)\,A_1^{p/\lambda_0}A_2^{1-p/\lambda_0}B.$$

Proof. In the first place, we have, for $\nu_1>\nu_0$,

$$\int_0^1 |S_{\nu_1}(x)-S_{\nu_0}(x)|\,dx$$

$$\leqslant \tfrac{1}{2}\sum_{\nu_0<\nu\leqslant\nu_1} 2^{\nu}\int_0^1\int_0^1 |f(x, t+2^{-\nu})-f(x, t)|\,|g(t)-g(t-2^{-\nu})|\,dx\,dt$$

$$\leqslant \tfrac{1}{2}\sum_{\nu_0+1}^{\infty} B2^{\nu(1-1/q)}A_2 2^{-\nu/\mu} \leqslant K(\lambda_0)\,A_2 B2^{-\nu_0/\lambda_0}.$$

It follows that $S(x)$ exists and that

$$\int_0^1 |S_\nu(x)-S(x)|\,dx \leqslant K(\lambda_0)\,A_2 B2^{-\nu/\lambda_0}. \tag{5.3}$$

On the other hand, for any fixed ν, we have

$$\int_0^1 |S_\nu(x+h)-S_\nu(x)|\,dx \tag{5.4}$$

$$= 2^{\nu}\int_0^1 \left|\int_0^1 \{f(x+h, t)-f(x, t)\}\{g(t)-g(t-2^{-\nu})\}\,dt\right| dx$$

$$\leqslant B2^{\nu(1-1/q)}\int_0^1\int_0^1 |f(x+h, t)-f(x, t)|\,dx\,dt \leqslant A_1 B2^{\nu(1-1/q)}h^{1/r}.$$

Hence, if we choose ν so that $2^{-\nu}$ lies between constant multiples of the expression

$$h^{p/r}A_1^{p}A_2^{-p},$$

the majorants in (5.3) and (5.4) both become of the form

$$KA_1^{p/\lambda_0}A_2^{1-p/\lambda_0}Bh^{1/\mu_0}$$

and the theorem follows at once.

We remark that, by a trivial simplification of the proof of (5.3), the additional hypothesis

$$\int_0^1 |f(x, y+h)-f(x, y)|\,dy \leqslant Ah^{1/p}$$

would have made $S_\nu(x)$ converge uniformly.

6. *An extension of the inequality* (1.1). With the help of (4.2), we are now able to derive from Theorem (5.2) a second proof of the inequality (1.1) which throws a little light on the question whether the inequality is valid for $1/\lambda = 1/p+1/q-1$. At the same time we generalize the inequality so as to make it apply to a Stieltjes Faltung of the form $\int f(x, t)\,dg(t)$.

(6.1) THEOREM. *Suppose that*

$$r>1, \quad p>1, \quad q>1, \quad 1/p+1/q-1=1/\lambda_0>0$$

and that $\mu>\mu_0=r\lambda_0/p$. *Then there exists* $K=K(\mu, p, q, r)<\infty$, *such that, if* $f(x, y)$ *and* $g(y)$ *have period* 1 *in* y *and if* $g(y)$ *is continuous, the upper bounds* A_1, A_2, B, C *of the sums*

$$\sum_i |f(x_i, y)-f(x_{i-1}, y)|^r, \quad \sum_j |f(x, y_j)-f(x, y_{j-1})|^p, \quad \sum_j |g(y_j)-g(y_{j-1})|^q,$$

$$\sum_i |S(x_i)-S(x_{i-1})|^\mu,$$

respectively, for $0=x_0\leqq x_1\leqq\ldots\leqq x_N$ *and all* y, *or for*

$$0=y_0\leqq y_1\leqq\ldots\leqq y_M=1$$

and all x, *satisfy the inequality*

$$C\leqq KA_1^{1/(r\lambda_0)}A_2^{1-1/(r\lambda_0)}B,$$

the function $S(x)$ *denoting the Stieltjes Faltung*

$$\int_0^1 f(x, y)\,dg(y),$$

which exists if A_2 *and* B *are finite.*

Given any $\eta>0$, we determine as in the proof of (4.2) a function $\phi(t)$, continuous and strictly increasing from 0 to 1 with t, which satisfies (4.7) and $\phi(t+1)=1+\phi(t)$. We denote further by $\varpi(u)$ an arbitrary monotone increasing function such that $\varpi(0)=0$, $\varpi(1)=1$, and we write

$$f^*(u, t)=f[\varpi(u), \phi(t)], \quad g^*(t)=g[\phi(t)],$$

$$S^*(u)=S[\varpi(u)]=\int_0^1 f[\varpi(u), y]\,dg(y)=\int_0^1 f^*(u, t)\,dg^*(t).$$

We then have, by (4.1),

$$\int_0^1 |f^*(u+h, t)-f^*(u)|\,du\leqq A_1h^{1/r}, \qquad \int_0^1 |f^*(u, t+h)-f^*(u, t)|\,dt\leqq A_2h^{1/p},$$

while, by (4.7), $|g^*(t+h)-g^*(t)| \leqslant (B+\eta)h^{1/q}$. Moreover, since $S^*(u)$ is a Riemann-Stieltjes integral, we have, for each u, $S_\nu^*(u) \to S(u)$, where

$$S_\nu^*(u) = 2^\nu \int_0^1 \{f^*(u, t)-f^*(u, t+2^{-\nu})\} g^*(t)\, dt$$

converges uniformly in view of the remark at the end of §5. Hence, by Theorem (5.2),

$$\int_0^1 |S^*(u+h)-S^*(u)|\, du \leqslant K A_1^{p/\lambda_0} A_2^{1-p/\lambda_0} B h^{1/\mu_0}, \tag{6.2}$$

and the conclusion follows at once from Theorem (4.2).

In the form in which we have stated Theorem (6.1) there is still an unnecessary restriction of continuity on the function g. This is disposed of by arguments that are not quite trivial. We require the following lemmas.

(6.3) *Let g_1 be a step function which agrees with g at the end points, and at some point or symbol x or $x\pm 0$, of each of its intervals of constancy. Then $V_q(g_1) \leqslant V_q(g)$.*

(6.4) *Let g_2 be a piecewise linear continuous function which agrees with g at the end points of its intervals of linearity. Then $V_q(g_2) \leqslant K V_q(g)$.*

The first of these lemmas is trivial. The second is established in [9], 254, §7, with $K = 8$. (Mr. Love informs me that he has proved (6.4) with $K = 1$ when g is real, or $K = 2$ when g is complex.)

(6.5) *In order to establish* (6.1) *without the restriction of continuity on g, it is sufficient to do so in the case in which g is a step function.*

To see this we observe that the Stieltjes Faltung $S(x)$ may always be expressed as the limit of a finite sum

$$\sum_{k=1}^{N} f(x, \eta_k)\{g(y_k)-g(y_{k-1})\}, \tag{6.6}$$

where the y_k are an increasing finite sequence, consisting of points y together with the symbols $y-0$ and $y+0$, while the η_k are points, and we have $y_{k-1} \leqslant \eta_k \leqslant y_k$. The y_k and η_k are chosen independently of x; we may suppose for instance that the points among the symbols y_k consist of the first n discontinuities of g together with the n points $y = l/n$ for $l = 0, 1, \ldots, n$, and that the η_k are the midpoints of (y_{k-1}, y_k). This follows easily from the theorems of [11], §10. Finally (6.6) is simply the Stieltjes Faltung of f with a certain step function g_1 of the type considered in (6.3).

(6.7) *In order to establish* (6.1) WITHOUT *the restriction of continuity on g, it is sufficient to do so* WITH *the restriction in the case in which g is piecewise linear.*

For when g is a step function, as we may suppose in view of (6.6), the values of $f(x, y)$ at the points y of constancy of g are irrelevant to the Stieltjes Faltung. By using (6.3) we may suppose f constant with respect to y in open strips which include the parallels to the x-axis through the discontinuities of g, and f being now of this special type, we may make g continuous and piecewise linear without affecting the Stieltjes Faltung. This affects the inequality that we are concerned with only to the extent of a factor K, by (6.4).

(6.1) and (6.7) imply that *in Theorem* (6.1) *we may remove the restriction of continuity on g.*

7. *Further results concerning the integrated Lipschitz class of a generalized Stieltjes Faltung.* We consider, in this section, only the case

$$f(x, y) = f(x+y),$$

where $f(x)$ is of period 1. We write, as previously,

$$S_\nu(x) = 2^\nu \int_0^1 \{f(x+t)-f(x+t+2^{-\nu})\} g(t)\, dt.$$

(7.1) THEOREM. *Let* $p>1$, $q>1$, $\lambda>1$, $1/\lambda \geqslant 1/p+1/q-1$, $\alpha>0$, $\beta>0$, $\alpha+\beta>1$, *and, for* $0<h<1$,

$$\left(\int_0^1 |f(x+h)-f(x)|^p\, dx\right)^{1/p} \leqslant Ah^\alpha, \quad \left(\int_0^1 |g(x+h)-g(x)|^q dx\right)^{1/q} \leqslant Bh^\beta.$$

Then there exists a function $S(x)$ *such that* $\int |S_\nu(x)-S(x)|^\lambda dx \to 0$ *as* $\nu \to \infty$, *and we have*

$$\left(\int_0^1 |S(x+h)-S(x)|^\lambda dx\right)^{1/\lambda} \leqslant KABh^{\alpha+\beta-1},$$

where K *depends only on* α, β. *Moreover, in the case* $1/p+1/q \leqslant 1$, $S_\nu(x)$ *converges uniformly to a function* $S(x)$ *and then*

$$|S(x+h)-S(x)| \leqslant KABh^{\alpha+\beta-1}.$$

Proof. In view of a well-known inequality ([7], 192, p. 143), we may suppose (by decreasing p and q if necessary) that $1/\lambda = 1/p+1/q-1$. We write as usual p', q', λ' for the exponents conjugate to p, q, λ, and we denote by $\phi(x)$ any function such that

$$\int_0^1 |\phi|^{\lambda'} dx = 1.$$

This being so, we observe that

$$\int_0^1 \{S_\nu(x)-S_{\nu-1}(x)\}\phi(x)\, dx$$

$$= -\tfrac{1}{2}2^\nu \int_0^1\int_0^1 \{f(x+t+2^\nu)-f(x+t)\}\{g(t)-g(t-2^{-\nu})\}\phi(x)\, dx\, dt$$

$$= -\tfrac{1}{2}2^\nu \int_0^1\int_0^1 a(x+t)\phi(x)\psi(t)\, dx\, dt, \text{ say,}$$

is majorized, on account of a well-known inequality of W. H. Young (special case of the analogue for integrals of [7] 274, p. 198, already cited on p. 453

above) by

$$\tfrac{1}{2}2^{\nu}\left(\int_0^1|a(x)|^p\,dx\right)^{1/p}\left(\int_0^1|\phi(x)|^{\lambda'}dx\right)^{1/\lambda'}\left(\int_0^1|\psi(x)|^q\,dx\right)^{1/q}\leqslant\tfrac{1}{2}AB2^{-(\alpha+\beta-1)\nu}.$$

It follows that, if $\nu_0<\nu_1$,

$$\left(\int_0^1|S_{\nu_1}(x)-S_{\nu_0}(x)|^{\lambda}dx\right)^{1/\lambda}\leqslant\tfrac{1}{2}AB\sum_{\nu_0<\nu\leqslant\nu_1}2^{-(\alpha+\beta-1)\nu},$$

so that the existence of a function $S(x)$ for which

$$\int_0^1|S_{\nu}(x)-S(x)|^{\lambda}dx\to 0$$

is assured, and we have

$$\left(\int_0^1|S_{\nu}(x)-S(x)|^{\lambda}dx\right)^{1/\lambda}\leqslant K(\alpha,\beta)AB2^{-(\alpha+\beta-1)\nu}. \tag{7.2}$$

Also, we easily find that

$$\left(\int_0^1|S_{\nu}(x+h)-S(x)|^{\lambda}dx\right)^{1/\lambda}\leqslant AB2^{(1-\beta)\nu}h^{\alpha}, \tag{7.3}$$

and, on choosing ν so that $2^{-\nu}$ lies between constant multiples of h, we derive from (7.2) and (7.3)

$$\left(\int_0^1|S(x+h)-S(x)|^{\lambda}dx\right)^{1/\lambda}\leqslant K(\alpha,\beta)ABh^{\alpha+\beta-1}.$$

Finally, in the case $1/p+1/q\leqslant 1$, we may suppose $1/p+1/q=1$ and we then observe that, by Hölder's inequality,

$$\begin{aligned}|S_{\nu}(x)-S_{\nu-1}(x)|&\leqslant\tfrac{1}{2}2^{\nu}\int_0^1|f(x+t+2^{-\nu})-f(x+t)||g(t)-g(t-2^{-\nu})|\,dt\\&\leqslant\tfrac{1}{2}AB2^{-(\alpha+\beta-1)\nu},\end{aligned}$$

while

$$\begin{aligned}|S_{\nu}(x+h)-S_{\nu}(x)|&\leqslant 2^{\nu}\int_0^1|f(x+t+h)-f(x+t)||g(t)-g(t-2^{-\nu})|\,dt\\&\leqslant AB2^{(1-\beta)\nu}h^{\alpha}.\end{aligned}$$

The conclusions follow by a trivial simplification of the previous argument.

We may regard as a corollary of Theorem (7.1) the following theorem of Hardy and Littlewood ([3], 627), which is obtained by choosing $g(x) = x/\theta$ for $0 \leqslant x < \theta$, where $0 < \theta < 1$ and $g(x) = 1$ for $\theta \leqslant x < 1$, the function g being continued by periodicity.

(7.4) *If* $\alpha > 1/p$, *the relation* $f \prec \mathrm{Lip}(\alpha, p)$ *implies* $f \sim f^*$, *where* $f^* \prec \mathrm{Lip}(\alpha - 1/p)$.

The function $g(x)$ defined above belongs, in fact, to $\mathrm{Lip}(1/q, q)$ for every $q \geqslant 1$, and in particular for $q = p'$. Moreover, we may suppose $f \prec W_p$ (*cf.* [10], § 6) without loss of generality. The expression

$$\int_0^1 f(x+t)\, dg(t) = \frac{1}{\theta} \int_x^{x+\theta} f(t)\, dt - f(x) \tag{7.5}$$

then exists as a Riemann-Stieltjes integral for almost all x, since the discontinuities of an $f \prec W_p$ are of the first kind and are enumerable. It therefore agrees almost everywhere with the generalized Stieltjes Faltung $S(x)$, and the latter belongs to the class $\mathrm{Lip}(\alpha - 1/p)$ by Theorem (7.1). It follows that f is equivalent to the function

$$f^*(x) = \frac{1}{\theta} \int_x^{x+\theta} f(t)\, dt - S(x),$$

which, as the difference of a function of $\mathrm{Lip}(1)$ with a function of $\mathrm{Lip}(\alpha - 1/p)$, belongs to the class $\mathrm{Lip}(\alpha - 1/p)$.

The same argument establishes various other results of similar type obtained by Hardy and Littlewood.

8. *Inequalities for multiple integrals.* Let $f(x, y)$ and $g(x, y)$ be of period 1 in each of the variables x and y, and let Ω denote a unit period square. We say that

$$\iint_\Omega f(x, y)\, dg(x, y) \tag{8.1}$$

exists in the Fréchet sense with the value I, if the number I is the unique limit, as $\delta \to 0$, of the upper and lower bounds of the sum

$$\sum_{i=1}^{N} \sum_{j=1}^{M} f(\xi_i, \eta_j)\{g(x_i, y_j) - g(x_{i-1}, y_j) - g(x_i, y_{j-1}) + g(x_{i-1}, y_{j-1})\}, \tag{8.2}$$

when

$$x_0 \leqslant \xi_1 \leqslant x_1 \leqslant \ldots \leqslant x_{N-1} \leqslant \xi_N \leqslant x_N = x_0 + 1,$$

$$y_0 \leqslant \eta_1 \leqslant y_1 \leqslant \ldots \leqslant y_{M-1} \leqslant \eta_M \leqslant y_M = y_0 + 1,$$

and $x_i - x_{i-1} \leqslant \delta$, $y_j - y_{j-1} \leqslant \delta$ for $i = 1, \ldots, N$ and for $j = 1, \ldots, M$.

The meshes of the subdivision and their representative points are here subject to conditions of regularity by arrangement illustrated by Fig. (i), whereas for double integrals in the (more restricted) Riemann-Stieltjes sense, a subdivision corresponding to the irregular arrangement illustrated by Fig (ii) is permitted.

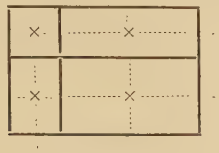

Fig. (i)

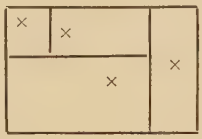

Fig. (ii)

It is convenient to introduce, for any function $F(x, y)$, the difference operators given by

$$\Delta_{h,k} F(\xi, \eta) = F(\xi+h, \eta+k) - F(\xi, \eta+k) - F(\xi+h, \eta) + F(\xi, \eta),$$

$$\Delta^2_{h,k} F(\xi, \eta) = \Delta_{h,k}\{\Delta_{h,k} F(x, y)\}_{x=\xi, y=\eta}$$

$$= (\Delta_{2h,2k} - 2\Delta_{h,2k} - 2\Delta_{2h,k} + 4\Delta_{h,k}) F(\xi, \eta),$$

where we make use of an identity easily verified. This being so, we define, by analogy with the corresponding one-dimensional theory of [14], the generalization of (8.1) which consists of the bilinear functional $I(f, g)$, the limit as $\mu, \nu \to \infty$ of the expression

$$S_{\mu,\nu} = 2^{\mu+\nu} \Delta_{2^{-\mu}, 2^{-\nu}} F(0, 0),$$

where

$$F(x, y) = \iint_\Omega f(x+u, y+v)\, g(u, v)\, du\, dv.$$

Here $S_{\mu,\nu}$ is an average of sums of the type (8.2); it is, in fact, the integral with respect to x, y over Ω of the function of x, y

$$\sum_{i=1}^{2^\mu} \sum_{j=1}^{2^\nu} f(x+i2^{-\mu}, y+j2^{-\nu})\, \Delta_{-2^{-\mu}, -2^{-\nu}} g(x+i2^{-\mu}, y+j2^{-\nu}),$$

which is a special case of (8.2). *Hence $I(f, g)$ is a generalization of the integral* (8.1) *in the Fréchet sense.*

We observe further that $S_{\mu,0}$ and $S_{0,\nu}$ vanish for all μ, ν by periodicity, and hence that we may write

$$S_{\mu_0,\nu_0} = \sum_{\mu=1}^{\mu_0} \sum_{\nu=1}^{\nu_0} s_{\mu,\nu}, \quad I(f, g) = \sum_{\mu=1}^{\infty} \sum_{\nu=1}^{\infty} s_{\mu,\nu},$$

where

$$\begin{aligned} s_{\mu,\nu} &= S_{\mu,\nu}-S_{\mu-1,\nu}-S_{\mu,\nu-1}+S_{\mu-1,\nu-1} \\ &= \tfrac{1}{4}\,2^{\mu+\nu}\{\Delta_{2^{-(\mu-1)},\,2^{-(\nu-1)}}-2\Delta_{2^{-\mu},\,2^{-(\nu-1)}}-2\Delta_{2^{-(\mu-1)},\,2^{-\nu}}+4\Delta_{2^{-\mu},\,2^{-\nu}}F(0,\,0) \\ &= \tfrac{1}{4}\,2^{\mu+\nu}\Delta^2_{2^{-\mu},\,2^{-\nu}}F(0,\,0) \\ &= \tfrac{1}{4}\,2^{\mu+\nu}\iint_\Omega \Delta_{2^{-\mu},\,2^{-\nu}}f(u,\,v)\,\Delta_{-2^{-\mu},\,-2^{-\nu}}g(u,\,v)\,du\,dv. \end{aligned}$$

Applying Hölder's inequality to this last expression, we deduce at once the following result.

(8.3) *Let f, g be functions (of period 1 in each of the variables x, y) for which*

$$\left(\iint_\Omega |\Delta_{h,\,k}f(u,\,v)|^p\,du\,dv\right)^{1/p} \leqslant Ah^{\alpha_1}k^{\alpha_2},$$

$$\left(\iint_\Omega |\Delta_{h,\,k}g(u,\,v)|^{p'}\,du\,dv\right)^{1/p'} \leqslant Bh^{\beta_1}k^{\beta_2},$$

where the exponents p, p' are conjugate and α_1, α_2, β_1, β_2 non-negative and subject to $\alpha_1+\beta_1>1$, $\alpha_2+\beta_2>1$. Then $I(f,\,g)$ exists and we have

$$|I(f,\,g)| \leqslant KAB,$$

where $K=K(\alpha_1,\,\beta_1,\,\alpha_2,\,\beta_2)$. The inequality remains valid in the limiting cases p or p' infinite.

We find, in fact, that $|s_{\mu,\nu}| \leqslant \frac{1}{4}\,2^{-(\alpha_1+\beta_1-1)\mu}\,2^{-(\alpha_2+\beta_2-1)\nu}AB$, so that the conclusion follows by summing over μ, ν.

We draw attention, in particular, to the following consequence of (8.3).

(8.4) *Let f, g be functions (of period 1 in each of the variables x, y) for which*

$$\iint_\Omega |f(u+h,\,v)-f(u,\,v)|\,du\,dv \leqslant A_1h^{\alpha_1},$$

$$\iint_\Omega |f(u,\,v+k)-f(u,\,v)|\,du\,dv \leqslant A_2k^{\alpha_2},$$

and

$$|\Delta_{h,\,k}g| \leqslant Bh^{\beta_1}k^{\beta_2},$$

where α_1, α_2, β_1, β_2 are non-negative and subject to

$$\frac{1-\beta_1}{\alpha_1}+\frac{1-\beta_2}{\alpha_2}<1.$$

Then $I(f, g)$ exists and we have

$$|I(f, g)|\leqslant KA_1^{\theta}A_2^{1-\theta}B,$$

where $K=K(\alpha_1, \beta_1, \alpha_2, \beta_2, \theta)$ for any θ such that

$$\frac{1-\beta_1}{\alpha_1}<\theta<1-\left(\frac{1-\beta_2}{\alpha_2}\right).$$

We have simply to observe that (since $0<\theta<1$)

$$\iint_{\Omega}|\Delta_{h,k}f(u, v)|\,du\,dv$$

$$=\left(\iint_{\Omega}|\Delta_{h,k}f(u, v)|\,du\,dv\right)^{\theta+(1-\theta)}$$

$$\leqslant K\left(\iint_{\Omega}|f(u+h, v)-f(u, v)|\,du\,dv\right)^{\theta}\left(\iint_{\Omega}|f(u, v+k)-f(u, v)|\,du\,dv\right)^{1-\theta}$$

$$\leqslant KA_1^{\theta}A_2^{1-\theta}h^{\alpha_1\theta}k^{\alpha_2(1-\theta)}.$$

We can therefore apply the case $p'=\infty$ of (8.3) with $KA_1^{\theta}A_2^{1-\theta}$ for A and $\alpha_1\theta$, $\alpha_2(1-\theta)$ for α_1, α_2, the conditions $\alpha_1\theta+\beta_1>1$, $\alpha_2(1-\theta)+\beta_2>1$ being satisfied.

With the help of (8.4) it is not difficult to establish the following inequality for multiple integrals in the Fréchet sense with respect to a function $g(x, y)$ of the form $g_1(x)g_2(y)$.

(8.5) THEOREM. *Suppose that $r>0$, $s>0$, $p\geqslant 1$, $q\geqslant 1$ and*

$$\frac{r}{p'}+\frac{s}{q'}<1, \qquad \frac{1}{p}+\frac{1}{q}\geqslant 1.$$

Then there exists $K(\theta)=K(p, q, r, s, \theta)<\infty$, such that, if $f(x, y)$ and $g_1(x)g_2(y)$ have period 1 in x and in y and g_1g_2 is continuous, the inequalities

$$\sum_{i=1}^{N}|f(x_i, y)-f(x_{i-1}, y)|^r\leqslant A_1{}^r, \qquad \sum_{j=1}^{M}|f(x, y_j)-f(x, y_{j-1})|^s\leqslant A_2{}^s,$$

$$\sum_{i=1}^{N}|g_1(x_i)-g_1(x_{i-1})|^p\leqslant B_1{}^p, \qquad \sum_{j=1}^{M}|g_2(y_j)-g_2(y_{j-1})|^q\leqslant B_2{}^q,$$

for all x_i $(i = 0, 1, 2, \ldots, N)$ *and* y, *and for all* y_j $(j = 0, 1, 2, \ldots, M)$ *and* x, *subject to*

$$0 = x_0 \leqslant x_1 \leqslant \ldots \leqslant x_N = 1, \quad 0 = y_0 \leqslant y_1 \leqslant \ldots \leqslant y_M = 1,$$

imply that the integral

$$I = \int_0^1 \int_0^1 f(x, y)\, dg_1(x)\, dg_2(y)$$

exists in the Fréchet sense and that we have

$$|I| \leqslant K(\theta) A_1^{\theta} A_2^{1-\theta} B_1 B_2, \tag{8.6}$$

where θ *is any number such that*

$$\frac{r}{p'} < \theta < 1 - \frac{s}{q'}.$$

We shall be content to sketch the proof in the case in which the integral I is known to exist. [The general case can then be dealt with by approximating to f by a suitable step function and by using (8.6) to obtain a bound for the approximation.] By Lemma (4.3), we can change our variables in such a way that (without affecting the conditions on f or the existence and value of I) the conditions on $g_1 g_2$ become of the Lipschitz type and imply

$$|\Delta_{h,k} g_1(x) g_2(y)| \leqslant (B_1 B_2 + \epsilon) h^{1/p} k^{1/q}.$$

The conditions on f certainly imply

$$\int_0^1 \int_0^1 |f(u+h, v) - f(u, v)|\, du\, dv \leqslant A_1 h^{1/r},$$

$$\int_0^1 \int_0^1 |f(u, v+k) - f(u, v)|\, du\, dv \leqslant A_2 k^{1/s},$$

on account of (4.1). The required inequality therefore follows from (8.4).

We remark in conclusion that Theorem (8.5) leads to a fresh proof of Theorem (6.1) when we employ the theorems on linear functionals of [9]. This method of proof does not, however, throw any fresh light on the question of its validity for $\mu = \mu_0$.

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ON THE LEVEL CURVES OF INTEGRAL AND MEROMORPHIC FUNCTIONS

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1. Let $f(z)$ be an integral function; any curve on which $|f(z)|$ is constant is called a level curve of $f(z)$. Some properties of level curves have been discussed by Pennycuick*, and in his paper he mentions a conjecture of J. M. Whittaker, viz.:

If the integral functions f and g are of constant modulus on a closed curve Γ, then

$$f(z) = K\{g(z)\}^{\alpha} \quad (\alpha > 0). \tag{1}$$

Valiron† has shown that this is true except when

$$f(z) = K\Big(w(z)-a\Big)^{\mu}\left(\frac{w(z)-b}{w(z)}\right)^{\nu}, \quad g(z) = K'\Big(w(z)-a\Big)^{\mu'}\left(\frac{w(z)-b}{w(z)}\right)^{\nu'}, \tag{2}$$

where $w(z)$ is an integral function without zeros, μ, μ', ν, ν' are positive integers (including zero), and K and K' are constants.

I propose to give a different proof which gives a similar, but less striking, theorem for meromorphic functions at the same time. The method is more direct; it shows the significance of the function $w(z)$ more clearly, and gives certain results about asymptotic values which are not quite obvious from (2).

2. Let $\Gamma = \Gamma(f_0)$ be a closed level curve of some meromorphic function $f_0(z)$ which is not a constant. There is no real loss of generality in supposing that Γ is a simple closed curve. For if not, we may consider one loop of Γ which forms a simple closed curve. It will obviously be sufficient to consider functions $f(z)$ such that $|f| = 1$ on Γ.

* K. Pennycuick, *Journal London Math. Soc.*, 11 (1936), 281–285.

† Valiron, *Comptes rendus*, 204 (1937), 402–404.

Let C_m denote the class of all *meromorphic* functions $f(z)$ such that $|f| = 1$ on Γ, and let C_i denote the class of all *integral* functions such that $|f| = 1$ on Γ. Polynomials and rational functions are included among integral and meromorphic functions respectively, but the constant 1 does not belong to C_m nor to C_i. The class C_m contains at least one member f_0; and, by putting

$$f(z) = \frac{f_0(z) - f_0(a)}{1 - \overline{f_0(a)} f_0(z)}, \tag{3}$$

we can obtain a function belonging to C_m with a zero at any prescribed point $z = a$ inside Γ. If $z = a$ is a pole of $f(z)$, we interpret (3) as $f(z) = 1/f_0(z)$. The class C_i may be empty.

THEOREM 1. *Let a be any point inside Γ, and let $w(z, a)$ and $f(z)$ be two functions belonging to C_m, both having zeros at $z = a$. Suppose that inside Γ $w(z, a)$ has not more zeros and poles all together, counted according to their multiplicity, than any other function of C_m which has a zero at a. Then $w(z, a)$ is a factor of $f(z)$, and in particular if $f(z)$ has the same number of poles and zeros as $w(z, a)$ inside Γ, $f(z) = e^{i\lambda} w(z, a)$, where λ is real. Further, if b is any other point inside Γ,*

$$w(z, b) = \frac{w(z, a) - w(b, a)}{1 - \overline{w(b, a)} w(z, a)}. \tag{3'}$$

When we say that $w(z, a)$ is a factor of $f(z)$, we mean that every zero or pole of $w(z, a)$ is a zero or pole, as the case may be, of $f(z)$ of at least as high order. It may be observed that $w(z, a)$ takes every value w_0 such that $|w_0| \leqslant 1$ at least once, but if $w(z, a)$ has no poles inside Γ, $w(z, a)$ may omit two values whose moduli are greater than one. Since no meromorphic function can omit more than two values, it follows from (3′) that the aggregate of functions $w(z, a)$ for different values of a contains at most two different integral functions, although $w(z, a)$ may coincide with one or other of these for several distinct values of a. If $w(z, a)$ has only one zero, and no pole inside Γ, it is the function which maps the interior of Γ on the unit circle so that the point a corresponds to the origin. In general the mapping function is a many valued function, and not a meromorphic function.

From Theorem 1 we deduce two main theorems.

THEOREM 2. *Suppose that C_i contains at least one member $g(z)$, and let a denote any zero of $g(z)$ inside Γ. Then $w(z, a)$ is an integral function; and if $w(z, a)$ takes every finite value at least once, then any function $f(z)$ belonging*

to C_i can be expressed in the form

$$f(z) = e^{i\lambda}\{w(z, a)\}^{\mu}, \tag{4}$$

where λ is real and μ a positive integer. If $w(z, a)$ does not take the value w_0, then any function belonging to C_i can be expressed in the form

$$f(z) = e^{i\lambda}\{w(z, a)\}^{\mu}\{w(z, b)\}^{\nu}, \tag{5}$$

where $w_0 = 1/\overline{w(b, a)}$, λ is real, and μ and ν are positive integers one of which may be zero.

This, of course, includes Valiron's result. Applying the method to meromorphic functions, we have

THEOREM 3. *Any function $f(z)$ belonging to C_m may be written in the form*

$$f(z) = e^{i\lambda} \prod_{\nu=1}^{N} \{w(z, a_\nu)\}^{\mu_\nu},$$

where λ is real, $a_1, a_2, \ldots, a_N$ are the zeros or poles of $f(z)$ inside Γ, and $\mu_1, \mu_2, \ldots . \mu_N$ are positive or negative integers.

3. *Proof of Theorem* 1. As we have already remarked, there is at least one member of C_m with a zero at a, and since the number of zeros and poles of any member of C_m inside Γ is finite, $w(z, a)$ certainly exists. Map the interior of Γ on $|\zeta| < 1$ by the function $\zeta = \zeta(z)$ so that the point a corresponds to the origin in the ζ plane. If we also make a fixed direction at a correspond to the positive real axis, the function $\zeta(z)$ is uniquely determined. Let $a = a_1, a_2, \ldots, a_n$ be the zeros or poles of $w(z, a)$ of orders $h_1, h_2, \ldots, h_n$ respectively inside Γ, and let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the corresponding points in the ζ plane. If a_ν is a pole, h_ν will be a negative integer, and since the point a corresponds to the origin, $\alpha_1 = 0$. Let T be the total number of zeros and poles of $w(z, a)$ inside Γ, *i.e.*

$$T = |h_1| + |h_2| + \ldots + |h_n|. \tag{6}$$

Since the mapping is (1, 1), $z(\zeta)$, the inverse of the mapping function, is regular for $|\zeta| < 1$, and so $\phi(\zeta) = w\{z(\zeta)\}$ is regular for $|\zeta| < 1$, and has zeros or poles of orders $h_1, h_2, \ldots, h_n$ at $\alpha_1, \alpha_2, \ldots, \alpha_n$ respectively. Also $\phi(\zeta)$ is continuous and $|\phi| = 1$ on $|\zeta| = 1$. Hence it is easy to show by the the maximum modulus principle that

$$w(z) = \phi(\zeta) = e^{i\lambda}\zeta^{h_1} \prod_{\nu=2}^{n} \left(\frac{\zeta - \alpha_\nu}{1 - \overline{\alpha}_\nu \zeta}\right)^{h_\nu}. \tag{7}$$

Similarly, since $f(z)$ also has a zero at $z = a$, we can write

$$f(z) = \psi(\zeta) = e^{i\lambda'} \zeta^{k_1} \prod_{\nu=2}^{m} \left(\frac{\zeta - \beta_\nu}{1 - \overline{\beta}_\nu \zeta} \right)^{k_\nu}, \tag{8}$$

where $\beta_2, \beta_3, \ldots, \beta_m$ are the other zeros and poles of $f(z)$. Multiplying (7) and (8) by suitable polynomials in ζ, not necessarily those which are printed in the denominators of (7) and (8), since some of the h_ν's and k_ν's may be negative, we obtain equations of the form

$$P_1(\zeta) w(z) = P_2(\zeta), \quad Q_1(\zeta) f(z) = Q_2(\zeta);$$

on arranging these in powers of ζ, we obtain

$$P(\zeta, z) = 0, \quad Q(\zeta, z) = 0,$$

where $P(\zeta, z)$, $Q(\zeta, z)$ are polynomials in ζ with coefficients in the field of meromorphic functions of z. These can be regarded as algebraic equations in ζ. The mapping function $\zeta = \zeta(z)$ certainly satisfies both of them, and so P and Q have a highest common factor $R(\zeta, z)$ which is a polynomial in ζ of degree not less than one with coefficients in the field of meromorphic functions.

We may, perhaps, remark that $P(\zeta, z)$ is a polynomial of degree T in ζ, and so for each fixed value of w, P has T roots. But since $w(z, a)$ may take the same value at several different points, these roots may correspond to different values of z, and the same is true for $f(z)$. The mapping function is unique, and the different roots of P and Q are accounted for by w and f taking the same value at several different points.

Now $w(a, a) = f(a) = 0$, and so, when $z = a$, $R(\zeta, z)$ must reduce to the highest common factor of

$$P(\zeta, a) = e^{i\lambda} \zeta^{h_1} \prod_{2}^{n}{}' (\zeta - \alpha_\nu)^{h_\nu} \prod_{2}^{n}{}'' (1 - \overline{\alpha}_\nu \zeta^{-h_\nu}),$$

$$Q(\zeta, a) = e^{i\lambda} \zeta^{k_1} \prod_{2}^{m}{}' (\zeta - \beta_\nu)^{k_\nu} \prod_{2}^{m}{}'' (1 - \overline{\beta}_\nu \zeta^{-k_\nu}),$$

where Π' denotes the product taken over all ν for which $h_\nu > 0$, or $k_\nu > 0$ and Π'' denotes the product taken over all ν for which $h_\nu < 0$ or $k_\nu < 0$.

Hence we may write

$$R(\zeta, a) = \zeta^{h_1'} \prod_{2}^{n}{}' (\zeta - \alpha_\nu)^{h_\nu'} \prod_{2}^{n}{}'' (1 - \overline{\alpha}_\nu \zeta^{-h_\nu'}),$$

where $0 < h_1' \leqslant h_1$, $0 \leqslant h_\nu' \leqslant h_\nu$, $\nu = 2, \ldots, n$, and consider the function

$$\chi(z) = \zeta^{h_1'} \prod_{2}^{n} \left(\frac{\zeta - \alpha_\nu}{1 - \overline{\alpha}_\nu \zeta} \right)^{h_\nu'}. \tag{9}$$

Since $\zeta(z)$ is regular and $|\zeta| < 1$ inside Γ, $\chi(z)$ is regular except perhaps at poles of $w(z, a)$ inside Γ. Multiplying (9) by the appropriate polynomial, we obtain a polynomial $S(\zeta, z)$ in ζ with coefficients which are regular inside Γ except for poles. It is of the same degree as $R(\zeta, z)$, and it is a factor of P and Q, as may be seen directly by dividing (7) and (8) by (9), and then multiplying by the appropriate polynomials. Hence $S(\zeta, z)$ differs from $R(\zeta, z)$ by a constant factor at most, and so, comparing the coefficients, we see that $\chi(z)$ is a meromorphic function. Further $|\chi| = 1$ when $|w| = 1$, *i.e.* when z lies on Γ, so that $\chi(z)$ belongs to C_m. Finally, since $h_1' > 0$, $\chi(z)$ has a zero of order h_1' at $z = a$, the point corresponding to $\zeta = 0$. It follows from our hypotheses that $\chi(z)$ has at least as many zeros and poles inside Γ as $w(z, a)$; it cannot have more, and so the number of zeros and poles of χ and w inside Γ is the same. It follows that $w(z, a) = K\chi(z)$, where K is a constant, and comparing coefficients of ζ^T, we see that $K = e^{i\lambda}$.

It follows from the contruction of $\chi(z)$ that every factor $\left(\frac{\zeta - a_\nu}{1 - \overline{a_\nu}\zeta}\right)$ on the right-hand side of (7) is contained in the right-hand side of (8) to at least as high order whether negative or positive, *i.e.* we may write $\beta_\nu = a_\nu$ and $|k_\nu| \geqslant |h_\nu'| = |h_\nu|$, $\nu = 2, 3, \ldots, n \leqslant m$. The mapping function is regular everywhere except perhaps for isolated singularities corresponding to zeros of $\phi'(\zeta)$, as may be seen by considering (7). We also see from (7) that every zero or pole of $w(z, a)$ whether inside or outside Γ must correspond to one or other of the points $0, a_2, a_3, \ldots, a_n, \infty, 1/\overline{a_2}, 1/\overline{a_3}, \ldots, 1/\overline{a_n}$; and, since $|k_\nu| \geqslant |h_\nu|$, it follows from (8) that $f(z)$ has a zero or pole, as the case may be, of at least as high order at each zero or pole of $w(z, a)$.

It remains to consider $w(z, b)$ the function belonging to C_m with a zero at $z = b$ which has not more zeros and poles inside Γ than any other such function. Now the function

$$W(z, b) = \frac{w(z, a) - w(b, a)}{1 - \overline{w(b, a)}\, w(z, a)}$$

belongs to C_m and has a zero at $z = b$. Further, putting

$$\zeta_1 = \frac{\zeta - \beta}{1 - \overline{\beta}\zeta}, \qquad \beta = \zeta(b)$$

in (7), we obtain a polynomial $U(\zeta_1, z)$ in ζ_1 of degree T with coefficients which are linear in $W(z, b)$. This can be rearranged so as to obtain the expression corresponding to (7) viz.

$$W(z, b) = \zeta_1^{k_1} \prod_{\nu=2}^{m} \left(\frac{\zeta_1 - \beta_\nu}{1 - \overline{\beta_\nu}\zeta_1}\right)^{k_\nu},$$

where $0, \beta_2, \beta_3, \ldots, \beta_m$ are points corresponding to zeros or poles of $W(z, b)$ inside Γ. It follows that

$$T = |k_1| + |k_2| + \ldots + |k_m|,$$

and so T', the number of zeros and poles of $w(z, b)$ inside Γ, is not greater than T. Reversing the argument, we see that T is not greater than T', and so $T = T'$, and $w(z, b) = W(z, b)$.

4. *Proof of Theorem* 2. Since $g(z)$ contains all the zeros and poles of $w(z, a)$ to at least as high order as $w(z, a)$, and $g(z)$ is an integral function, so is $w(z, a)$. Hence the function

$$G_1(z) = g(z)/w(z, a)$$

is also an integral function belonging to C_i, or a constant; and repeating the argument, we see that there is a positive integer μ such that

$$G_\mu(z) = g\{(z)/(w(z, a)\}^\mu$$

is either an integral function belonging to C_i without a zero at $z = a$, or else a constant whose modulus is one. In the latter case

$$g(z) = e^{i\lambda}\{w(z, a)\}^\mu,$$

where λ is real.

If $G_\mu(z)$ belongs to C_i, it has at least one zero at $z = b$, say, inside Γ. It follows that $w(z, b)$ is also an integral function different from $w(z, a)$ which implies that $w(z, a)$ does not take the value $1/\overline{w(b, a)}$. As before we can choose a positive integer ν such that

$$F_\nu(z) = G_\mu(z)/\{w(z, b)\}^\nu$$

is either a function of C_i which has no zero at $z = b$ or a constant. But now if $F_\nu(z)$ is an integral function it has a zero at some point $z = c$, and $w(z, c)$ is an integral function different from $w(z, a)$ and $w(z, b)$, and a reference to (3′) shows that this implies that $w(b, a)$ is not equal to $w(c, a)$. It follows that $w(z, a)$ omits both the values $1/\overline{w(b, a)}$ and $1/\overline{w(c, a)}$ which is impossible, and so $F_\nu(z)$ is a constant. Since $|g| = 1$ on Γ, we have

$$g(z) = e^{i\lambda}\{w(z, a)\}^\mu \{w(z, b)\}^\nu,$$

where λ is real, and μ and ν are integers.

Let $f(z)$ be any other member of C_i with a zero at $z = a'$, and suppose that $w(z, a)$ takes every value at least once. Then either $w(z, a') \equiv w(z, a)$ or $w(z, a')$ is not an integral function, which contradicts the first part of the theorem. It follows as above that $w(z, a') = w(z, a)$ and that (4) holds. If

$w(z, a)$ omits the value $w_0 = 1/\overline{w(b, a)}$. $w(z, b)$ is also an integral function, and the zeros of $f(z)$ may coincide with those of $w(z, b)$. Hence we see that (5) holds.

The same method gives Theorem 3. Since $f(z)$ may now have poles, we can admit any function of the form $\{w(z, a_\nu)\}^{h_\nu}$, where h_ν is a positive or negative integer as a factor of $f(z)$.

5. *Asymptotic values.* If $f(z)$ belongs to C_m and has a finite asymptotic value f_0 associated with a path of determination γ, it follows from (8) that ζ tends to a finite point ζ_0 as $z \to \infty$ along γ. Since any other function $F(z)$ belonging to C_m can be expressed in the form

$$F(z) = \prod_{\nu=1}^{m'} \left(\frac{\zeta - \beta_\nu'}{1 - \overline{\beta_\nu'}\,\zeta} \right)^{k_\nu'},$$

it follows that $F(z) \to F_0$ (possibly ∞) as $z \to \infty$ along γ, if $|f_0| = 1$, $|F_0| = 1$. If $|f_0| \neq 1$, suppose that $f(z)$ and $F(z)$ have no poles inside Γ. Then if $|f_0| < 1$, $|\zeta_0| < 1$, and so $|F_0| < 1$. But if $|f_0| > 1$, $|\zeta_0| > 1$, and $|F_0| > 1$. In the latter case F_0 is finite or infinite according as ζ_0 is not or is equal to one of the numbers $1/\overline{\beta_\nu'}$, $\nu = 1, 2, \ldots, m'$. Summing up, we have

THEOREM 4. *Let $f(z)$ and $F(z)$ be any two functions belonging to C_m, and suppose that $f(z)$ has a finite asymptotic value f_0, approached as $z \to \infty$ along a path γ. Then $F(z) \to F_0$ (possibly ∞) as $z \to \infty$ along γ, and if $|f_0| = 1$, $|F_0| = 1$. If, in addition, $f(z)$ and $F(z)$ have no poles inside Γ, then $|F_0| \gtrless 1$ according as $|f_0| \gtrless 1$.*

It may be observed that there are branches of level curves which form simple closed curves which contain both zeros and poles. For example, if K is large and positive we can find a simple closed curve on which $|z \tan z| = K$ and $|z| > |z_0(K)|$. This curve obviously includes both poles and zeros of $z \tan z$.

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ON THE ABSOLUTE SUMMABILITY OF FOURIER SERIES

By J. M. HYSLOP.

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1. It has been shown* by Zygmund that, if in the range $(0, 2\pi)$ the periodic function $f(x)$ is of bounded variation and belongs to Lip α† where $0 < \alpha < 1$, the Fourier series of $f(x)$ is absolutely convergent for all values of x. More recently Bosanquet‡ has proved that, if

$$\phi(t) = \tfrac{1}{2}\{f(x+t)+f(x-t)-2f(x)\}$$

is of bounded variation in the range $0 \leqslant t \leqslant 2\pi$, the Fourier series of $f(t)$ is absolutely summable (C, δ), $\delta > 0$, at the point $t = x$.

These theorems naturally suggest the problem of the behaviour of the Fourier series of $f(x)$ when $f(x)$ merely satisfies a Lipschitz condition. One answer to the question is provided by the well-known theorem§ of Bernstein, to the effect that, if $f(x)$ belongs to Lip α, the Fourier series of $f(x)$ converges absolutely for all values of x when $\alpha > \frac{1}{2}$, but not necessarily when $\alpha \leqslant \frac{1}{2}$. In this paper I propose to extend Bernstein's theorem by showing that, if $f(x)$ belongs to Lip α, where $0 < \alpha \leqslant \frac{1}{2}$, the Fourier series of $f(x)$ is absolutely summable (C, δ) for all values of x if $\delta > \frac{1}{2} - \alpha$.

I begin by recalling the definition‖ of absolute Cesàro summability¶.

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* Zygmund, **7** and **8**, 136.

† The function $f(x)$ is said to belong to the Lipschitz class of order α, or more briefly to Lip α, for $a \leqslant x \leqslant b$, if

$$|f(x+h)-f(x)| = O(|h^\alpha|),$$

uniformly for $a \leqslant x \leqslant b$.

‡ Bosanquet, **2**, **3**.

§ Bernstein, **1**.

‖ Kogbetliantz, **6**.

¶ See Bosanquet and Hyslop, **4**, for a comprehensive list of references on absolute summability.

If $s_n^{(\delta)}$ is the n-th Cesàro mean of order δ for the series Σu_n, that is, if

$$s_n^{(\delta)} = \frac{\sum_{\nu=0}^{n} E_{n-\nu}^{(\delta)} u_\nu}{E_n^{(\delta)}} = \frac{\sum_{\nu=0}^{n} E_{n-\nu}^{(\delta-1)} U_\nu}{E_n^{(\delta)}},$$

where $U_n = \sum_{\nu=0}^{n} u_\nu$, and $E_\nu^{(\delta)}$ is defined formally by the relation

$$\sum_{\nu=0}^{\infty} E_\nu^{(\delta)} x^\nu = (1-x)^{-\delta-1},$$

the series Σu_n, or the sequence U_n, is said to be absolutely summable (C, δ), or summable $|C, \delta|$, when the series

$$\Sigma |s_n^{(\delta)} - s_{n-1}^{(\delta)}|$$

is convergent.

The two lemmas* which follow are of great importance.

LEMMA 1. *If the series Σu_n is summable $|C, \delta|$ it is also summable $|C, \delta'|$ for $\delta' > \delta \geqslant 0$.*

LEMMA 2. *If $\tau_n^{(\delta)}$ denotes the n-th Cesàro mean of order δ for the sequence nu_n, then*

$$\tau_n^{(\delta)} = n\{s_n^{(\delta)} - s_{n-1}^{(\delta)}\}.$$

It follows at once from Lemma 2 that a necessary and sufficient condition that the series Σu_n should be summable $|C, \delta|$ is that the series $\Sigma n^{-1} |\tau_n^{(\delta)}|$ should be convergent.

2. THEOREM 1. *If the function $f(x)$ is periodic and belongs to* Lip α, *where $0 < \alpha \leqslant \frac{1}{2}$, then the Fourier series of $f(x)$ is summable $|C, \delta|$ for all values of x when $\delta > \frac{1}{2} - \alpha$.*

By Lemma 1, we may suppose throughout that $0 < \delta < \frac{1}{2}$.

If the Fourier series of $f(x)$ is

$$\tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx),$$

then that of $\frac{1}{2}\{f(x+t)+f(x-t)\}$ is

$$\tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \cos nt.$$

* Kogbetliantz, 6.

Hence

$$u_n = a_n \cos nx + b_n \sin nx = \frac{2}{\pi}\int_0^\pi \phi(t)\cos nt\,dt,$$

and

$$nu_n = \frac{2}{\pi}\int_0^\pi \phi(t)\frac{d}{dt}\sin nt\,dt.$$

It follows that

$$\tau_n^{(\delta)} = \frac{2}{\pi}\int_0^\pi \phi(t)\frac{d}{dt}\left\{\frac{1}{E_n^{(\delta)}}\sum_{k=0}^{n} E_k^{(\delta-1)}\sin(n-k)t\right\}dt,$$

and, since $E_n^{(\delta)} \sim \dfrac{n^\delta}{\Gamma(\delta+1)}$, we have*

$$\sum_{n=1}^{\infty} n^{-1}|\tau_n^{(\delta)}| < A\sum_{n=1}^{\infty} n^{-1-\delta}\left|\int_0^\pi \phi(t)\left\{\sum_{k=0}^{n}(n-k)E_k^{(\delta-1)}\cos(n-k)t\right\}dt\right|$$

$$\leqslant A\left\{\sum_{n=1}^{\infty} S_1(n) + \sum_{n=1}^{\infty} S_2(n)\right\},$$

where

$$S_1(n) = n^{-\delta}\left|\int_0^\pi \phi(t)\left\{\sum_{k=0}^{\infty} E_k^{(\delta-1)}\cos(n-k)t\right\}dt\right|,$$

$$S_2(n) = n^{-1-\delta}\left|\int_0^\pi \phi(t)\left\{\sum_{k=n+1}^{\infty} nE_k^{(\delta-1)}\cos(n-k)t + \sum_{k=0}^{n} kE_k^{(\delta-1)}\cos(n-k)t\right\}dt\right|.$$

We now write

$$S_2(n) \leqslant S_{2,1}(n) + S_{2,2}(n),$$

where the integral in $S_{2,1}(n)$ extends over $(0, n^{-1})$ and that in $S_{2,2}(n)$ over (n^{-1}, π). Since $E_k^{(\delta-1)}$ steadily decreases and $kE_k^{(\delta-1)}$ steadily increases as k increases, the absolute value of each of the sums in $S_{2,1}(n)$ is less than $An^\delta t^{-1}$. Hence

$$\sum_{n=1}^{\infty} S_{2,1}(n) < A\sum_{n=1}^{\infty} n^{-1-\delta}\int_0^{n^{-1}} n^\delta t^{-1}|\phi(t)|\,dt$$

$$< A\sum_{n=1}^{\infty} n^{-1}\int_0^{n^{-1}} t^{\alpha-1}dt$$

$$< A\sum_{n=1}^{\infty} n^{-1-\alpha} < A.$$

* Here A denotes a positive constant and has not necessarily the same value each time it occurs.

We must now show that $S_{2,2}$ behaves in a similar way. We have

$$S_{2,2}(n) = n^{-1-\delta}\Big|\int_{n-1}^{\pi}\phi(t)(2\sin\tfrac{1}{2}t)^{-1} \times\Big[\sum_{k=n+1}^{\infty} nE_k^{(\delta-1)}\{\sin(k-n+\tfrac{1}{2})t-\sin(k-n-\tfrac{1}{2})t\} + \sum_{k=0}^{n} kE_k^{(\delta-1)}\{\sin(n-k+\tfrac{1}{2})t-\sin(n-k-\tfrac{1}{2})t\}\Big]dt\Big|,$$

and summation by parts gives

$$S_{2,2}(n) = n^{-1-\delta}\Big|\int_{n-1}^{\pi}\phi(t)(2\sin\tfrac{1}{2}t)^{-1} \times\Big[-nE_{n+1}^{(\delta-1)}\sin\tfrac{1}{2}t+n\sum_{k=n+1}^{\infty}\{E_k^{(\delta-1)}-E_{k+1}^{(\delta-1)}\}\sin(k-n+\tfrac{1}{2})t + nE_n^{(\delta-1)}\sin\tfrac{1}{2}t+\sum_{k=0}^{n-1}\{(k+1)E_{k+1}^{(\delta-1)}-kE_k^{(\delta-1)}\}\sin(n-k-\tfrac{1}{2})t\Big]dt$$

$$\leqslant S_{2,2,1}(n)+S_{2,2,2}(n)+S_{2,2,3}(n),$$

where

$$S_{2,2,1}(n) = n^{-\delta}\Big|\int_{n-1}^{\pi}\tfrac{1}{2}\phi(t)\{E_n^{(\delta-1)}-E_{n+1}^{(\delta-1)}\}dt\Big|,$$

$$S_{2,2,2}(n) = n^{-\delta}\Big|\int_{n-1}^{\pi}\phi(t)(2\sin\tfrac{1}{2}t)^{-1} \times\Big[\sum_{k=n+1}^{\infty}\{E_k^{(\delta-1)}-E_{k+1}^{(\delta-1)}\}\sin(k-n+\tfrac{1}{2})t\Big]dt\Big|,$$

$$S_{2,2,3}(n) = n^{-1-\delta}\Big|\int_{n-1}^{\pi}\phi(t)(2\sin\tfrac{1}{2}t)^{-1} \times\Big[\sum_{k=1}^{n}\{kE_k^{(\delta-1)}-(k-1)E_{k-1}^{(\delta-1)}\}\sin(n-k+\tfrac{1}{2})t\Big]dt\Big|.$$

Since $$E_k^{(\delta-1)}-E_{k+1}^{(\delta-1)} = O(k^{\delta-2}),$$

we have $$\sum_{n=1}^{\infty} S_{2,2,1}(n) < A\sum_{n=1}^{\infty} n^{-2} < A.$$

Since the expression $E_k^{(\delta-1)}-E_{k+1}^{(\delta-1)}$ steadily decreases as k increases, the absolute value of the sum in $S_{2,2,2}(n)$ is less than $An^{\delta-2}t^{-1}$. Hence

$$\sum_{n-1}^{\infty} S_{2,2,2}(n) < A\sum_{n=1}^{\infty} n^{-2}\int_{n^{-1}}^{\pi} t^{-2}|\phi(t)|\,dt$$

$$< A\sum_{n=1}^{\infty} n^{-2}\int_{n^{-1}}^{\pi} t^{\alpha-2}\,dt$$

$$< A\sum_{n=1}^{\infty} n^{-1-\alpha}+A\sum_{n=1}^{\infty} n^{-2} < A.$$

Using the relation

$$kE_k^{(\delta-1)}-(k-1)\,E_{k-1}^{(\delta-1)} = \delta E_{k-1}^{(\delta-1)},$$

we may write

$$S_{2,2,3}(n) \leqslant S_{2,2,3,1}(n)+S_{2,2,3,2}(n),$$

where

$$S_{2,2,3,1}(n) = \delta n^{-1-\delta}\left|\int_{n^{-1}}^{\pi}\phi(t)(2\sin\tfrac{1}{2}t)^{-1}\left\{\sum_{k=0}^{\infty} E_k^{(\delta-1)}\sin(n-k-\tfrac{1}{2})t\right\}dt\right|,$$

$$S_{2,2,3,2}(n) = \delta n^{-1-\delta}\left|\int_{n^{-1}}^{\pi}\phi(t)(2\sin\tfrac{1}{2}t)^{-1}\left\{\sum_{k=n}^{\infty} E_k^{(\delta-1)}\sin(n-k-\tfrac{1}{2})t\right\}dt\right|.$$

Arguing in much the same way as in the case of $S_{2,2,2}(n)$, we at once obtain

$$\sum_{n=1}^{\infty} S_{2,2,3,2}(n) < A.$$

Also

$$\left|\sum_{k=0}^{\infty} E_k^{(\delta-1)}\sin(n-k-\tfrac{1}{2})t\right| = \left|\mathfrak{I}\left\{\sum_{k=0}^{\infty} E_k^{(\delta-1)}e^{(n-\frac{1}{2})it}e^{-kit}\right\}\right|$$

$$= |\mathfrak{I}\{e^{(n-\frac{1}{2})it}(1-e^{-it})^{-\delta}\}|$$

$$\leqslant |(1-e^{-it})^{-\delta}| < At^{-\delta},$$

for $0<t<\pi$. It follows that

$$\sum_{n=1}^{\infty} S_{2,2,3,1}(n) < A\sum_{n=1}^{\infty} n^{-1-\delta}\int_{n^{-1}}^{\pi} t^{-1-\delta}|\phi(t)|\,dt$$

$$< A\sum_{n=1}^{\infty} n^{-1-\delta}\int_{n^{-1}}^{\pi} t^{\alpha-1-\delta}\,dt$$

$$< A\sum_{n=1}^{\infty} n^{-1-\alpha}+A\sum_{n=1}^{\infty} n^{-1-\delta} < A.$$

It has thus been proved that the series $\sum_{n=1}^{\infty} S_2(n)$ is convergent. We proceed to prove that the same is true for the series $\sum_{n=1}^{\infty} S_1(n)$.

We write

$$\int_0^\pi \phi(t) \left\{ \sum_{k=0}^{\infty} E_k^{(\delta-1)} \cos(n-k)t \right\} dt = \int_0^\pi \phi(t) p(t) \cos nt\, dt + \int_0^\pi \phi(t) q(t) \sin nt\, dt$$

$$= p_n + q_n,$$

where $$p(t) = \sum_{k=0}^{\infty} E_k^{(\delta-1)} \cos kt, \qquad q(t) = \sum_{k=0}^{\infty} E_k^{(\delta-1)} \sin kt.$$

Now $p(t)$ and $q(t)$ are continuous for $\eta \leqslant t \leqslant \pi$ and their absolute values when $0 < t \leqslant \eta$ are each less than $At^{-\delta}$. The constants p_n and q_n are thus the Fourier coefficients of an even and an odd function respectively, and each of these functions belongs to the Lebesgue class L^2. It follows from Parseval's theorem that

$$\sum_{n=1}^{\infty} {p_n}^2 \sin^2 nh < A \int_0^\pi |\phi(t+h)p(t+h) - \phi(t-h)p(t-h)|^2 dt$$

$$\leqslant 2A\{I_1(h) + I_2(h)\},$$

where $$I_1(h) = \int_0^\pi \{p(t+h)\}^2 \{\phi(t+h) - \phi(t-h)\}^2 dt,$$

$$I_2(h) = \int_0^\pi \{\phi(t-h)\}^2 \{p(t+h) - p(t-h)\}^2 dt.$$

Now, taking h to be positive, we have

$$\phi(t+h) - \phi(t-h) = \tfrac{1}{2}[\{f(x+t+h) - f(x+t-h)\} - \{f(x-t+h) - f(x-t-h)\}]$$

$$< Ah^\alpha,$$

and therefore $$I_1(h) < Ah^{2\alpha} \int_0^\pi t^{-2\delta} dt = O(h^{2\alpha}),$$

since $0 < \delta < \frac{1}{2}$. It should be noted that it is only in this part of the proof that we require to use the full hypothesis.

We now split up $I_2(h)$ into two parts $I_{2,1}(h)$ and $I_{2,2}(h)$, where

$$I_{2,1}(h) = \int_{-h}^{h} \{\phi(t)\}^2 \{p(t+2h) - p(t)\}^2 dt,$$

$$I_{2,2}(h) = \int_{h}^{\pi-h} \{\phi(t)\}^2 \{p(t+2h) - p(t)\}^2 dt.$$

We then have

$$I_{2,1}(h) \leqslant 2\int_{-h}^{h} \{\phi(t)\}^2 [\{p(t+2h)\}^2 + \{p(t)\}^2]\,dt$$

$$= O\left\{\int_{-h}^{h} t^{2\alpha}(t+2h)^{-2\delta}\,dt\right\} + O\left\{\int_{-h}^{h} t^{2\alpha-2\delta}\,dt\right\}$$

$$= O(h^{2\alpha-2\delta+1}),$$

and

$$I_{2,2}(h) = 4h^2\int_{h}^{\pi-h} \{\phi(t)\}^2 \{p'(t+2\theta h)\}^2\,dt \quad (0<\theta<1),$$

$$= O\left\{h^2\int_{h}^{\pi} t^{2\alpha}(\sin\tfrac{1}{2}t)^{-2\delta-2}\,dt\right\}$$

$$= O\left\{h^2\int_{h}^{\pi} t^{2\alpha-2\delta-2}\,dt\right\}$$

$$= O(h^{2\alpha-2\delta+1}) + O(h^2).$$

It follows that, when h tends to zero,

$$\sum_{n=1}^{\infty} p_n^{\,2}\sin^2 nh = O(h^{2\alpha}),$$

and it may be proved similarly that

$$\sum_{n=1}^{\infty} q_n^{\,2}\sin^2 nh = O(h^{2\alpha}).$$

Let $h = \pi/(2N)$. Then we obtain

$$\sum_{n=1}^{N} p_n^{\,2}\sin^2\frac{n\pi}{2N} = O(N^{-2\alpha}),$$

and, writing $N = 2^{\nu}$, we at once deduce that

$$\sum_{n=2^{\nu-1}+1}^{2^{\nu}} p_n^{\,2} = O(2^{-2\nu\alpha}).$$

Applying Schwarz's inequality, we have

$$\sum_{n=2^{\nu-1}+1}^{2^{\nu}} n^{-\delta}|\,p_n| \leqslant \left\{\sum_{n=2^{\nu-1}+1}^{2^{\nu}} p_n^{\,2}\right\}^{\frac{1}{2}} \left\{\sum_{n=2^{\nu-1}+1}^{2^{\nu}} n^{-2\delta}\right\}^{\frac{1}{2}}$$

$$= O\{2^{-\nu\alpha}\}\,O\{2^{\nu(\frac{1}{2}-\delta)}\}$$

$$= O\{2^{-\nu(\alpha+\delta-\frac{1}{2})}\}.$$

A similar relation also holds in the case of q_n.

It now follows that

$$\sum_{n=1}^{\infty} S_1(n) \leqslant \sum_{n=1}^{\infty} n^{-\delta}|p_n| + \sum_{n=1}^{\infty} n^{-\delta}|q_n|$$

$$= \sum_{\nu=1}^{\infty} \left\{ \sum_{n=2^{\nu-1}+1}^{2^{\nu}} (n^{-\delta}|p_n| + n^{-\delta}|q_n|) \right\}$$

$$< A \sum_{\nu=1}^{\infty} 2^{-\nu(\alpha+\delta-\frac{1}{2})} < A,$$

since $\delta > \frac{1}{2} - \alpha$.

3. It will now be shown that this theorem provides us with a "best possible" result.

LEMMA 3. *A necessary condition* that the series Σu_n should be summable $|C, \delta|$ is that the series $\Sigma n^{-\delta}|u_n|$ should be convergent.*

THEOREM 2. *There exists a function belonging to* Lip α, *where* $0 < \alpha < 1$, *whose Fourier series is not summable* $|C, \frac{1}{2} - \alpha|$ *for almost all values of* x.

It is known† that the series

$$\sum_{n=2}^{\infty} n^{-\frac{1}{2}-\alpha} \cos(n \log n + nx)$$

is the Fourier series of a function which belongs to Lip α. For this series, when $\delta = \frac{1}{2} - \alpha$,

$$\sum_{n=2}^{\infty} n^{-\delta}|u_n| = \sum_{n=2}^{\infty} n^{-1}|\cos(n \log n + nx)|,$$

which certainly diverges for almost all values of x, since its convergence in a set of positive measure would imply, by the Lusin-Denjoy theorem‡, the convergence of the series

$$\sum_{n=2}^{\infty} n^{-1}\{|\cos(n \log n)| + |\sin(n \log n)|\}.$$

The result therefore follows from Lemma 3.

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* Kogbetliantz, 6.

† Hardy and Littlewood, 5. See also Zygmund, 8, 116–119.

‡ Zygmund, 8, 131.

[*Added November* 12, 1937]. Dr. L. S. Bosanquet has written to point out to me that the word "almost" may be deleted from the conclusion of Theorem 2. Although the theorem as stated was quite sufficient for my purposes, the more general theorem is of considerable interest. I here reproduce his simple proof, the idea of which is taken from a paper by him and H. Kestelman.

It is known (*cf.* the references given above) that, when $0 < \alpha < 1$ and $a > 0$, the series

$$\sum_{n=2}^{\infty} n^{-\frac{1}{2}-\alpha} \cos(an \log n + nx) \tag{1}$$

converges uniformly in $0 \leqslant x \leqslant 2\pi$ to a function which belongs to Lip α. For this series, when $\delta = \frac{1}{2} - \alpha$,

$$\begin{aligned}
\sum_{n=2}^{\infty} n^{-\delta} |u_n| &= \sum_{n=2}^{\infty} n^{-1} |\cos(an \log n + nx)| \\
&\geqslant \sum_{n=2}^{\infty} n^{-1} \cos^2(an \log n + nx) \\
&= \tfrac{1}{2} \sum_{n=2}^{\infty} n^{-1} \{1 + \cos(2an \log n + 2nx)\} \\
&\geqslant \tfrac{1}{2} \sum_{n=2}^{\infty} n^{-\alpha - \frac{1}{2}} \left| \sum_{n=2}^{\infty} n^{-1} \cos(2an \log n + 2nx) \right| = \infty,
\end{aligned}$$

since the last series is (1) with $\alpha = \frac{1}{2}$, a replaced by $2a$, and x replaced by $2x$; and so it converges for every value of x.

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NOTE ON THE ABSOLUTE CESÀRO SUMMABILITY OF POWER SERIES

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1. Let $$f(z) = \sum_{n=0}^{\infty} c_n z^n = \sum_{n=0}^{\infty} c_n r^n e^{ni\theta}$$

be a power series regular for $|z| = r < 1$. We say that $f(z)$ belongs to the (complex) class Lip (k, p), where $p \geqslant 1$, $0 < k \leqslant 1$, if

$$M_p(r, f') \equiv \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f'(re^{i\theta})|^p \, d\theta\right)^{1/p} = O\left((1-r)^{-1+k}\right)$$

as $r \to 1-0$. It was proved by Hardy and Littlewood* that, *if* $1 < p \leqslant 2$, $kp > 1$, *and* $f(z)$ *belongs to* Lip (k, p), *the series* $\Sigma c_n e^{ni\theta}$ *is absolutely convergent.*

Hardy and Littlewood's theorem is still valid if $p > 2$ and $k > \frac{1}{2}$. For any function belonging to Lip (k, p), for $p > 2$, belongs to Lip $(k, 2)$. On the other hand, this theorem ceases to be true (i) if $p > 2$ and $\frac{1}{2} \geqslant k > 1/p$, or (ii) if $p > 1$ and $0 < kp \leqslant 1$. Hardy and Littlewood justified (i) by giving the function

$$f(z) = \sum_{n=1}^{\infty} n^{-\frac{1}{2}-k} e^{ai n \log n} z^n \quad (k > 0, \ a > 0), \tag{1}$$

which belongs to Lip (k, p) for all p. But the series $\Sigma n^{-\frac{1}{2}-k}$ is divergent for $k \leqslant \frac{1}{2}$. To justify (ii), we may take the function

$$f(z) = \log \frac{1}{1-z} = \sum_{n=1}^{\infty} \frac{z^n}{n}, \tag{2}$$

* Hardy and Littlewood, *Math. Zeitschrift*, 28 (1928), 612–634.

which belongs to $\mathrm{Lip}\,(1/p,\ p)$ for every $p>1$. But the series Σn^{-1} is divergent.

2. The object of this note is to study the absolute Cesàro summability of the series $\Sigma c_n e^{ni\theta}$ in these circumstances. We denote by $s_n^{\alpha}(\theta)$ $(\alpha \geqslant 0)$ the n-th Cesàro mean of order α of the sequence $\sum_{\nu=0}^{n} c_\nu e^{\nu i\theta}$, and say that the series $\Sigma c_n e^{ni\theta}$ is *absolutely summable* (C, α), or *summable* $|C, \alpha|$, if the series $\Sigma\{s_n^{\alpha}(\theta)-s_{n-1}^{\alpha}(\theta)\}$ is absolutely convergent. In fact, we shall prove the following theorems.

THEOREM 1. *If $p>2$, $\frac{1}{2} \geqslant k > 1/p$, and $f(z)$ belongs to* $\mathrm{Lip}\,(k, p)$, *then, for all values of θ, the series $\Sigma c_n e^{ni\theta}$ is summable $|C, \alpha|$ for every $\alpha > \frac{1}{2}-k$.*

THEOREM 2. *If $p>1$, $0<kp\leqslant 1$, and $f(z)$ belongs to* $\mathrm{Lip}\,(k, p)$, *and if θ satisfies*

$$\int_0^t |f'(re^{i\theta+i\phi})|^p\, d\phi = O\left(\frac{|t|}{(1-r)^{p(1-k)}}\right) \tag{3}$$

for $0<1-r\leqslant|t|\leqslant\pi$, then the series $\Sigma c_n e^{ni\theta}$ is summable $|C, \alpha|$ for every $\alpha > (1/p)-k$ or for every $\alpha > \frac{1}{2}-k$ according as $p\leqslant 2$ or $p>2$.*

3. Before proving these theorems, it is convenient to state some known results as lemmas. Lemma 5 is simply another form of the theorem of Hardy and Littlewood quoted in § 1.

LEMMA 1†. *Let $\tau_n^{\alpha}(\theta)$ denote the n-th Cesàro mean of order α of the sequence $nc_n e^{ni\theta}$. Then*

$$\tau_n^{\alpha}(\theta) = n\{s_n^{\alpha}(\theta)-s_{n-1}^{\alpha}(\theta)\}.$$

LEMMA 2†. *If a series is summable $|C, \alpha|$, it is also summable $|C, \beta|$ for every $\beta \geqslant \alpha$.*

LEMMA 3†. *If the series $\Sigma n^{-1}\tau_n^{\alpha}(\theta)$ is summable $|C, \beta|$, the series $\Sigma c_n e^{ni\theta}$ is summable $|C, \alpha+\beta|$.*

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* The condition that $f(z)$ belongs to $\mathrm{Lip}\,(k, p)$ is also included in (3). Some condition such as (3) is necessary, as is shown by the function (2), since Σn^{-1} is not summable $|C|$.

† Kogbetliantz, *Bull. des Sci. Math.* (2), 49 (1925), 234–256.

LEMMA 4†. *Suppose* $F(z)=\sum_{n=0}^{\infty} b_n z^n$ *regular for* $|z|<1$. *Let* $\alpha>0$ *and*

$$F_\alpha(z)=\frac{1}{\Gamma(\alpha)}\int_0^z (z-u)^{\alpha-1}F(u)\,du = z^\alpha \sum_{n=0}^{\infty}\frac{\Gamma(n+1)}{\Gamma(n+1+\alpha)}\,b_n z^n,$$

where $(z-u)^{\alpha-1}$ *and* z^α *have their principal values and the path of integration is rectilinear. If* $p>0$, $\alpha<1-k$, *and* $M_p(r, F)=O\left((1-r)^{-1+k}\right)$, *then*

$$M_p(r, F_\alpha)=O\left((1-r)^{-1+k+\alpha}\right).$$

LEMMA 5. *If* $1<p\leqslant 2$, $kp>1$, *and* $M_p(r, F)=O\left((1-r)^{-1+k}\right)$, *then the series* $\Sigma n^{-1}b_n$ *is absolutely convergent.*

4. *Proof of Theorem* 1. By Lemma 1, it is plainly sufficient to show that the series $\Sigma n^{-1}\tau_n{}^\alpha(\theta)$ is absolutely convergent for every $\alpha>\frac{1}{2}-k$.

By Lemma 2, we can suppose, without loss of generality, that

$$\tfrac{1}{2}-k<\alpha<\min\left(\tfrac{1}{2}-\frac{1}{p},\ 1-k\right).$$

Let
$$F(z)=\frac{ze^{i\theta}f'(ze^{i\theta})}{(1-z)^\alpha}=\sum_{n=1}^{\infty}A_n{}^\alpha\tau_n{}^\alpha(\theta)z^n,$$

where
$$A_n{}^\alpha=\frac{\Gamma(n+1+\alpha)}{\Gamma(n+1)\,\Gamma(1+\alpha)}.$$

Then
$$F_\alpha(z)=\frac{z^\alpha}{\Gamma(1+\alpha)}\sum_{n=1}^{\infty}\tau_n{}^\alpha(\theta)z^n.$$

Since, by Hölder's inequality, we have

$$M_2{}^2(r, F)\leqslant\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{|f'(re^{i\theta+i\phi})|^2}{|1-re^{i\phi}|^{2\alpha}}\,d\phi$$

$$\leqslant\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|f'(re^{i\theta+i\phi})|^p\,d\phi\right)^{2/p}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{d\phi}{|1-re^{i\phi}|^{2\alpha p/(p-2)}}\right)^{(p-2)/p}$$

and $2\alpha p/(p-2)<1$, it follows that

$$M_2(r, F)\leqslant O(1)\,M_p(r, f')=O\left((1-r)^{-1+k}\right).$$

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* Hardy and Littlewood, *Math. Zeitschrift*, 34 (1931), 403–439.

Hence, by Lemma 4, we obtain

$$M_2(r, F_\alpha) = O\Big((1-r)^{-1+k+\alpha}\Big).$$

Since $\alpha > \frac{1}{2}-k$ or $k+\alpha > \frac{1}{2}$, it follows from Lemma 5 that the series $\Sigma n^{-1}\tau_n{}^\alpha(\theta)$ is absolutely convergent.

5. Let $u(\theta)$ be a real function which is integrable in the sense of Lebesgue and periodic with period 2π. Then the following theorem is a direct corollary of Theorem 1.

THEOREM 3. *If $p > 2$, $\frac{1}{2} \geqslant k > 1/p$, and $u(\theta)$ belongs to* Lip (k, p), *i.e. $u(\theta)$ satisfies*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |u(\theta+h)-u(\theta)|^p\, d\theta = O(h^{kp})$$

as $h \to 0$, then, for all values of θ, the Fourier series of $u(\theta)$ is summable $|C, \alpha|$ for every $\alpha > \frac{1}{2}-k$.*

For, suppose that the Fourier series of $u(\theta)$ is

$$\tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta).$$

Let
$$c_0 = \tfrac{1}{2}a_0, \quad c_n = a_n - ib_n,$$

and
$$f(z) = f(re^{i\theta}) = \sum_{n=0}^{\infty} c_n r^n e^{ni\theta}.$$

Then $f(z)$ is regular for $|z| = r < 1$. If $u(\theta)$ belongs to Lip (k, p), $f(z)$ belongs to the complex class Lip (k, p)†. Hence, for all values of θ, the series $\Sigma c_n e^{ni\theta}$, and therefore the Fourier series of $u(\theta)$, is summable $|C, \alpha|$ for every $\alpha > \frac{1}{2}-k$.

6. *Proof of Theorem* 2. Suppose first that $1 < p \leqslant 2$. Let

$$0 < \alpha < \min(1/p,\ 1-k).$$

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* This result has also been obtained, for the class Lip k, by Dr. J. M. Hyslop, who shows that it is false for $\alpha = \frac{1}{2}-k$, by giving the example $u(\theta) = \Sigma n^{-\frac{1}{2}-k}\cos(n\log n + n\theta)$, which belongs to Lip k, for $k > 0$ [and so to Lip (k, p) for every p], but is not summable $|C, \frac{1}{2}-k|$, if $0 < k \leqslant \frac{1}{2}$. In the same way, the function (1) shows that Theorem 1 is false with $\alpha = \frac{1}{2}-k$. For, if the series (1) is summable $|C, \alpha|$, it follows that $\Sigma n^{-\frac{1}{2}-k-\alpha} e^{ai n\log n} z^n$ is summable $|C, 0|$, *i.e.* $\Sigma n^{-\frac{1}{2}-k-\alpha}|z^n|$ is convergent, and this is false if $|z| = 1$, $\alpha = \frac{1}{2}-k$.

† Hardy and Littlewood, *Math. Zeitschrift*, 28 (1928), 612–634.

Let $F(z)$ and $F_a(z)$ have the same meanings as they have in the proof of Theorem 1. Then

$$M_p{}^p(r, F) \leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{|f'(re^{i\theta+i\phi})|^p}{|1-re^{i\theta}|^{\alpha p}}\, d\phi \leqslant K\int_{-\pi}^{\pi} \frac{|f'(re^{i\theta+i\phi})|^p}{\{(1-r)^2+\phi^2\}^{\frac{1}{2}\alpha p}}\, d\phi,$$

where K is independent of r. Let

$$G(r, t) = \int_0^t |f'(re^{i\theta+i\phi})|^p\, d\phi.$$

Integrating by parts, we have

$$M_p{}^p(r, F) \leqslant K\left[\frac{G(r, \phi)}{\{(1-r)^2+\phi^2\}^{\frac{1}{2}\alpha p}}\right]_{-\pi}^{\pi} + K\int_{-\pi}^{\pi} \frac{\phi G(r, \phi)\, d\phi}{\{(1-r)^2+\phi^2\}^{\frac{1}{2}\alpha p+1}}$$

$$\leqslant K(1-r)^{-(1-k)p} + K\int_{-\pi}^{\pi} \frac{\phi G(r, \phi)\, d\phi}{\{(1-r)^2+\phi^2\}^{\frac{1}{2}\alpha p+1}}.$$

Write $$I = \int_0^{\pi} \frac{\phi G(r, \phi)\, d\phi}{\{(1-r)^2+\phi^2\}^{\frac{1}{2}\alpha p+1}} = \int_0^{1-r} + \int_{1-r}^{\pi} = I_1 + I_2.$$

Since

$$I_1 \leqslant G(r, 1-r)\int_0^{1-r} \frac{\phi\, d\phi}{(1-r)^{\alpha p+2}} \leqslant K(1-r)^{-(1-k)p+1-\alpha p} \leqslant K(1-r)^{-(1-k)p}$$

and

$$I_2 \leqslant K(1-r)^{-(1-k)p}\int_{1-r}^{\pi} \phi^{-\alpha p}\, d\phi \leqslant K(1-r)^{-(1-k)p}\{K+K(1-r)^{1-\alpha p}\}$$

$$\leqslant K(1-r)^{-(1-k)p}$$

for $1-\alpha p > 0$, we have $I = O\left((1-r)^{-(1-k)p}\right)$. Similarly, we can show that

$$\int_{-\pi}^{0} \frac{\phi G(r, \phi)\, d\phi}{\{(1-r)^2+\phi^2\}^{\frac{1}{2}\alpha p+1}} = O\left((1-r)^{-(1-k)p}\right).$$

Hence $$M_p(r, F) = O\left((1-r)^{-1+k}\right)$$

and, by Lemma 4, we have

$$M_p(r, F_a) = O\left((1-r)^{-1+k+\alpha}\right).$$

Hence, by Lemma 5, the series $\Sigma n^{-1}\tau_n{}^{\alpha}(\theta)$ is absolutely convergent for $\alpha+k>1/p$ or $\alpha>1/p-k$.

Now, suppose that $p>2$. Let $0<\alpha<\min(1/p, \frac{1}{2}-k)$. By the argument used above we can show that

$$M_p(r, F_\alpha)=O\left(1-r\right)^{-1+k+\alpha}\Big).$$

Hence, by Theorem 1, the series $\Sigma n^{-1}\tau_n{}^{\alpha}(\theta)$ is summable $|C, \beta|$ for $\beta>\frac{1}{2}-k-\alpha$. Hence, by Lemma 3, the series $\Sigma c_n e^{ni\theta}$ is summable $|C, \beta+\alpha|$, *i.e.* is summable $|C, \gamma|$, for $\gamma=\beta+\alpha>\frac{1}{2}-k$. This completes the proof.

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ÜBER EINE KLASSE VON ABELSCHEN GRUPPEN

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Es sei R ein Hauptidealring, d.h. ein Ring*, in dem jedes Ideal ein Hauptideal ist. Ein R-Modul $\mathfrak{R}$ ist eine additiv geschriebene Abelsche Gruppe, die R als Operatorenbereich besitzt. Ein solcher Modul heisst endlich, wenn seine Elemente linear in der Gestalt

$$a_1 u_1 + \ldots + a_n u_n, \quad a_1, \ldots, a_n \in R$$

durch endlich viele Elemente $u_1, \ldots, u_n$ dargestellt werden können. In diesem Fall besitzt er eine linear unabhängige Basis. Die Länge dieser Basis hängt nur vom Modul ab und heisst der Rang des Moduls.

Es sei p eine im folgende fest gehaltene Primzahl, und S eine Menge von Primzahlen, die p nicht enthält. Es sei I der Ring aller rationalen Zahlen, deren Nenner als Produkte von Primzahlen aus S darstellbar sind, und P der Ring aller rationalen Zahlen der Form a/p^m ($a \in I$, $m \geqslant 0$). Es sei F der rationale Zahlkörper, $\bar{P}$ der p-adische† Zahlkörper und $\bar{I}$ der Ring der ganzen p-adischen Zahlen. I und P sind Hauptidealringe.

Diese Untersuchung beschäftigt sich mit den Abelschen Gruppen, die I-Untermoduln eines endlichen P-Moduls sind. Wir werden alle diese Gruppen explizit aufstellen und jede davon durch Invarianten charakterisieren. Als Anwendungen der Theorie werden wir die Automorphismengruppen einer speziellen Klasse dieser Gruppen bestimmen und ein volles Invariantensystem der Untergruppen eines endlichen F-Moduls konstruieren.

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* Wegen dieses und anderer vorkommenden algebraischen Begriffe verweise ich auf das bekannte van der Waerdensche Buch *Moderne Algebra* insbesondere Kap. 3 und 15.

† Eine elementare Darstellung der Theorie der p-adischen Zahlen steht in C. Chevalley, "Sur la théorie du corps de classes", Chap. V, *Journal of the Faculty of Science, Tokyo*, 2, 9.

Es seien $u_1, \ldots, u_n$ Unbestimmte und $\mathfrak{P} = Pu_1 + \ldots + Pu_n$ der aus $u_1, \ldots, u_n$ konstruierte P-Modul. Dieser Modul lässt sich in den $\bar{P}$-Modul $\bar{\mathfrak{P}} = \bar{P}u_1 + \ldots + \bar{P}u_n$ einbetten. Unter einem P-Modul wird im folgenden stets ein Modul dieser Art verstanden.

Wir führen den Begriff der Konvergenz in einem $\bar{P}$-Modul

$$\bar{\mathfrak{P}} = \bar{P}u_1 + \ldots + \bar{P}u_n$$

ein.

Definition. Eine unendliche Folge

$$(u^\nu) = a_1^\nu u_1 + \ldots + a_n^\nu u_n \quad (\nu = 0, 1, 2, 3, \ldots)$$

von Elementen aus $\mathfrak{P}$ heisse konvergent zum Element $u = a_1 u_1 + \ldots + a_n u_n$, wenn für $1 \leqslant r \leqslant n$ jede der Folgen (a_r^ν) p-adisch gegen a_r konvergiert.

Geht man von der Basis $u_1, \ldots, u_n$ zu einer anderen Basis über, so verifiziert man leicht, dass die Folge (u^ν) auch in bezug auf diese Basis konvergiert, oder anderes gesagt: Die Konvergenz einer Folge ist von der Basiswahl unabhängig. Ist $\mathfrak{T}$ irgendeine Teilmenge des Moduls $\bar{\mathfrak{P}}$, so möge $\bar{\mathfrak{T}}$ die Abschliessung von $\mathfrak{T}$ in $\bar{\mathfrak{P}}$ durch Bildung aller p-adischen Grenzwerte bedeuten.

Es möge $\mathfrak{G}$ eine Gruppe bezeichnen, die I-Untermodul eines endlichen P-Moduls ist. Es sei $P\mathfrak{G}$ die Gruppe, die aus allen endlichen Summen $a_1 x_1 + \ldots + a_n x_n$ $(a_1, \ldots, a_n \in P,\ x_1, \ldots, x_n \in \mathfrak{G})$ besteht. Diese Gruppe ist offenbar der kleinste $\mathfrak{G}$-enthaltende P-Modul, und ist folglich ein endlicher P-Modul. In gleicher Weise sei $\bar{P}\mathfrak{G}$ die Gruppe, die aus allen endlichen Summen $c_1 x_1 + \ldots + c_n x_n$ $(c_1, \ldots, c_n \in \bar{P})$, besteht. Ist $P\mathfrak{G} = Pu_1 + \ldots + Pu_m$, so ist $\bar{P}\mathfrak{G} = \bar{P}u_1 + \ldots + \bar{P}u_m$. $u_1, \ldots, u_m$ sind linear unabhängig in bezug auf $\bar{P}$.

Satz 1. *Eine abgeschlossene Untergruppe $\mathfrak{H}$ eines endlichen $\bar{P}$-Moduls hat die Struktur*

$$\bar{P}v_1 + \ldots + \bar{P}v_k + \bar{I}v_{k+1} + \ldots + \bar{I}v_n,$$

wo $v_1, \ldots, v_n$ linear unabhängig in bezug auf $\bar{P}$ sind. k und n sind invariant gegenüber Isomorphie.

Beweis. Es sei x ein Element aus $\mathfrak{H}$. Für ganzrationale r ist $rx \in \mathfrak{H}$. Da $\mathfrak{H}$ abgeschlossen ist, folgt daraus: $\bar{I}x \in \mathfrak{H}$, d.h. $\mathfrak{H}$ ist ein $\bar{I}$-Modul.

(1) $\mathfrak{H}$ enthalte keinen $\bar{P}$-Modul. Da $\mathfrak{H}$ in einem endlichen $\bar{P}$-Modul liegt, gibt es n linear unabhängige Elemente $u_1, \ldots, u_n$ aus $\mathfrak{H}$ mit möglichst

grossem n. Offenbar ist

$$\mathfrak{H} \leqslant \bar{P}u_1 + \ldots + \bar{P}u_n.$$

Es sei $\mathfrak{H}_0 = \bar{I}u_1 + \ldots + \bar{I}u_n$ und $\mathfrak{H}_\nu = p^{-\nu}\mathfrak{H}_0 \cap \mathfrak{H}$ $(\nu = 0, 1, 2, 3, \ldots)$. Dabei ist $\mathfrak{H}_0 \leqslant \mathfrak{H}_1 \leqslant \mathfrak{H}_2 \leqslant \ldots$.

Angenommen es sei $\mathfrak{H}_0 < \mathfrak{H}_1 < \ldots < \mathfrak{H}_\nu < \ldots$. Dann lässt sich eine Folge x_ν $(\nu = 1, 2, 3, \ldots)$ aus $\mathfrak{H}_0$ bestimmen, so dass $p^{-\nu}x_\nu$ in $\mathfrak{H}_\nu$, aber nicht in $\mathfrak{H}_{\nu-1}$ liegt. Da $\mathfrak{H}_0$ eine kompakte Teilmenge von $\mathfrak{H}$ ist, konvergiert eine Teilfolge $x_{\nu'}$ gegen ein Element x von $\mathfrak{H}_0$. Sei $\nu_0 > 0$. Da $p^{\nu_0}\mathfrak{H}_0$ eine Umgebung (in bezug auf $\mathfrak{H}_0$) des Nullelements ist, gibt es eine Zahl ν_1, sodass für $\nu' > \nu_1$

$$x_{\nu'} - x \in p^{\nu_0}\mathfrak{H}_0, \quad \text{d.h.} \quad p^{-\nu_0}x_{\nu'} - p^{-\nu_0}x \in \mathfrak{H}_0.$$

Daraus folgt erstens $x \neq 0$, zweitens, dass $p^{-\nu_0}x$ in $\mathfrak{H}$ liegt. Mithin ist $\bar{I}p^{-\nu_0}x \in \mathfrak{H}$. Da ν_0 beliebig war, ist also $\bar{P}x \in \mathfrak{H}$, im Widerspruch zur Voraussetzung, dass $\mathfrak{H}$ keinen $\bar{P}$-Modul enthält.

Es gibt daher eine Zahl ν_2 mit $\mathfrak{H}_{\nu_2} = \mathfrak{H}_{\nu_2+1}$. Diese Gleichung bedeutet nichts anderes als die Tatsache, dass $\mathfrak{H}$ keine Glieder von $p^{-\nu_2-1}\mathfrak{H}_0$ mit genauem Nenner p^{ν_2+1} enthält. Hieraus folgt, dass $\mathfrak{H}$ keine Glieder von $p^{-(\nu_2+\nu)}\mathfrak{H}_0$ $(\nu = 2, 3, 4, \ldots)$ mit genauem Nenner grösser als p^{ν_2} besitzt, oder, was dasselbe ist,

$$\mathfrak{H}_{\nu_2} = \mathfrak{H}_{\nu_2+1} = \mathfrak{H}_{\nu_2+2} = \ldots = \mathfrak{H}.$$

Da nun $\mathfrak{H}_0 \leqslant \mathfrak{H} \leqslant p^{-\nu_2}\mathfrak{H}_0$, ist $\mathfrak{H}$ endlicher $\bar{I}$-Modul vom Range n, und also in der Form $\mathfrak{H} = \bar{I}v_1 + \ldots + \bar{I}v_n$ darstellbar.

(2) $\mathfrak{H}$ enthalte den maximalen $\bar{P}$-Modul $\bar{P}v_1 + \ldots + \bar{P}v_k$.

Da $\bar{P}\mathfrak{H}$ der kleinste $\mathfrak{H}$-enthaltende $\bar{P}$-Modul ist, ist $\bar{P}\mathfrak{H}$ endlich und in der Form

$$\bar{P}\mathfrak{H} = \bar{P}v_1 + \ldots + \bar{P}v_k + \bar{P}u_{k+1} + \ldots + \bar{P}u_n$$

darstellbar.

Der Durschnitt $\mathfrak{Q} = \mathfrak{H} \cap \bar{P}u_{k+1} + \ldots + \bar{P}u_n$ enthält keinen $\bar{P}$-Modul und ist abgeschlossen. Daher ist nach (1)

$$\mathfrak{Q} = \bar{I}v_{k+1} + \ldots + \bar{I}v_n.$$

Es sei x ein Element aus $\mathfrak{H}$. Aus der Darstellung

$$x = s + t \quad (s \in \bar{P}v_1 + \ldots + \bar{P}v_k,\ t \in \bar{P}u_{k+1} + \ldots + \bar{P}u_n)$$

folgt $s \in \mathfrak{H}$, also $t \in \mathfrak{Q}$. Mithin ist

$$\mathfrak{H} \leqslant \bar{P}v_1 + \ldots + \bar{P}v_k + \mathfrak{Q}.$$

Trivialerweise ist

$$\mathfrak{H} \geqslant \bar{P}v_1 + \ldots + \bar{P}v_k + \mathfrak{Q},$$

und somit ist bewiesen:

$$\mathfrak{H} = \bar{P}v_1 + \ldots + \bar{P}v_k + \bar{I}v_{k+1} + \ldots + \bar{I}v_n.$$

Die Zahl n ist die grösste Anzahl von linearen unabhängigen Elementen aus $\mathfrak{H}$, und die Zahl k ist der Rang des maximalen $\bar{P}$-Moduls von $\mathfrak{H}$. Daraus folgt, dass beide der Zahlen n und k invariant gegenüber Isomorphie sind. Damit ist alles bewiesen.

Satz 2. *Ein I-Untermodul $\mathfrak{G}$ eines endlichen P-Moduls hat die Struktur*

$$\mathfrak{G} = Pu_1 + \ldots + Pu_n \cap \bar{P}v_1 + \ldots + \bar{P}v_k + \bar{I}v_{k+1} + \ldots + \bar{I}v_n,$$

wo $v_j = \sum_i m_{ij} u_i$, m_{ij} *aus* $\bar{P}$, $|m_{ij}| \neq 0$.

Beweis. Es sei $\bar{\mathfrak{G}}$ die Abschliessung von $\mathfrak{G}$ innerhalb des endlichen $\bar{P}$-Moduls $\bar{P}\mathfrak{G}$. Wir zeigen, dass $\mathfrak{G} = P\mathfrak{G} \cap \bar{\mathfrak{G}}$ ist. Sei dazu x ein Element aus $P\mathfrak{G} \cap \bar{\mathfrak{G}}$. x ist also als Limes einer Folge x_ν ($\nu = 0, 1, 2, 3, \ldots$) von Elementen aus $\mathfrak{G}$ darstellbar. Es sei $P\mathfrak{G} = Pu_1 + \ldots + Pu_n$. Da $\mathfrak{G} \leqslant P\mathfrak{G}$, gibt es die Darstellungen:

$$x = a_1 u_1 + \ldots + a_n u_n$$

$$x_\nu = a_1^{\nu} u_1 + \ldots + a_n^{\nu} u_n \quad (\nu = 0, 1, 2, 3, \ldots), \quad a_1, \ldots, a_n, a_1^{\nu}, \ldots, a_n^{\nu} \in P$$

Jedes Element aus $P\mathfrak{G}$ kann offenbar durch Multiplikation mit einer genügend hohen Potenz von p in ein Element aus $\mathfrak{G}$ verwandelt werden. Daher gibt es eine Zahl ν_0, sodass

$$p^{\nu_0} u_1, \ldots, p^{\nu_0} u_n \in \mathfrak{G}.$$

Da $p^{\nu_0} \bar{I}$ eine Umgebung der Null ist, gibt es eine Zahl ν_1 derart, dass für $\nu \geqslant \nu_1$

$$a_i^{\nu} - a_i \in p^{\nu_0} \bar{I}, \quad 1 \leqslant i \leqslant n.$$

Daher ist

$$a_i^{\nu} - a_1 \in p^{\nu_0} \bar{I} \cap P = p^{\nu_0} I, \quad 1 \leqslant i \leqslant n,$$

d.h. $$x_\nu - x \in Ip^{\nu_0} u_1 + \ldots + Ip^{\nu_0} u_n \leqslant \mathfrak{G} \quad \text{für} \quad \nu \geqslant \nu_1.$$

Also liegt auch x in $\mathfrak{G}$. Damit ist $P\mathfrak{G} \cap \bar{\mathfrak{G}} \leqslant \mathfrak{G}$ bewiesen. Andererseits ist $\mathfrak{G} \leqslant P\mathfrak{G} \cap \bar{\mathfrak{G}}$, und folglich ist $\mathfrak{G} = P\mathfrak{G} \cap \bar{\mathfrak{G}}$.

Nach Satz 1 lässt sich $\bar{\mathfrak{G}}$ in der Form

$$\bar{\mathfrak{G}} = \bar{P}v_1 + \ldots + \bar{P}v_k + \bar{I}v_{k+1} + \ldots + \bar{I}v_n$$

ausdrücken. Aus $\bar{\mathfrak{G}} \leqslant \bar{P}\bar{\mathfrak{G}} = \bar{P}u_1 + \ldots + \bar{P}u_n$ folgt, dass $v_1, \ldots, v_n$ linear von $u_1, \ldots, u_n$ abhängen mit Koeffizienten aus $\bar{P}$. Da $v_1, \ldots, v_n$ und $u_1, \ldots, u_n$ beide linear unabhängig in bezug auf $\bar{P}$ sind, ist $|m_{ij}| \neq 0$.

SATZ 3. *Für*

$$\mathfrak{G} = (Pu_1 + \ldots + Pu_n) \cap (\bar{P}v_1 + \ldots + \bar{P}v_k + \bar{I}v_{k+1} + \ldots + \bar{I}v_n) = U \cap \bar{V}$$

$$(v_j = \Sigma m_{ij} u_i, \quad |m_{ij}| \neq 0, \quad m_{ij} \text{ aus } \bar{P}),$$

ist $P\mathfrak{G} = U$ *und* $\bar{\mathfrak{G}} = \bar{V}$.

Beweis. Es sei (m'_{ij}) die zu (m_{ij}) inverse Matrix. Es gibt eine Potenz p^r, sodass $p^r m'_{ij} \in \bar{I}$ $(1 \leqslant i \leqslant n, \ 1 \leqslant j \leqslant n)$. Dann sind $p^r u_1, \ldots, p^r u_n \in \bar{V}$, und folglich sind $p^r u_1, \ldots, p^r u_n \in \mathfrak{G}$. Daraus folgt $U = p^r U \leqslant P\mathfrak{G}$. $P\mathfrak{G} \leqslant U$ ist klar, also ist $P\mathfrak{G} = U$.

Ferner ist $\bar{P}\mathfrak{G} = \bar{P}u_1 + \ldots + \bar{P}u_n$. Da $|m_{ij}| \neq 0$, folgt

$$\bar{P}\mathfrak{G} = \bar{P}v_1 + \ldots + \bar{P}v_n.$$

Daher ist $\bar{V}$ innerhalb $\bar{P}\mathfrak{G}$ abgeschlossen. Nun ist wie vorher $\bar{\mathfrak{G}}$ die Abschliessung von $\mathfrak{G}$ innerhalb $\bar{P}\mathfrak{G}$. Aus $\mathfrak{G} \leqslant \bar{V}$ folgt dann $\bar{\mathfrak{G}} \leqslant \bar{V}$. Es sei x ein Element aus $\bar{V}$. Da $\bar{U} \geqslant \bar{V}$, gibt es eine gegen x konvergierende Folge x_ν aus U. Da $\bar{V}$ offenbar eine Umgebung des Nullelements von $\bar{P}\mathfrak{G}$ ist, gibt es eine Zahl ν_0, so dass

$$x_\nu - x \in \bar{V} \quad \text{für} \quad \nu \geqslant \nu_0 \quad \text{gilt.}$$

Also ist $x_\nu \in \bar{V}$ für $\nu \geqslant \nu_0$, und folglich ist $x_\nu \in \bar{V} \cap U = \mathfrak{G}$ für $\nu \geqslant \nu_0$. Daher ist $x \in \bar{\mathfrak{G}}$. Daraus ergibt sich $\bar{\mathfrak{G}} = \bar{V}$, womit alles bewiesen ist.

SATZ 4. *Sind* $\mathfrak{G}$, $\mathfrak{G}_1$ *isomorphe I-Untermoduln von endlichen P-Moduln, so lässt sich die Isomorphie auf die zugehörigen* $\bar{P}$*-Moduln auf genau eine Weise fortsetzen. Dabei entsprechen sich* $P\mathfrak{G}$, $\bar{\mathfrak{G}}$ *und* $P\mathfrak{G}_1$, $\bar{\mathfrak{G}}_1$.

Beweis. Ist x ein Element von $\mathfrak{G}$, so bezeichnen wir das x entsprechende Element von $\mathfrak{G}_1$ mit x'. Der Isomorphismus $\mathfrak{G} \cong \mathfrak{G}_1$ lässt sich zu einem Isomorphismus $P\mathfrak{G} \cong P\mathfrak{G}_1$ fortsetzen, wenn man dem Element ax $(a \in P)$ von $P\mathfrak{G}$ das Element ax' von $P\mathfrak{G}_1$ zuordnet. Es sei nun $v_1, \ldots, v_n$ eine P-Basis von $P\mathfrak{G}$ und $v_1', \ldots, v_n'$ die ihr entsprechende Basis von $P\mathfrak{G}_1$. Da $v_1, \ldots, v_n$ eine $\bar{P}$-Basis von $\bar{P}\mathfrak{G}$ und $v_1', \ldots, v_n'$ eine $\bar{P}$-Basis von $\bar{P}\mathfrak{G}_1$ ist, lässt sich der Isomorphismus $P\mathfrak{G} \cong P\mathfrak{G}_1$ zu einem Isomorphismus

$\bar{P}\mathfrak{G} \simeq \bar{P}\mathfrak{G}_1$ fortsetzen, wenn man dem Element

$$c_1 v_1 + \ldots + c_n v_n \quad (c_1, \ldots, c_n \in \bar{P})$$

von $\bar{P}\mathfrak{G}$ das Element

$$c_1 v_1 + \ldots + c_n v_n$$

von $\bar{P}\mathfrak{G}_1$ zuordnet. Dabei entspricht einer konvergenten Folge (x_ν) $(\nu = 0, 1, 2, 3, \ldots)$ aus $\bar{P}\mathfrak{G}$ eine konvergente Folge (x_ν') aus $\bar{P}\mathfrak{G}_1$ und umgekehrt. Überdies wird der Limes einer Folge auf den Limes der ihr entsprechenden Folge abgebildet. Daher wird die Gruppe $\bar{\mathfrak{G}}$ auf die Gruppe $\bar{\mathfrak{G}}_1$ abgebildet, und folglich ist $\bar{\mathfrak{G}} \simeq \bar{\mathfrak{G}}_1$. Damit ist der Satz bewiesen.

Definition. Für jede natürliche Zahl n sei $\mathfrak{A}(n)$ die multiplikative Gruppe aller derjenigen n-reihigen Matrizes A über P, die die Eigenschaft haben, dass die Zuordnung

$$(u_1, \ldots, u_n) \to (u_1, \ldots, u_n) A$$

einen umkehrbaren Automorphismus des P-Moduls $Pu_1 + \ldots + Pu_n$ definiert, also: die Determinante $|A|$ ist eine Einheit von P, d.h. ist A bezüglich P unimodular.

Definition. Für jede ganze Zahl k, wobei $0 \leqq k \leqq n$ ist, sei $\mathfrak{B}(n, k)$ die multiplikative Gruppe aller n-reihigen Matrizes B über $\bar{P}$, die die Eigenschaft haben, dass die Zuordnung

$$(u_1, \ldots, u_n) \to (u_1, \ldots, u_n) B$$

einen umkehrbaren Automorphismus des $\bar{I}$-Moduls

$$\bar{P}u_1 + \ldots + \bar{P}u_k + \bar{I}u_{k+1} + \ldots + \bar{I}u_n$$

definiert.

Man kann leicht beweisen: Die Gruppe $\mathfrak{B}(n, k)$ besteht aus allen Matrizes der Form

$$B = \begin{pmatrix} B_1 & D_1 \\ 0 & C_1 \end{pmatrix},$$

wobei B_1 eine k-reihige reguläre Matrix über $\bar{P}$, C_1 eine $(n-k)$-reihige unimodulare Matrix über $\bar{I}$, und D_1 eine beliebige Matrix über $\bar{P}$ mit k Zeilen und $n-k$ Spalten ist.

Bezüglich der zwei Gruppen $\mathfrak{A}(n)$ und $\mathfrak{B}(n, k)$ führen wir jetzt eine Klasseneinteilung der Menge aller n-reihigen regulären Matrizes über $\bar{P}$ ein. Zwei solche Matrizes M_1, M_2 heissen äquivalent, wenn es eine Matrix A aus

$\mathfrak{A}(n)$ und eine Matrix B aus $\mathfrak{B}(n, k)$ gibt, für welche die Gleichung

$$M_1 = AM_2B$$

besteht. Diese Aequivalenz ist reflexiv, symmetrisch und transitiv, wie man leicht nachweist. Man kann daher in üblicher Weise die Menge aller n-reihigen regulären Matrizes in zu einander fremde Klassen von äquivalenten Matrizes einteilen. Die Klasse, die die Matrix M enthält besteht aus allen Matrizes AMB, wobei die Matrix A und die Matrix B, jede unabhängig von der anderen, die Gruppen $\mathfrak{A}(n)$ bzw. $\mathfrak{B}(n, k)$ durchlaufen.

Ist die Gruppe $\mathfrak{G}$ ein I-Untermodul eines endlichen P- Moduls, so ist die Gruppe $\bar{\mathfrak{G}}$ eindeutig durch $\mathfrak{G}$ bestimmt. Nach Satz 1 ist $\bar{\mathfrak{G}}$ in der Form

$$\bar{\mathfrak{G}} = \bar{P}v_1 + \ldots + \bar{P}v_k + \bar{I}v_{k+1} + \ldots + \bar{I}v_n$$

darstellbar, wobei die Zahlen n und k Invarianten von $\bar{\mathfrak{G}}$ sind. Ist nun $\mathfrak{G}'$ eine zu $\mathfrak{G}$ isomorphe Gruppe, so ist nach Satz 4 $\bar{\mathfrak{G}} \simeq \bar{\mathfrak{G}}'$. Hieraus folgt, dass die Zahlen n und k durch $\mathfrak{G}$ allein bestimmt sind und invariant gegenüber Isomorphie sind. Wir bezeichnen sie daher mit $n(\mathfrak{G})$ und $k(\mathfrak{G})$.

Durch die in Satz 2 angegebene Darstellung

$$\mathfrak{G} = (Pu_1 + \ldots + Pu_n) \cap (\bar{P}v_1 + \ldots + \bar{P}v_k + \bar{I}v_{k+1} + \ldots + \bar{I}v_n)$$

ist eine reguläre Matrix $M = (m_{ij})$ bestimmt. Es sei

$$\mathfrak{G} = (Pu_1' + \ldots + Pu_n') \cap (\bar{P}v_1' + \ldots + \bar{P}v_k' + \bar{I}v_{k+1}' + \ldots + \bar{I}v_n')$$

eine zweite Darstellung dieser Art und $M' = (m_{ij}')$ die dadurch bestimmte Matrix. Sind A und B die Matrizes, für die

$$(u_1, \ldots, u_n) = (u_1', \ldots, u_n')A$$

$$(v_1', \ldots, v_n') = (v_1, \ldots, v_n)B$$

gilt, so ist $AMB = M'$. Da nach Satz 3

$$P\mathfrak{G} = Pu_1 + \ldots + Pu_n = Pu_1' + \ldots + Pu_n'$$

und

$$\bar{\mathfrak{G}} = \bar{P}v_1 + \ldots + \bar{P}v_k + \bar{I}v_{k+1} + \ldots + \bar{I}v_n = \bar{P}v_1' + \ldots + \bar{P}v_k' + \bar{I}v_{k+1}' + \ldots + \bar{I}v_n',$$

folgt $A \in \mathfrak{A}(n)$, $B \in \mathfrak{B}(n, k)$ also $M \sim M'$. Die Matrix M liegt also in einer Klasse, die von der speziellen Wahl der Basen unabhängig ist. Diese Klasse ist daher durch die Gruppe $\mathfrak{G}$ allein bestimmt. Wir bezeichnen sie mit $\mathfrak{m}(\mathfrak{G})$.

Nun lässt sich nach Satz 4 der Isomorphismus $\mathfrak{G} \simeq \mathfrak{G}'$ zu einem Isomorphismus $\bar{P}\mathfrak{G} \simeq \bar{P}\mathfrak{G}'$ fortsetzen, wobei $P\mathfrak{G} \simeq P\mathfrak{G}'$, $\bar{\mathfrak{G}} \simeq \bar{\mathfrak{G}}'$. Sind

$u_1'', \ldots, u_n''$ und $v_1'', \ldots, v_n''$ die Basen, die den Basen $u_1, \ldots, u_n$ und $v_1, \ldots, v_n$ entsprechen, so ist

$$\mathfrak{G}' = Pu_1'' + \ldots + Pu_n'' \cap \bar{P}v_1'' + \ldots + \bar{P}v_k'' + \bar{I}v_{k+1}'' + \ldots + \bar{I}v_n''$$

und

$$(u_1'', \ldots, u_n'')M = (v_1'', \ldots, v_n''),$$

d.h. die Matrix M liegt in der Klasse $\mathfrak{m}(\mathfrak{G}')$, und somit $\mathfrak{m}(\mathfrak{G}') = \mathfrak{m}(\mathfrak{G})$. Also ist $\mathfrak{m}(\mathfrak{G})$ invariant gegenüber Isomorphie.

Ist nun umgekehrt M eine vorgegebene Matrix aus der Klasse $m(\mathfrak{G})$ der Gruppe $\mathfrak{G}$, so kann man durch passende Wahl der Basen von $P\mathfrak{G}$ und $\bar{\mathfrak{G}}$ stets eine Darstellung

$$\mathfrak{G} = (Pu_1 + \ldots + Pu_n) \cap (\bar{P}v_1 + \ldots + \bar{P}v_k + \bar{I}v_{k+1} + \ldots + \bar{I}v_n)$$

finden, deren zugehörige Matrix M ist. Es sei $\mathfrak{G}'$ eine zweite Gruppe mit $n(\mathfrak{G}) = n(\mathfrak{G}')$, $k(\mathfrak{G}) = k(\mathfrak{G}')$ und $\mathfrak{m}(\mathfrak{G}) = \mathfrak{m}(\mathfrak{G}')$. Da M zu $\mathfrak{m}(g')$ gehört, gibt es eine Darstellung

$$\mathfrak{G}' = (Pu_1' + \ldots + Pu_n') \cap (\bar{P}v_1' + \ldots + \bar{P}v_k' + \bar{I}v_{k+1}' + \ldots + \bar{I}v_n)$$

mit der zugehörigen Matrix M.

Die Zuordnung $u_1 \to u_1'$, $u_2 \to u_2'$, $\ldots$, $u_n \to u_n'$ definiert einen Isomorphismus von $\bar{P}\mathfrak{G}$ auf $\bar{P}\mathfrak{G}'$. Dabei entsprechen sich $v_1, \ldots, v_n$ und $v_1', \ldots, v'_n$. Daher ist mit Rücksicht auf Satz 3 $\bar{\mathfrak{G}} \cong \bar{\mathfrak{G}}'$. Folglich ist

$$P\mathfrak{G} \cap \bar{\mathfrak{G}} \cong P\mathfrak{G}' \cap \bar{\mathfrak{G}}',$$

d.h. $\mathfrak{G} \cong \mathfrak{G}'$. Damit haben wir bewiesen: Eine Gruppe $\mathfrak{G}$ ist durch das Invariantensystem $n(\mathfrak{G})$, $k(\mathfrak{G})$, $\mathfrak{m}(\mathfrak{G})$ vollständig charakterisiert.

Zu vorgegebenen Zahlen n und k, mit $0 \leqslant k \leqslant n$, und einer zu den Zahlen zugehörigen Klasse $\mathfrak{m}$ von Matrizes gibt es offenbar eine Gruppe $\mathfrak{G}$ mit $n(\mathfrak{G}) = n$, $k(\mathfrak{G}) = k$, und $\mathfrak{m}(\mathfrak{G}) = \mathfrak{m}$.

Wir fassen diese Ergebnisse zusammen in

Satz 5. *Für die I-Untermoduln der endlichen P-Moduln sind* $n(\mathfrak{G})$, $k(\mathfrak{G})$ *und die Klasse* $\mathfrak{m}(\mathfrak{G})$ *von* (m_{ij}) *ein volles Invariantensystem bezüglich Isomorphie. Zwischen diesen Invarianten bestehen die einzigen Relationen:*

$$0 \leqslant k \leqslant n \text{ (ganzzahlig)}, \quad |m_{ij}| \neq 0 \quad (m_{ij} \epsilon \bar{P},\ i, j = 1, \ldots n).$$

Satz 6. $\mathfrak{G}$ *ist ein endlicher I-Modul für* $k(\mathfrak{G}) = 0$ *und ein endlicher P-Modul für* $k(\mathfrak{G}) = n(\mathfrak{G})$.

Beweis. Im Falle $k(\mathfrak{G}) = 0$ ist nach Satz 2

$$\mathfrak{G} = (Pu_1 + \ldots + Pu_n) \cap (\bar{I}v_1 + \ldots + \bar{I}v_n).$$

Da $v_j = \sum_i m_{ij} u_i$ mit $m_{ij} \epsilon \bar{P}$, gibt es eine Potenz p^r, sodass

$$p^r m_{ij} \epsilon \bar{I}, \quad 1 \leqslant i \leqslant n, \quad 1 \leqslant j \leqslant n.$$

Daher ist $$\bar{I}v_1 + \ldots + \bar{I}v_n \leqslant p^{-r}(\bar{I}u_1 + \ldots + \bar{I}u_n).$$

Daraus folgt $$\mathfrak{G} \leqslant p^{-r}(Iu_1 + \ldots + Iu_n).$$

Da I ein Hauptidealring ist, ist daher $\mathfrak{G}$ ein endlicher I-Modul.
Im Falle $k(\mathfrak{G}) = n(\mathfrak{G})$ ist

$$\mathfrak{G} = (Pu_1 + \ldots + Pu_n) \cap (\bar{P}v_1 + \ldots + \bar{P}v_n).$$

Da $\bar{P}$ ein Körper ist, ist

$$\bar{P}u_1 + \ldots + \bar{P}u_n = \bar{P}v_1 + \ldots + \bar{P}v_n.$$

Folglich ist $\mathfrak{G} = Pu_1 + \ldots + Pu_n$, womit alles bewiesen ist.

Die Automorphismengruppen

Wir untersuchen jetzt die Automorphismengruppen der Gruppen $\mathfrak{G}$ im Falle, wo $n(\mathfrak{G}) = 2$ und die Menge S leer ist. Da $0 \leqslant k(\mathfrak{G}) \leqslant n(\mathfrak{G})$, gibt es nur die drei Möglichkeiten $k(\mathfrak{G}) = 0$, $k(\mathfrak{G}) = 1$, $k(\mathfrak{G}) = 2$ für die Zahl $k(\mathfrak{G})$. Nach Satz 6 sind die Fälle $k(\mathfrak{G}) = 0$ und $k(\mathfrak{G}) = 2$ völlig trivial; also setzen wir $k(\mathfrak{G}) = 1$ voraus. Wenn S leer ist, wird I der Ring der ganzen rationalen Zahlen und P der Ring derjenigen rationalen Zahlen, deren Nenner Potenzen von p sind.

SATZ 7. *Jede Gruppe $\mathfrak{G}$ lässt sich in der Form*

$$\mathfrak{G} = (Pu_1 + Pu_n) \cap \left(\bar{P}(u_1 + cu_2) + \bar{I}u_2\right)$$

darstellen, wo c eine passend gewählte p-adische Zahl ist.

Beweis. Es sei

$$M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$$

eine Matrix aus der Klasse $m(\mathfrak{G})$. Wenn man die Matrix M linksseitig mit einer Matrix aus $\mathfrak{A}(2)$ oder rechtsseitig mit einer Matrix aus $\mathfrak{B}(2, 1)$ multipliziert erhält man definitionsgemäss eine zu M äquivalente Matrix.

Durch linksseitige Multiplikation kann man die zwei Zeilen von M vertauschen oder eine davon mit irgendeiner Potenz von p multiplizieren. Durch rechtsseitige Multiplikation lässt sich irgendein Multiplum der ersten Spalte von der zweiten abziehen und jede Spalte mit einer p-adischen Einheit multiplizieren.

Da $|M| \neq 0$, folgt, dass eine der Zahlen m_{11}, m_{21} von Null verschieden ist. Durch Vertauschung der Zeilen erreicht man daher $m_{11} \neq 0$. Man zieht die mit m_{12}/m_{11} multiplizierte erste Spalte von der zweiten ab. Dabei wird $m_{12} = 0$, und folglich $m_{22} \neq 0$. Sei also $m_{22} = \epsilon p^k$, wo ϵ eine p-adische Einheit ist. Man multipliziert die zweite Zeile mit p^{-k} und die zweite Spalte mit ϵ^{-1}. Hierdurch wird $m_{22} = 1$. In gleicher Weise kann m_{11} zu 1 gemacht werden. Es gibt also eine zu M äquivalente Matrix der Form

$$\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}.$$

Die Gruppe $\mathfrak{G}$ lässt sich daher folgendermassen darstellen:

$$\mathfrak{G} = Pu_1 + Pu_2 \cap \bar{P}(u_1 + cu_2) + \bar{I}u_2.$$

Damit ist der Satz bewiesen.

Satz 8. *Ist für*

$$\mathfrak{G} = Pu_1 + Pu_2 \cap \bar{P}(u_1 + cu_2) + \bar{I}u_2$$

die Zahl c rational, so ist

$$\mathfrak{G} = Pw_1 + Iw_2.$$

Beweis. Es sei $c = m/n$ mit $(m, n) = 1$. Es gibt ganzrationale a und b mit $am + bn = 1$. Wir setzen $w_1 = nu_1 + mu_2$, $w_2 = au_1 - bu_2$. Dabei ist

$$Pu_1 + Pu_2 = Pw_1 + Pw_2$$

und

$$\bar{P}(u_1 + cu_2) + \bar{I}u_2 = \bar{P}w_1 + \bar{I}w_2.$$

Daraus folgt

$$\mathfrak{G} = Pw_1 + Iw_2,$$

womit der Satz bewiesen ist.

Im Falle, wo die Zahl c rational ist, ist also die Gruppe $\mathfrak{G}$ trivial. Wir schliessen diesen Fall aus und setzen im folgenden voraus, dass c irrational ist.

Wir brauchen nun das folgende zahlentheoretische Hilfsmittel. Es sei K ein quadratischer Unterkörper des p-adischen Körpers $\bar{P}$ und $w(\mu)$ die p-adische Bewertungsfunktion der Zahlen μ aus K. Da $w(\mu) \geqslant 0$ für ganzrationale μ, folgt unter Benutzung der Relationen

$$w(\mu_1 + \mu_2) \geqslant \min\big(w(\mu_1), w(\mu_2)\big)$$

$$w(\mu_1 \mu_2) = w(\mu_1) + w(\mu_2),$$

dass $w(\mu) \geqq 0$ für alle ganzen Zahlen μ aus K. Diejenigen ganzen Zahlen μ aus K, für die $w(\mu) > 0$ gilt, bilden ein Primideal $\mathfrak{p}$. Mithin ist $w(\mu) = 0$ für jede ganze Zahl μ, die zu $\mathfrak{p}$ prim ist. Ist nun μ eine ganze Zahl aus K mit der Idealdarstellung $(\mu) = \mathfrak{p}\mathfrak{A}$, $(\mathfrak{p}, \mathfrak{A}) = 1$, so ist offenbar $w(\mu)$ der kleinste positive Wert, den die Funktion $w(\mu)$ annehmen kann, d.h. $w(\mu) = 1$. Folglich ist für $(\mu) = \mathfrak{p}^r\mathfrak{A}$ ($\mathfrak{p}$ zu $\mathfrak{A}$ prim) $w(\mu) = r$. Die p-adische Bewertung von K wird also durch das Primideal $\mathfrak{p}$ vermittelt. Die Zahlen μ aus K, für die $(\mu) = \mathfrak{p}^r$, $r \gtreqless 0$ gilt, bilden eine multiplikative Gruppe, die durch K allein bestimmt ist. Wir bezeichnen sie daher mit $G(K)$.

Da jede quadratische Gleichung innerhalb des komplexen Zahlkörpers lösbar ist, folgt, dass es stets einen Körper K' von komplexen Zahlen gibt, der zu K isomorph ist. Folglich kann der Einheitensatz durch Isomorphie auf K übertragen werden. Im Falle, wo die Diskriminante von K positiv ist (d.h. wo K' reell ist), gibt es eine Einheit ϵ, durch die sämtliche Einheiten in der Form $\pm\epsilon^r$, $r \gtreqless 0$ darstellbar sind. Ist dagegen die Diskriminante von K negativ, also K' imaginär, so ist die Einheitengruppe von K von endlicher Ordnung.

Nach der elementaren Theorie gibt es ein Hauptideal $(\mu_1) = \mathfrak{p}^m$, $m > 0$. Ist die Zahl m möglichst klein gewählt werden, so erzeugt jede Zahl μ aus $G(K)$ ein Hauptideal $(\mu) = \mathfrak{p}^{rm}$, $r \gtreqless 0$. Da die Erzeugenden eines Hauptideals durch Multiplizieren mit Einheiten auseinander hervorgehen, lässt sich jede Zahl μ aus $G(K)$ in der Form

$$\mu_1^r \epsilon \quad \left(r \gtreqless 0, \quad \epsilon \text{ eine Einheit}\right)$$

darstellen. Aus den beiden Resultaten schliessen wir: Die Gruppe $G(K)$ enthält eine endliche Anzahl von Elementen endlicher Ordnung, und die Basiszahl der Elemente unendlicher Ordnung ist entweder eins oder zwei, je nachdem die Diskriminante von K negativ oder positiv ist.

Aus $w(p) = 1$ folgt $\mathfrak{p} | (p)$. Da die Primzahl p innerhalb K nach der Henselschen Theorie in zwei verschiedene Primideale zerfällt, ist $(p) = \mathfrak{p}\mathfrak{p}'$, $\mathfrak{p} \neq \mathfrak{p}'$.

SATZ 9. *Für*

$$\mathfrak{G} = Pu_1 + Pu_2 \cap \bar{P}(u_1 + cu_2) + \bar{I}u_2$$

sei K der Körper, der durch Adjunktion der Zahl c zum rationalen Zahlkörper entsteht. Ist K nicht quadratisch, so gibt es nur die zwei Automorphismen $x \to x$ und $x \to -x$ von $\mathfrak{G}$. Ist dagegen K quadratisch, so ist die Automorphis-

mengruppe von $\mathfrak{G}$ zu einer Untergruppe von $G(K)$ von endlichem Index isomorph.

Beweis. Wenn wir Satz 4 auf Automorphismen spezialisieren, erhalten wir: Jeder Automorphismus von $\mathfrak{G}$ lässt sich zu einem Automorphismus von $\bar{P}\mathfrak{G}$ fortsetzen, wobei $P\mathfrak{G}$ und $\bar{\mathfrak{G}}$ beide in sich selbst übergehen. Es sei T ein Automorphismus von $\mathfrak{G}$. Da nach Satz 3 $\bar{\mathfrak{G}} = \bar{P}(u_1+cu_2)+\bar{I}u_2$, ist offenbar

$$T\,\bar{P}(u_1+cu_2) = \bar{P}(u_1+cu_2).$$

Demnach ist

$$T(u_1+cu_2) = \mu(u_1+cu_2) \quad (\mu \in \bar{P}).$$

Es sei

$$Tu_1 = \alpha u_1+\beta u_2,$$

$$Tu_2 = \gamma u_1+\delta u_2.$$

Da T ein Automorphismus von $P\mathfrak{G} = Pu_1+Pu_2$ ist, sind

$$\alpha, \beta, \gamma, \delta \in P \quad \text{und} \quad \alpha\delta-\gamma\beta = \pm p^r, \quad r \gtreqless 0.$$

Durch Vergleich der Koeffizienten erhält man

$$\mu = \alpha+\gamma\,.\,c,$$

$$\mu\,.\,c = \beta+\delta\,.\,c.$$

Da c irrational ist, folgt, dass α, β, γ, δ durch Angabe von μ eindeutig bestimmt sind. Ist nun T_1 ein zweiter Automorphismus von $\mathfrak{G}$ und μ_1 die Zahl, für die

$$T_1(u_1+cu_2) = \mu_1(u_1+cu_2)$$

gilt, so folgt, dass

$$TT_1(u_1+cu_2) = \mu\,.\,\mu_1(u_1+cu_2).$$

Also ist die Automorphismengruppe von $\mathfrak{G}$ zu einer multiplikativen Gruppe von Zahlen aus K isomorph.

Es sei $\gamma = 0$. Hieraus folgt $\beta = 0$, $\alpha = \delta$ und somit ist $\alpha = \delta = \pm p^{r_1}$, $r_1 \gtreqless 0$. Die Zuordnung $u_1 \to \pm p^{r_1}u_1$, $u_2 \to \pm p^{r_1}u_2$ definiert einen Automorphismus von $\bar{\mathfrak{G}}$. Daher ist $r_1 = 0$, d.h. entweder $\alpha = \delta = +1$ oder $\alpha = \delta = -1$. In diesem Fall ist also T entweder der identische Automorphismus $x \to x$ oder der Automorphismus $x \to -x$.

Es sei $\gamma \neq 0$. Durch Elimination der Zahl c erhält man

$$\mu^2-(\alpha+\delta)\mu+\alpha\delta-\beta\gamma = 0.$$

Hieraus folgt, dass $c = \mu - \alpha/\gamma$ quadratisch ist. Folglich ist K quadratisch. Im Falle, wo K nicht quadratisch ist, folgt also $\gamma = 0$. Daher gibt es nur die zwei Automorphismen von $\mathfrak{G}$, wie wir eben bewiesen haben. Damit ist der erste Teil des Satzes bewiesen.

Nun sei K quadratisch. Da $\alpha, \beta, \gamma, \delta \epsilon P$, gibt es eine Potenz p^{r_2} mit $p^{r_2}\mu$ ganz. Aus $\alpha\delta - \beta\gamma = \pm p^r$ folgt, dass, für eine genügend hohe Potenz p^{r_3} auch $p^{r_3}\mu^{-1}$ ganz ist. Daher hat μ innerhalb K die Darstellung

$$(\mu) = \mathfrak{p}^{r_4}\mathfrak{p}'^{r_5}, \quad r_4, r_5 \gtrless 0.$$

Die Relationen

$$T(u_1 + cu_2) = \mu(u_1 + cu_2),$$

$$Tu_2 = \gamma(u_1 + cu_2) + (\delta - \gamma c)u_2$$

definieren einen Automorphismus von $\bar{\mathfrak{G}}$. Daher ist $\delta - \gamma c$ eine p-adische Einheit, und folglich ist das Ideal $\mathfrak{p}$ in der reduzierten Idealdarstellung von $\delta - \gamma c$ nicht enthalten. Nun ist $\delta - \gamma c$ die zu μ konjugierte Zahl, wie man leicht verifiziert. Mithin kommt das zu $\mathfrak{p}$ konjugierte Primideal $\mathfrak{p}'$ in der Idealdarstellung von μ nicht vor. Also ist $(\mu) = \mathfrak{p}^{r_4}$, d.h. μ gehört der Gruppe $G(K)$ an.

Wie gehen jetzt in umgekehrter Richtung von einer Zahl μ' von $G(K)$ aus. Genau wie vorher sind rationale Zahlen α', β', γ', δ' durch die Relationen

$$\mu' = \alpha' + \gamma' c,$$

$$\mu c = \beta' + \delta' c$$

eindeutig bestimmt. In entsprechender Weise ist $G(K)$ zu einer Gruppe von Automorphismen des Moduls $Fu_1 + Fu_2$ (F der rationale Zahlkorper) isomorph. Dabei entspricht der Zahl μ' einen Automorphismus S, für den

$$Su_1 = \alpha' u_1 + \beta' u_2,$$

$$Su_2 = \gamma' u_1 + \delta' u_2$$

gilt.

Die additive Zahlengruppe $I + Ic$ enthält eine Untergruppe der Hauptordnung $\mathfrak{o}$ von K von endlichem Index r_6. Folglich ist

$$r_6\mathfrak{o} \leqslant I + Ic.$$

Ferner sei r_7 ein Multiplum von r_6, für das $r_6^{-1} r_7 c$ ganz ist. Da nun jedes Element von $G(K)$ sich durch Multiplikation mit einer genügend hohen

Potenz von p in eine ganze Zahl verwandeln lässt, ist

$$r_7 G(K) \leqslant P + Pc,$$

und

$$r_7 G(K)c \leqslant P + Pc.$$

d.h. für jede Zahl μ' aus $G(K)$ sind $r_7\alpha', r_7\beta', r_7\gamma', r_7\delta' \epsilon P$, wo $\mu' = \alpha' + \gamma' c$, $\mu' = \beta' + \delta' c$. Daher ist für jedes S

$$r_7 S(Pu_1 + Pu_2) \leqslant Pu_1 + Pu_2.$$

Hieraus folgt

$$r_7(Pu_1 + Pu_2) \leqslant S(Pu_1 + Pu_2) \leqslant r_7^{-1}(Pu_1 + Pu_2).$$

Also gibt es nur endlich viele Möglichkeiten für $S(Pu_1 + Pu_2)$. Es existieren daher zwei ganze rationale Zahlen r_8 und r_9 für jedes S, so dass

$$S^{r_8}(Pu_1 + Pu_2) = S^{r_8 + r_9}(Pu_1 + Pu_2).$$

Also ist

$$s^{r_9}(Pu_1 + Pu_2) = Pu_1 + Pu_2.$$

Nach dem oben Bewiesenen ist die Zahl $\delta' - \gamma' c$ eine p-adische Einheit. Da

$$S(u_1 + cu_2) = \mu'(u_1 + cu_2),$$

$$Su_2 = \gamma'(u_1 + cu_2) + (\delta' - \gamma' c)u_2,$$

ist also S ein Automorphismus von $\overline{\mathfrak{G}}$. Aus den beiden Resultaten ergibt sich dann

$$S^{r_9}(Pu_1 + Pu_2) \cap \left(\bar{P}(u_1 + cu_2) + \bar{I}u_2\right) = (Pu_1 + Pu_2) \cap \left(\bar{P}(u_1 + cu_2) + \bar{I}u_2\right)$$

d.h. $S^{r_9}\mathfrak{G} = \mathfrak{G}$.

Wir haben einerseits bewiesen, dass eine Untergruppe von $G(K)$ zur Automorphismengruppe von $\mathfrak{G}$ isomorph ist und andererseits, dass eine Potenz jedes Elements von $G(K)$ zu dieser Untergruppe gehört. Da $G(K)$ durch endlich viele Elemente erzeugt werden kann, folgt also, dass diese Untergruppe einen endlichen Index hat. Damit ist alles bewiesen.

Die Untergruppen der endlichen F-Moduln.

Es sei $\mathfrak{G}$ eine Untergruppe eines endlichen F-Moduls. Es sei q irgendeine Primzahl und R_q der Ring aller rationalen Zahlen, deren Nenner zu q prim sind. Es bezeichne $R_q\mathfrak{G}$ die Menge aller endlichen Summen

$$a_1 x_1 + \ldots + a_m x_m,\ a_1, \ldots, a_m \epsilon R_q, \quad x_1, \ldots, x_m \epsilon \mathfrak{G}.$$

$R_q\mathfrak{G}$ ist eine Gruppe, die innerhalb des zu $\mathfrak{G}$ zugehörigen F-Moduls definiert ist.

SATZ 10. *$\mathfrak{G}$ ist der Durchschnitt aller Gruppen $R_q\mathfrak{G}$, d.h.*

$$\mathfrak{G} = \underset{q}{D}(R_q\mathfrak{G}).$$

Beweis. Sei x ein Element des Durchschnitts $\underset{q}{D}(R_q\mathfrak{G})$. Da $x \in R_q\mathfrak{G}$, gibt es eine Darstellung

$$x = a_1x_1+\ldots+a_nx_n, \quad a_1, \ldots, a_n \in R_q, \; x_1, \ldots, x_n \in \mathfrak{G}.$$

Es gibt eine ganzrationale Zahl r, die zu q prim ist und für die

$$ra_1, \; ra_2, \; \ldots, \; ra_n$$

alle ganz sind. Somit ist $rx \in \mathfrak{G}$. Sind $q_1, \ldots, q_m$ die verschiedene Primteiler von r, so gibt es in derselben Weise ganzrationale Zahlen $r_1, \ldots, r_m$, die bzw. zu $q_1, \ldots, q_m$ prim sind und für die

$$r_1x, \; r_2x, \; \ldots, \; r_mx \in \mathfrak{G}.$$

Da $(r, r_1, \ldots, r_m) = 1$, gibt es ganzrationale $k, k_1, \ldots, k_m$, für die

$$x = (kr+k_1r_1+\ldots+k_mr_m) \in \mathfrak{G}.$$

Mithin ist $\underset{q}{D}(R_q\mathfrak{G}) \leqslant \mathfrak{G}$.

Trivialerweise ist $\mathfrak{G} \leqslant \underset{q}{D}(R_q\mathfrak{G})$

und folglich $\underset{q}{D}(R_q\mathfrak{G}) = \mathfrak{G}$.

Damit ist der Satz bewiesen.

Wie im Satz 4 lässt sich leicht beweisen:

SATZ 11. *Sind $\mathfrak{G}$, $\mathfrak{G}'$ isomorphe Gruppen, so lässt sich jeder Isomorphismus $\mathfrak{G} \cong \mathfrak{G}'$ auf genau eine Weise zu einem Isomorphismus $R_q\mathfrak{G} \cong R_q\mathfrak{G}'$ für jede Primzahl q fortsetzen.*

Wir betrachten nun eine Gruppe $R_q\mathfrak{G}$. Diese Gruppe kann als R_q-Untermodul des endlichen F-Moduls $F\mathfrak{G}$ aufgefasst werden. Setzt man $p = q$ und S die Menge aller Primzahlen mit Ausnahme von q, so wird I der Ring R_q und P der Körper F. Also kann man die Sätze 1-5 auf $R_q\mathfrak{G}$ anwenden. Die Invarianten $n(R_q\mathfrak{G})$ und $k(R_q\mathfrak{G})$ von $R_q\mathfrak{G}$ sind nach Satz 11 zugleich Invarianten von $\mathfrak{G}$. Die Zahl $n(R_q\mathfrak{G})$ ist offenbar der Rang des kleinsten $\mathfrak{G}$-enthaltenden F-Moduls und ist daher von q unabhängig. Falls die Gruppe $\mathfrak{G}$ ein I-Untermodul eines endlichen P-Moduls ist, ist diese

Invariante die Zahl $n(\mathfrak{G})$. Wir bezeichnen sie daher mit $n(\mathfrak{G})$ für jede Gruppe $\mathfrak{G}$, die Untergruppe eines endlichen F-Moduls ist.

In Verbindung mit einer Gruppe $\mathfrak{G}$ und ihren Invarianten $n(\mathfrak{G}), k(R_q\mathfrak{G})$ führen wir nun eine Klasseneinteilung von gewissen Mengen von Matrizes ein, die der früheren Klasseneinteilung entspricht. Es sei A_q eine $n(\mathfrak{G})$-reihige reguläre Matrix über dem q-adischen Zahlkörper und $\mathfrak{A}$ die Menge $(A_2, A_3, A_5, \ldots)$, in der eine und nur eine Matrix A_q zu jeder Primzahl zugeordnet wird. Es bezeichne $\mathfrak{M}$ die Menge aller möglichen Mengen $\mathfrak{A}$.

Definition. Zwei Elemente $(A_2, A_3, A_5, \ldots)$ $(A_2', A_3', A_5', \ldots)$ der Menge $\mathfrak{M}$ heissen äquivalent, wenn es eine $n(\mathfrak{G})$-reihige reguläre Matrix A über F und ein Element $(B_2, B_3, B_5, \ldots)$ aus $\mathfrak{M}$ gibt, sodass

$$A_q' = AA_qB_q, \quad B_q \in \mathfrak{B}\big(n(\mathfrak{G}),\ k(R_q\mathfrak{R})\big)$$

für jede Primzahl q.

Diese Aequivalenz ist reflexiv, symmetrisch und transitiv und teilt daher die Menge $\mathfrak{M}$ in zu einander fremde Klassen von äquivalenten Matrizes ein.

Es sei jetzt F_q der q-adische Zahlkörper und I_q der Ring aller ganzen q-adischen Zahlen. Jede Gruppe $\overline{R_q\mathfrak{G}}$ lässt sich nach Satz 1 in der Form

$$\overline{R_q\mathfrak{G}} = F_q v_1^q + \ldots + F_q v_k^q + I_q v_{k+1}^q + \ldots + I_q v_n^q \quad [n = n(\mathfrak{G}),\ k = k(R_q\mathfrak{G})]$$

darstellen. Jeder Gruppe $R_q\mathfrak{G}$ ordnen wir nun eine solche Basis $v_1^q, \ldots, v_n^q$ zu. Es sei $u_1, \ldots, u_n$ irgendeine F-Basis des F-Moduls $F\mathfrak{G}$. Jeder Primzahl q ordnen wir diejenige Matrix A_q zu, für die

$$(v_1^q, \ldots, v_n^q) = (u_1, \ldots, u_n)A_q$$

gilt. A_q ist nach Satz 2 eine reguläre n-reihige Matrix über dem Körper F_q. Hierdurch ist ein Element $(A_2, A_3, A_5, \ldots)$ aus $\mathfrak{M}$ bestimmt. Genau wie im Beweise des Satzes 5 beweist man nun erstens, dass die Klasse $\mathfrak{c}$ dieses Elements unabhängig von der Wahl der Basen ist und zweitens, dass isomorphe Gruppen dieselbe Klasse $\mathfrak{c}$ bestimmen. Die Klasse $\mathfrak{c}$ ist daher eine Invariante der Gruppe $\mathfrak{G}$ und wird deswegen mit $\mathfrak{c}(\mathfrak{G})$ bezeichnet.

Wir wollen jetzt beweisen, dass diese Invarianten ein volles Invariantensystem für die Untergruppen der endlichen F-Moduln sind. Seien dazu $\mathfrak{G}, \mathfrak{G}'$ zwei Gruppen mit $n(\mathfrak{G}) = n(\mathfrak{G}')$, $k(R_q\mathfrak{G}) = k(R_q\mathfrak{G}')$ für jede Primzahl q und $\mathfrak{c}(\mathfrak{G}) = \mathfrak{c}(\mathfrak{G}')$. Zu jedem Element $(A_2, A_3, A_5, \ldots)$ aus der Klasse $\mathfrak{c}(\mathfrak{G})$ gibt es genau wie im Satz 5 ein Basissystem

$$u_1, \ldots, u_\mu,$$

$$v_1^q, \ldots, v_\mu^q,$$

wobei $$F\mathfrak{G} = Fu_1 + \ldots + Fu_n \quad [n = n(\mathfrak{G})],$$

$$\overline{R_q\mathfrak{G}} = F_q v_1^q + \ldots + F_q v_k^q + I_q v_{k+1}^q + \ldots + I_q v_n^q \quad [k = k(R_q\mathfrak{G})],$$

und

$$(u_1, \ldots, u_n) A_q = (v_1^q, \ldots, v_n^q) \quad \text{für} \quad q = 2, 3, 5, \ldots.$$

Da $\mathfrak{c}(\mathfrak{G}) = \mathfrak{c}(\mathfrak{G}')$, existiert für die Gruppe $\mathfrak{G}'$ ein entsprechendes Basissystem

$$u_1', \ldots, u_n',$$

$$v_1'^q, \ldots, v_1'^q,$$

mit $$(u_1', \ldots, u_n') A_q = (v_1'^q, \ldots, v_n'^q) \quad (q = 2, 3, 5, \ldots).$$

Der durch die Zuordnung $u_1 \to u_1'$, $u_2 \to u_2'$, ..., $u_n \to u_n'$ definierte Isomorphismus des F-Moduls $F\mathfrak{G}$ auf den F-Modul $F\mathfrak{G}'$ bildet wie im Satz 5 die Gruppe $R_q\mathfrak{G}$ auf die Gruppe $R_q\mathfrak{G}'$ ab. Mit Rücksicht auf Satz 10 folgt nun, dass die Gruppe $\mathfrak{G}$ in die Gruppe $\mathfrak{G}'$ übergeht, d.h. $\mathfrak{G} \cong \mathfrak{G}'$. Damit haben wir bewiesen:

Satz 12. *Für die Untergruppen $\mathfrak{G}$ der endlichen F-Moduln sind $n(\mathfrak{G})$, $k(R_q\mathfrak{G})$ $(q = 2, 3, 5, \ldots)$ und die Klasse $\mathfrak{c}(\mathfrak{G})$ ein volles Invariantensystem bezüglich Isomorphie.*

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ON THE SYSTEM NORMALIZERS OF A SOLUBLE GROUP

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1. *Introduction.*

The present paper is a continuation of the theory of soluble groups which we have begun to develop in the four following notes:

I. "A note on soluble groups", *Journal London Math. Soc.*, 3 (1928), 98–105.

II. "A characteristic property of soluble groups", *ibid.*, 12 (1937), 198–200.

III. "Complemented groups", *ibid.*, 12 (1937), 201–204.

IV. "On the Sylow systems of a soluble group", *Proc. London Math. Soc.* (2), 43 (1937), 316–323.

The soluble groups may be characterized from among the groups of finite order in two quite distinct ways. First, they are the groups whose derived series ends in 1, or (what is the same) whose chief factors are all Abelian. This manner of defining them may be described for obvious reasons as a *commutatorial* one. Secondly, as was shown in I and II, they are the groups which can be expressed as products of permutable prime-power subgroups, or (what is the same) whose Sylow subgroups all have complements. This second definition has reference to what we may call *Sylow structure*, meaning the manner in which a group contains its Sylow subgroups.

In this connection, we recall the well known theorem of Burnside according to which the groups of finite order which coincide with their

own hypercentrals are the same as those whose Sylow subgroups are all self-conjugate. Here again we have two equivalent definitions of a class of groups, one in terms of commutator structure, the other in terms of Sylow structure.

These two contrasted aspects of group structure, reflected (in the case of the soluble groups) by the two definitions quoted above, we shall attempt in the present note to bring into a more intimate relationship. An important step in this direction, valid for all groups of finite order, has been taken recently by Otto Grün†, who shows how to construct a Sylow subgroup of the derived group of G, starting from a Sylow subgroup of G. The results given below, however, depend essentially on the notion of a *Sylow system* introduced in IV (*cf.* §2 below); and since only soluble groups possess Sylow systems, they are (with a few exceptions of which Theorem 3.7 and Theorem 9.3 are the most important) definitely restricted to soluble groups, or, at the most, to groups with a self-conjugate soluble subgroup. The connection of the present theory with the theorem of Grün is by no means immediate.

The object of this paper is to describe the properties of a new class of subgroups, the normalizers of Sylow systems, or *system normalizers*, and especially their relations with the self-conjugate subgroups. We may regard the system normalizers of a soluble group as the most convenient representatives of its Sylow structure, while the self-conjugate subgroups represent its commutator structure.

The most precise results which we have obtained are the avoidance and covering theorems‡ of §§6 and 7. These are sufficient to determine exactly the order of the meet of any system normaliser with any self-conjugate subgroup. In particular, we deduce in §8 that the number of Sylow systems of a soluble group G is equal to the product of the orders of its "eccentric" chief factors. Further, Theorem 9.1 and Theorem 3.4 show the meet and join of the system normalizers of G to be respectively the hypercentral of G and G itself.

In §§3 and 4, we deal with the intrinsic properties of the system normalizers (Theorem 3.3), their relations with one another (Theorem 4.3 to Theorem 4.6) and with certain types of non-self-conjugate subgroup (Theorem 3.5, Theorem 4.7, and Theorem 4.8). The most interesting of these results is probably the abstract definition of the system normalizers provided by Theorem 4.8.

† "Beiträge zur Gruppentheorie, I", *Journal für Math.*, 174 (1936), 1–14.

‡ These have been given, in a more special form, in the note "Solvable groups", *Bull. American Math. Soc.*, 42 (1936), 642–643.

As will be shown later, the theorems of the present paper also provide a convenient method for the construction of all the soluble groups of a given order.

2. *Definitions related to Sylow structure.*

For the various features which constitute the "Sylow structure" of a soluble group, we shall retain the terms used in IV.

Except where the contrary is definitely stated, G will always denote a soluble group of order $p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r}$, where the p's are different primes and the α's are positive.

By I, there exists in G at least one subgroup S_i of index $p_i^{\alpha_i}$, such a subgroup being called a *Sylow p_i-complement* of G. If

$$S_1, S_2, \dots, S_r \tag{1}$$

is any complete set of Sylow complements of G, one for each of the r primes p_i, then the 2^r distinct meets which can be formed from the set (1), including the "null meet" G, we call a *Sylow system* of G. The members of a Sylow system are all *S-subgroups* of G, that is, their order and index are coprime. By I, two S-subgroups of the same order are always conjugate. And a Sylow system contains just one member from each of the 2^r classes of conjugate S-subgroups.

In particular, a Sylow system contains a complete set of Sylow subgroups

$$S_1', S_2', \dots, S_r'; \tag{2}$$

where S_i', of order $p_i^{\alpha_i}$, is the meet of the S_j's other than S_i.

It is shown in IV that any two members of a given Sylow system are permutable. And an alternative definition is as follows: a Sylow system consists of all 2^r products formed from any complete set of permutable Sylow subgroups, such as (2), the "null product" 1 being included. (The hypothesis of permutability is essential, otherwise the products would not be subgroups.)

If $\mathfrak{S}$ is any Sylow system of G, we define the *system normalizer*, $N(\mathfrak{S}) = N$, associated with $\mathfrak{S}$, to consist of all elements of G which transform every member of $\mathfrak{S}$ into itself. Thus N is the meet of the normalizers of the complete set of Sylow complements (1) which define $\mathfrak{S}$; or alternatively, it is the meet of the normalizers of the complete set of permutable Sylow subgroups (2) which belong to $\mathfrak{S}$.

3. *First properties of the system normalizers.*

THEOREM 3.1. *The system normalizers form a characteristic class of conjugate subgroups of G.*

For by IV, any two Sylow systems are conjugate in G. Therefore, any two system normalizers are conjugate. Any automorphism of G merely permutes its Sylow systems. Hence the class of conjugate subgroups formed by the system normalizers is invariant under every automorphism of G, that is to say, it is characteristic (in the sense of Frobenius).

THEOREM 3.2. *The index ϖ of a system normalizer in G is equal to the number of distinct Sylow systems of G. If the number of Sylow p_i-complements of G is ϖ_i, then*

$$\varpi = \varpi_1 \varpi_2 \dots \varpi_r.$$

The first statement is an immediate corollary of (3.1), the second of the definition of a Sylow system. $\varpi = \varpi(G)$ is an important invariant of the soluble group G; its value is given in a different form in § 8.

THEOREM 3.3. *A system normalizer $N(\mathfrak{S})$ is the direct product of its Sylow subgroups N_i:*

$$N(\mathfrak{S}) = N_1 \times N_2 \times \dots \times N_r,$$

where N_i is the meet of S_i' with the normalizer of S_i.

Here, the S_i and S_i' are, respectively, the Sylow complements and subgroups of the system $\mathfrak{S}$. Since S_i' is the meet of the S_j's other than S_i, and N_i is contained in S_i', it follows that N_i is contained in (and therefore normalizes) each of these $r-1$ Sylow complements. It also normalizes S_i, by definition. Hence N_i belongs to $N = N(\mathfrak{S})$.

Now let N_i' be any Sylow p_i-subgroup of N. Then N_i' normalizes S_i'; and since its order is a power of p_i, while S_i' is a Sylow p_i-subgroup of G, it follows that N_i' is contained in S_i'. N_i' is also contained in the normalizer of S_i. Thus N_i' is contained in N_i. But the order of N_i is a power of p_i; and as we have seen, N_i belongs to N. Hence $N_i' = N_i$, and N has only one Sylow p_i-subgroup, viz. N_i. This being true for all i, the theorem follows.

THEOREM 3.4. *The join of all the system normalizers of G is G itself. No proper self-conjugate subgroup of G can contain any system normalizer of G.*

By Theorem 3.1, the join of the system normalisers is a self-conjugate subgroup of G. It is therefore sufficient to prove that a given system normalizer $N = N(\mathfrak{S})$ cannot be contained in a given maximal self-conjugate subgroup K of G.

Let the index of K in G be p_i. Then S_i, together with all its conjugates under G, is contained in K. Hence the normalizer M_i of S_i in G satisfies the relation $G = KM_i$. Thus M_i contains cosets of S_i which have no element in common with K. In such a coset, S_i' possesses just one element. And such an element, common to S_i' and the normalizer of S_i, belongs to the Sylow p_i-subgroup of N, by Theorem 3.3. Since it does not belong to K, N cannot lie in K.

In contrast to Theorem 3.4, we have

THEOREM 3.5. *Every maximal subgroup H of G which is not self-conjugate contains at least one system normalizer of G.*

For, by a well known theorem due to Galois, the index of H in G is the power of a prime, say $p_i^{\lambda_i}$. Any Sylow p_i-complement S_i of H is, therefore, at the same time a Sylow p_i-complement of G. And it will be sufficient to prove that H contains the normalizer of S_i in G.

Denote by L the meet of H with its conjugates in G. Then G/L has one and only one least self-conjugate subgroup, say K/L, which is of order $p_i^{\lambda_i}$. If K'/K is any least self-conjugate subgroup of G/K, the order p_j^{μ} of K'/K must be prime to p_i, that is, $i \neq j$. For otherwise, K/L would belong to the central of K'/L, whereas actually (by the theorem of Galois referred to above) we know that every element of G not in K transforms K/L according to an outer automorphism.

Let M be the meet of H with K'. The order of M/L is then p_j^{μ}, and evidently M is the meet of S_iL with K'. Since L and K' are both self-conjugate in G, it follows that the normalizer N' of S_i is contained in the normalizer H of M. This is the result required.

We have shown incidentally that *every system normalizer of H contains at least one system normalizer of G.*

More precisely, S_i has $p_i^{\lambda_i}$ times as many conjugates in G as in H, that is,

$$\varpi_i(G) = p_i^{\lambda_i}\varpi_i(H).$$

These $\varpi_i(G)$ conjugates of S_i in G obviously fall into $p_i^{\lambda_i}$ sets of $\varpi_i(H)$, each set forming the complete class of conjugate Sylow p_i-complements of one of the $p_i^{\lambda_i}$ distinct conjugates of H.

On the other hand, for $j \neq i$, each of the $\varpi_j(G)$ Sylow p_j-complements of G meets H in a Sylow p_j-complement of H. But in general, a given

Sylow p_j-complement of H will arise in this way from more than one Sylow p_j-complement of G. Thus we usually have $\varpi_j(G) = p_j^{\lambda_j}\varpi_j(H)$, with some $\lambda_j > 0$. And then obviously

$$\prod_{j \neq i} p_j^{\lambda_j}$$

gives the index of a system normalizer of G in a system normalizer of H or one of its conjugates. By Theorem 3.3, the latter is a direct product of prime-power groups; and it is well known that every proper subgroup of such a group is distinct from its own normalizer. Hence, *in general a system normalizer is not its own normalizer.*

The simplest example of this phenomenon is the octahedral group. Here the system normalizers are of order 2, and are contained self-conjugately in subgroups of order 4. Thus, in general there will be fewer distinct system normalizers than Sylow systems.

We have mentioned this point because of the familiar fact that (in any group of finite order) the normalizer of an S-subgroup is its own normalizer. In the case of soluble groups, the normalizers of S-subgroups have many other interesting properties, and we may take the present opportunity of mentioning a few of them.

It will be convenient if we give first the logical interrelations of the properties in question.

THEOREM 3.6. *Let M be any subgroup of finite index in the arbitrary group G. Then there hold between the statements* (i)–(iv) *below the equivalences*

$$\text{(i)} + \text{(ii)} \sim \text{(i)} + \text{(iii)} \sim \text{(iv)};$$

where we have

(i) *every subgroup of G containing M is its own normalizer in G;*

(ii) *M is not contained in the meet of any two distinct conjugate subgroups of G;*

(iii) *if M and M' are conjugate in G, they are conjugate in their join;*

(iv) *in any transitive permutational representation of G, M leaves at most one symbol invariant.*

Further, if the statements (i)–(iv) *hold for a given subgroup M of G, they also hold for every subgroup of G containing M.*

The last sentence is obvious so far as (iv) is concerned; and it therefore follows also for the first three statements, in view of the above equivalences.

To prove the first equivalence, let K be any subgroup of G containing M. Then it results from (i) that the number of conjugates of M in K is precisely equal to the index $(K:M)$. Now let λ be the number of distinct conjugates of K in G which contain M; and let μ be the number of distinct classes of conjugate subgroups of K which contain representatives conjugate to M in G. The number of conjugates of M in G being $(G:M)$, it follows that the number of times a conjugate of M is contained in a conjugate of K is equal to $\lambda(G:M)$. But the latter number is also equal to $\mu(G:K)(K:M)$, since each of the $(G:K)$ conjugates of K contains just μ classes of $(K:M)$ members each, all conjugate to M in G. But $(G:M)=(G:K)(K:M)$. Hence $\lambda=\mu$.

So far we have used only the statement (i). But (ii) merely means that $\lambda=1$ for all K. While (iii) means that $\mu=1$ for all K. Thus, given (i), (ii) and (iii) each imply the other.

To prove the second equivalence, we need merely note that (iv) is the result of expressing (i) and (ii) in the language of permutation groups.

For if M is contained in a subgroup K which differs from its own normalizer, then in the transitive representation of G determined by K, the latter (and hence M also) will leave more than one symbol invariant. On the other hand, if M is contained in both K_1 and K_2, where K_1 and K_2 are distinct and conjugate, then in the transitive representation of G determined by K_1, M will leave at least two symbols invariant.

Conversely, suppose we have a transitive representation of G in which M leaves invariant the symbols x_1 and x_2. Let H_i denote the subgroup of all elements of G leaving x_i invariant. Then M lies in both H_1 and H_2. If H_1 coincides with H_2, it is distinct from its normalizer, and M contravenes (i). If $H_1 \neq H_2$, then M contravenes (ii), since H_1 and H_2 are conjugate.

Thus Theorem 3.6 is completely proved.

THEOREM 3.7†. *Let H be a self-conjugate soluble subgroup of the arbitrary group G; and let M be the normalizer in G of the S-subgroup S of H. Then the statements* (i)–(iv) *of Theorem* 3.6 *are true. They still hold even when H is not soluble, provided S is a Sylow subgroup of H.*

Owing to the equivalences of Theorem 3.6, it will be sufficient to prove (i) and (iii).

† [*Inserted* 9 *November*, 1937. The case $G=H$, S a Sylow subgroup of H, is known, at least so far as (i) and (iii) are concerned; *cf.* Zassenhaus, *Lehrbuch der Gruppentheorie*, I, 191.]

Let K be any subgroup of G containing M, and let P be any element from the normalizer of K. Then $T = P^{-1}SP$ is (like S) an S-subgroup of H, and (also like S) it lies in K. Hence both S and T are S-subgroups (of the same order) of the meet L of H with K. Therefore S and T are conjugate in L, in any case, if they are Sylow subgroups; and by the conjugacy property for the S-subgroups of a soluble group, in the case when H (and therefore L) is soluble. Thus there exists an element Q of L such that $T = Q^{-1}SQ$. It follows that PQ^{-1} lies in the normalizer M of S. Since both L and M lie in K, we deduce that P lies in K. That is, K is its own normalizer.

Next let M' be any conjugate of M in G, $M' = R^{-1}MR$; and let J be the join of M and M'. Now S and $R^{-1}SR$ are S-subgroups of H both contained in J; hence they are S-subgroups of the meet L of H with J. As before, they are conjugate in L, so that $R^{-1}SR = U^{-1}SU$ for some U of L. But $U^{-1}MU$ must then coincide with M', for it is the normalizer of $U^{-1}SU$ in G. Since L lies in J, it follows that M and M' are conjugate in J, and Theorem 3.7 is proved.

The familiar theorem that the normalizer of an S-subgroup is its own normalizer depends merely on the fact that an S-subgroup is necessarily characteristic in its normalizer. On the other hand, Theorem 3.7 depends essentially on the conjugacy theorems, as the above proof clearly shows.

4. *Relative system normalizers.*

Let the soluble group G be contained as a self-conjugate subgroup in the larger, not necessarily soluble, group L. Then the set of all elements of L which transform into itself a given Sylow system $\mathfrak{S}$ of G will be called *a system normalizer of G in (or relative to) L*, and denoted by $N_L(\mathfrak{S})$.

The system normalizers of G in L form a single class of conjugate subgroups in L; indeed they are all conjugate under G. Clearly also

$$N(\mathfrak{S}) = N_G(\mathfrak{S}) = N_L(\mathfrak{S}) \wedge G$$

and

$$L = N_L(\mathfrak{S})\, G.$$

The index of $N_L(\mathfrak{S})$ in L is equal to the index of $N(\mathfrak{S})$ in G, and is the same as the number of distinct Sylow systems of G, viz. $\varpi(G)$.

When it is desirable to make the distinction, we may call $N_G(\mathfrak{S})$ an *absolute* system normalizer of G, and the $N_L(\mathfrak{S})$ having $L \neq G$ *relative* system normalizers.

In the present section, we are concerned mainly with the case in which L also is a soluble group: and to preserve our previous conventions,

we make a slight change of notation and denote L and G by G and H, respectively.

Then let $\mathfrak{T}$ be any Sylow system of the self-conjugate subgroup H of G. By IV, $\mathfrak{T}$ is the meet of H with a suitably chosen Sylow system $\mathfrak{S}$ of G. And it is clear that $N_G(\mathfrak{T})$ contains $N_G(\mathfrak{S})$.

THEOREM 4.1. *The meet M_i of the normalizer of T_i in G with S_i' is a Sylow p_i-subgroup of the group $M = N_G(\mathfrak{T})$. The M_i are all permutable and determine a Sylow system $\mathfrak{M}$ of M.*

Here T_i is the Sylow p_i-complement of H in system $\mathfrak{T}$, and S_i' the Sylow p_i-subgroup of G in system $\mathfrak{S}$. (If the order of H is prime to p_i, then naturally $T_i = H$. This and similar obvious conventions are adopted throughout.)

This theorem is the natural generalization of Theorem 3.3.

We may show first that M_i leaves $\mathfrak{T}$ invariant. It leaves T_i invariant by definition. On the other hand, for $j \neq i$, M_i belongs to S_i' and therefore to S_j. And since H is self-conjugate, every element of S_j leaves $T_j = H \wedge S_j$ invariant.

Thus M_i belongs to M. If the number of Sylow p_i-complements in H is $p_i^{\beta_i}$, and we write $\gamma_i = \alpha_i - \beta_i$, then, by Theorem 3.2, $p_i^{\gamma_i}$ is the order of a Sylow p_i-subgroup of M. But for the same reason, the normalizer of T_i in G is of index $p_i^{\beta_i}$, and hence M_i is of order $p_i^{\gamma_i}$. Thus M_i is a Sylow p_i-subgroup of M.

Both M_i and M_j lie in the S-subgroup $S_i' S_j'$ of G. Hence the order of their join involves only the primes p_i and p_j. Since they are Sylow subgroups of M, they must therefore be permutable. The products of the M_i thus form a Sylow system $\mathfrak{M}$ of M.

We may note in passing that

$$\mathfrak{M} = M \wedge \mathfrak{S}; \tag{4.2}$$

that is, $\mathfrak{M}$ may be formed by taking the meets of M with the members of $\mathfrak{S}$.

We shall now prove one of the most important properties of relative system normalizers. With the notation of Theorem 4.1, this is

THEOREM 4.3. $N_M(\mathfrak{M}) = N_G(\mathfrak{S})$. *Or, the absolute system normalizers of any relative system normalizer (relative to G) are all absolute system normalizers of G.*

The general statement follows from the particular result owing to the conjugacy property of system normalizers, viz. Theorem 3.1.

Every element of $N_G(\mathfrak{S})$ leaves invariant both S_i' and the normalizer of T_i, therefore also their meet M_i. Thus $N_G(\mathfrak{S})$ is contained in $N_G(\mathfrak{M})$; or since it belongs to M, in $N_M(\mathfrak{M})$. On the other hand, every element of $N_M(\mathfrak{M})$ leaves invariant both M_i and also T_i', the latter since $M = N_G(\mathfrak{T})$. But it is easy to see that $T_i' M_i = S_i'$. Thus $N_M(\mathfrak{M})$ leaves all the S_i' invariant, and so belongs to $N_G(\mathfrak{S})$. Combining these two results, we obtain $N_M(\mathfrak{M}) = N_G(\mathfrak{S})$.

THEOREM 4.4. *If M is a system normalizer of H relative to G, then $\varpi(G) = \varpi(H)\,\varpi(M)$; where $\varpi(G)$, $\varpi(H)$, and $\varpi(M)$ denote the numbers of Sylow systems of G, H and M, respectively.*

For, with our previous notation, $\varpi(G)$ and $\varpi(H)$ are the indices of $N_G(\mathfrak{S})$ and $N_G(\mathfrak{T})$, respectively, in G. Also, by Theorem 4.3, $\varpi(M)$ is the index of $N_G(\mathfrak{S})$ in $M = N_G(\mathfrak{T})$.

If we take $H = G$, M becomes an absolute system normalizer of G, and Theorem 4.4 gives $\varpi(M) = 1$. This means that M has only one Sylow system; hence it is the direct product of its Sylow subgroups. Thus we have proved anew the result of Theorem 3.3.

If we denote by $\varpi_i(G)$ the number of Sylow p_i-complements of G, then Theorem 4.4 gives immediately the corollary

$$\varpi_i(G) = \varpi_i(H)\,\varpi_i(M). \tag{4.41}$$

By repeated applications of Theorem 4.4, we obtain the following factorization property of $\varpi(G)$:

THEOREM 4.5. *Let*

$$G = G_0 \geqslant G_1 \geqslant G_2 \geqslant \ldots \geqslant G_r = 1$$

be any chain of subgroups of the soluble group G such that (for each $\alpha = 1, 2, \ldots, r$) G_α is self-conjugate in $G_{\alpha-1}$. Further, let M_α denote a system normalizer of G_α relative to $G_{\alpha-1}$. Then

$$\varpi(G) = \varpi(M_1)\,\varpi(M_2) \ldots \varpi(M_r),$$

and

$$\varpi_i(G) = \varpi_i(M_1)\,\varpi_i(M_2) \ldots \varpi_i(M_r).$$

Note that, since $G_r = 1$, $M_r = G_{r-1}$.

The next result is a generalization of Theorem 4.3 and reduces to the latter in the case $G = L$.

THEOREM 4.6. *Let G and H be self-conjugate soluble subgroups of the arbitrary group L, and let G contain H. Let $\mathfrak{S}$ be any Sylow system of G,*

and $\mathfrak{T}$ its meet with H. Finally, let M and $\mathfrak{M}$ be the meets of G and $\mathfrak{S}$, respectively, with $K = N_L(\mathfrak{T})$. Then $N_K(\mathfrak{M}) = N_L(\mathfrak{S})$.

Obviously $M = N_G(\mathfrak{T})$, and, if $G = L$, then $M = K$, and we obtain Theorem 4.3 as stated above.

In the general case, we know already from Theorem 4.3 that $N_M(\mathfrak{M}) = N_G(\mathfrak{S})$. But $N_M(\mathfrak{M})$ and $N_G(\mathfrak{S})$ are precisely the meets of $N_K(\mathfrak{M})$ and $N_L(\mathfrak{S})$, respectively, with G. On the other hand, since H is self-conjugate in L and $\mathfrak{T}$ the meet of H with $\mathfrak{S}$, it follows that every element of $N_L(\mathfrak{S})$ leaves $\mathfrak{T}$ invariant. In other words, $N_L(\mathfrak{S})$ belongs to K; and it therefore also leaves invariant the meet $\mathfrak{M}$ of $\mathfrak{S}$ with K. Hence $N_L(\mathfrak{S})$ is contained in $N_K(\mathfrak{M})$. Finally, from the conjugacy property of Sylow systems, every coset of G in L must contain elements from $N_L(\mathfrak{S})$, and hence also elements from $N_K(\mathfrak{M})$. But as we have seen these subgroups have the same meet with G. If the order of this meet is α, every coset of G in L must contain exactly α members from each of $N_K(\mathfrak{M})$ and $N_L(\mathfrak{S})$. And as the former of these contains the latter, these α elements must be the same in both cases. Thus $N_K(\mathfrak{M})$ and $N_L(\mathfrak{S})$ coincide and the theorem is proved.

The proof holds even when the order of L is infinite.

THEOREM 4.7. *Let $N = N_K(\mathfrak{M})$ be any system normalizer, relative to the arbitrary group K, of the soluble self-conjugate subgroup M of K. Then there exists a chain of subgroups*

$$N = N_0 < N_1 < \ldots < N_l = K$$

stretching from N to K and such that each is maximal but not self-conjugate in the next.

(The case $l = 0$, $N = K$ is not excluded.) Since the index of N in K is equal to $\varpi(M)$ and is therefore finite, we may argue by induction over this index.

Let M' be any self-conjugate subgroup of K contained in M, let $\mathfrak{M}'$ be the meet of $\mathfrak{M}$ with M', $K' = N_K(\mathfrak{M}')$, M'' the meet of K' with M, and finally $\mathfrak{M}''$ the meet of $\mathfrak{M}$ with K'. Then Theorem 4.6 gives

$$N_{K'}(\mathfrak{M}'') = N_K(\mathfrak{M}) = N.$$

Thus K' is a system normalizer relative to K, while N is a system normalizer relative to K'. And accordingly our induction hypothesis allows us to ignore every case in which $N < K' < K$.

Hence we may assume that for every possible choice of M' we have either $N = K'$ or else $K' = K$. If $N = K'$, we may replace M by M' and

$\mathfrak{M}$ by $\mathfrak{M}'$ and proceed as before. On the other hand, $K' = K$ only if M' is the direct product of its Sylow subgroups. If M itself is the direct product of its Sylow subgroups, then $N = K$ and there is nothing to prove. Thus we may suppose that M is not, but M' is, the direct product of its Sylow subgroups. Further, we can always choose M' to be of index a power of a prime in M, say of index p^α. But then the Sylow subgroups of M which are of order prime to p are all self-conjugate, therefore characteristic, and therefore self-conjugate in K. Hence N reduces to the normalizer in K of some Sylow p-subgroup of M. By Theorem 3.7, every subgroup of K containing N is its own normalizer. Hence every chain of subgroups connecting N with K in which each member is maximal in the next must have also the property that no member is self-conjugate in the next. Such a chain certainly exists, and so the theorem is proved.

From Theorem 4.7 we shall deduce a purely abstract characterization of the absolute system normalizers of a soluble group. This consists of a certain minimal property and makes no reference to the Sylow systems.

We may recall that a subgroup M of a group G is said to be minimal with respect to a certain property (a), if it has the property (a) but is such that no proper subgroup of M has the property (a). Then we have

THEOREM 4.8. *A subgroup M of the soluble group G is an absolute system normalizer of G if and only if it is minimal with respect to the following property (a): M can be connected with G by a chain of subgroups*

$$M = M_0 < M_1 \ldots < M_l = G,$$

each of which is maximal but not self-conjugate in the next.

That the absolute system normalizers have the property (a) is a particular case of Theorem 4.7. That they are minimal with respect to this property follows from the proof of Theorem 3.5, which shows rather more than is strictly required, viz. that if M has property (a), its absolute system normalizers must all contain absolute system normalizers of G. So *a fortiori* must M itself. Hence no proper subgroup of an absolute system normalizer of G can possibly have property (a). And further, a subgroup M with property (a) can be minimal with respect to this property only if it coincides with an absolute system normalizer of G. This proves the result.

We may remark in connection with Theorem 4.8 that not every subgroup M which contains an absolute system normalizer of G need have property (a). For example, the six system normalizers of the octahedral group G are of order 2 and are contained in pairs by three subgroups M of

order 4. But the only proper subgroups of G which properly contain an M are its Sylow subgroups of order 8, and in these the M is necessarily self-conjugate, so that no chain such as that of Theorem 4.8 can exist.

Another example bearing on the condition for a subgroup M to contain a system normalizer of G is the following, which shows that it is not sufficient (as might be supposed from Theorem 3.7 and § 9) for M to be its own normalizer. This counter-example is provided by the direct product $G = G_1 \times G_2$, where G_1 and G_2 are tetrahedral groups. G contains twenty-four tetrahedral subgroups having no elements in common with either G_1 or G_2. These twenty-four subgroups correspond 1-1 to the twenty-four isomorphisms of G_1 with G_2 in the obvious way, and each of them is its own normalizer in G. But the system normalizers of G are of order 9 and so cannot be contained in any tetrahedral subgroup.

5. *Definitions related to commutator structure.*

We now pass to the main theme of the paper, the relations between the system normalizers of a soluble group and its chief series. If we are to describe these relations succinctly and without intolerable circumlocutions, it will first be necessary to introduce certain definitions, some of which will, no doubt, be familiar. In the present section, G denotes any group, not necessarily soluble or even of finite order.

If H and K are any two self-conjugate subgroups of G and $H > K$, we call H/K a *factor* of G.

If there exists no self-conjugate subgroup L of G such that $H > L > K$, then we call H/K a *chief factor* of G. Otherwise H/K is *composite*.

If

$$H = H_0 > H_1 > \ldots > H_l = K \tag{5.1}$$

is any chain of self-conjugate subgroups of G, stretching from H to K, we say that the factor H/K can be *decomposed* into the *component factors* H_i/H_{i+1}, $i = 0, 1, \ldots, l-1$. A composite factor can in general be decomposed in many ways. If G is finite, it can always be decomposed into chief factors. In particular, if $G = H$ and $K = 1$, and if the factors H_i/H_{i+1} are all chief factors, (5.1) is a *chief series* of G.

Definition 5.2. The set of all elements of G which transform H/K according to the identical automorphism forms a self-conjugate subgroup C, called the *centralizer* of H/K in G. C may also be defined as the greatest subgroup of G containing K and such that the commutator subgroup

$[C, H]$ lies in K; or (what is the same) such that in G/K the elements of C/K are all permutable with those of H/K.

If the centralizer of H/K is G, we call H/K a *central factor* of G. More generally, if L is any subgroup of G, we say that H/K is *central for* (*or relative to* L) whenever L is contained in the centralizer of H/K. Thus H/K is central for L, if and only if the commutator of every element of L with every element of H lies in K.

Any factor of G which can be decomposed into factors which are all central for L is said to be *hypercentral for* L. When $L = G$, we call it a *hypercentral factor* of G. Every finite group G contains a characteristic subgroup, $Z^* = Z^*(G)$, which is uniquely determined by the following property: H being a self-conjugate subgroup of G, $H/1$ is a hypercentral factor of G if and only if H is contained in Z^*. (This group Z^* is the "limit" of the upper central series of G:

$$1 = Z_0 \leqslant Z_1 \leqslant Z_2 \leqslant \ldots,$$

where the Z_i are defined by the rule that Z_i/Z_{i-1} is the central of G/Z_{i-1}.) Following Remak, we shall call Z^* *the hypercentral of* G.

A chief factor of G which is not central for L will be called *eccentric for* L, and a factor which can be decomposed into chief factors all of which are eccentric for L will be called *hypereccentric for* L. If $L = G$, we speak of eccentric chief factors and hypereccentric factors of G, in the two cases.

The definitions of hypercentral and hypereccentric are independent of the manner of decomposition of the factors into chief factor components, by the Jordan-Hölder theorem in its operator form.

Finally, to describe the possible incidence relations which may hold between a factor and a subgroup, as opposed to the commutatorial relations covered by definition 5.2, we need:

Definition 5.3. The subgroup L is said to *cover* the factor H/K if it contains at least one element from every coset of K in H, that is, if the join of K with the meet of L and H coincides with H.

At the other extreme, L is said to *avoid* the factor H/K if it contains no element of H which does not belong to K, that is, if the meet of H with the join of L and K coincides with K.

If $$G = G_0 > G_1 > \ldots > G_l = 1$$

is any chain of self-conjugate sub-groups of G, and if we write $L_i = L \wedge G_i$, then each L_i is self-conjugate in L, and the order of L is the product of the orders of the factors L_i/L_{i+1} of L, $i = 0, 1, \ldots, l-1$. If L covers G_i/G_{i+1},

then $L_i/L_{i+1} \simeq G_i/G_{i+1}$. While if L avoids G_i/G_{i+1}, then $L_i = L_{i+1}$. Thus for a subgroup which either covers or avoids each of the factors G_i/G_{i+1}, the order of the subgroup is equal to the product of the orders of those of the factors which it covers.

6. *Avoidance theorems.*

Once more, let G be a soluble group.

THEOREM 6.1. *Every system normalizer of G avoids every hypereccentric factor of G, and, in particular, it avoids every eccentric chief factor of G.*

This result is a particular case of

THEOREM 6.2. *If G is self-conjugate in a larger group L, then every system normalizer of G relative to L avoids every factor of L which is hypereccentric for G.*

Here L need not be soluble†.

If $\mathfrak{S}$ is any Sylow system of G and $N = N_L(\mathfrak{S})$, then it is sufficient to prove that N avoids any given chief factor H/K of L which is eccentric for G. For, if H/K were hypereccentric for G, it could be decomposed into chief factors of L each of which is eccentric for G; and, if N avoids each of these components, it must avoid H/K itself.

If H/K is eccentric for G, its centralizer C in L cannot contain G. Let $D = C \wedge G$. Then we can choose the subgroup D', self-conjugate in L, contained in G, containing D and such that D'/D is a chief factor of L. Since G is soluble, the orders of H/K and D'/D are both prime-powers; that of the former, since L/G is central for G and H/K eccentric. Moreover, as in the proof of Theorem 3.5, these two orders are coprime. Let them be p_i^{λ} and p_j^{μ}, respectively, so that $i \neq j$. Finally, if S_j' is the Sylow p_j-subgroup of G which belongs to $\mathfrak{S}$, let N_j be its normalizer in L. Then N_j contains N, and it will therefore be sufficient to show that N_j avoids H/K.

Suppose not, and let P be an element, contained in H but not in K, which transforms S_j' into itself. Since the order of D'/D is a power of p_j, every element of D' is of the form QR, with Q in S_j' and R in D. The commutator of P with Q lies in S_j', since P normalizes S_j'; and it also lies

† For convenience, we suppose it to be finite. The extension of the proof to the case of infinite L presents no difficulties.

in H, since P lies in H. Hence it must lie in K, since the orders of S_j' and H/K are coprime. Again, the commutator of P with R lies in K, since R lies in C and P in H. Hence the commutator of P with every element QR of D' lies in K. But the elements of H with this property form a self-conjugate subgroup of L containing K, which, since P is not in K and H/K is a chief factor of L, must coincide with H. Thus D' belongs to the centralizer of H/K. This is a contradiction, and thus Theorem 6.2 is proved.

An easy corollary of the avoidance theorem is

THEOREM 6.3. *The absolute system normalizers of G are contained in the hypercentrals of their own normalizers in G.*

Indeed, it is clear that any subgroup of G which avoids every eccentric chief factor in a given chief series of G will be contained in the hypercentral of its normalizer.

7. *Covering theorems.*

In the opposite direction to the results of the preceding section, we have

THEOREM 7.1. *Every system normalizer of G covers every hypercentral factor of G, and, in particular, it covers every central factor of G.*

This is a particular case of

THEOREM 7.2. *If G is self-conjugate in a larger group L, then every system normalizer of G relative to L covers every factor of L which is hypercentral for G.*

As before L need not be soluble , or even of finite order.

We shall obtain the proof by considering the effect of homomorphisms of L. If θ is any such homomorphism, we denote the effect of θ on any element, subgroup, or system of subgroups of L by writing θ as a multiplier after the symbol which represents the element, etc., concerned. Then $G\theta$ is a self-conjugate soluble subgroup of $L\theta$, and $\mathfrak{S}\theta$ is a Sylow system of $G\theta$. (It is understood that if only s of the r primes dividing the order of G enter into that of $G\theta$, then the 2^s distinct members of $\mathfrak{S}\theta$ are each repeated 2^{r-s} times. This is irrelevant to the argument.)

We shall find it convenient to generalize the above situation slightly *by taking for $\mathfrak{S}$ any system of permutable S-subgroups of G*, but not necessarily a Sylow system. $\mathfrak{S}\theta$ will then be a system of permutable S-subgroups of

$G\theta$, the relation of permutability being preserved by homomorphisms. With this understanding, the crucial result on which Theorem 7.2 depends may be stated as follows.

THEOREM 7.3. *If N is the normalizer of $\mathfrak{S}$ in L, then $N\theta$ is the normalizer of $\mathfrak{S}\theta$ in $L\theta$.*

Let L_0 denote the self-conjugate subgroup of L consisting of all the elements P for which $P\theta = 1$. Thus $L\theta \simeq L/L_0$.

We shall consider first the special case in which the meet $L_0 \wedge G = 1$. In this case, every element of L_0 is permutable with every element of G. Hence N contains L_0. $P\theta$ transforms $\mathfrak{S}\theta$ into itself if and only if P transforms $L_0\mathfrak{S}$ into itself. (By $L_0\mathfrak{S}$ we mean the system formed by the products of L_0 with the members of $\mathfrak{S}$.) But $\mathfrak{S}$ is the meet of $L_0\mathfrak{S}$ with G, and since G is self-conjugate in L, every such P must also transform $\mathfrak{S}$ into itself. Thus P belongs to N whenever $P\theta$ leaves $\mathfrak{S}\theta$ invariant. The converse being obvious, the result follows.

We may take next another particular case, that in which L_0 is a prime-power subgroup of G. Let the order of L_0 be a power of the prime p_i, and let S denote the join (or, owing to the permutability, we may also say the product) of all those members of $\mathfrak{S}$ whose orders are prime to p_i. If there are none, we may put $S = 1$. Let P be any element of L transforming $L_0\mathfrak{S}$ into itself, that is, such that $P\theta$ belongs to the normalizer of $\mathfrak{S}\theta$. Then every member of $\mathfrak{S}$ whose order involves p_i must contain L_0 and is therefore left invariant by P. On the other hand, if $P^{-1}SP = S'$, we must have $L_0S = L_0S'$. S and S' are obviously S-subgroups of the same order of the group L_0S. They are therefore conjugate in L_0S; or, more precisely, conjugate under L_0. Thus there exists an element Q of L_0 such that $Q^{-1}SQ = S'$. Then PQ^{-1} leaves invariant both $L_0\mathfrak{S}$ and also S. Hence it must leave invariant the system of meets $L_0\mathfrak{S} \wedge S$. But these are precisely the members of $\mathfrak{S}$ whose orders are prime to p_i. Since Q obviously leaves invariant the members of $\mathfrak{S}$ which contain L_0, it follows that PQ^{-1} leaves all members of $\mathfrak{S}$ invariant, that is to say, it belongs to N. Hence P belongs to L_0N, and $P\theta$ to $N\theta$. The normalizer of $\mathfrak{S}\theta$ must therefore be contained in $N\theta$. Since, on the other hand, every element of $N\theta$ obviously transforms $\mathfrak{S}\theta$ into itself, the theorem is proved in the present case.

To conclude the proof of Theorem 7.3, we need show only that every homomorphism θ of L may be expressed as the resultant of a finite sequence of homomorphisms θ_i, each of which is of one or other of the two particular kinds already considered. If $\theta = \theta_1\theta_2 \dots \theta_\lambda$, and if we write $\phi_i = \theta_1\theta_2 \dots \theta_i$, and denote by L_i the self-conjugate subgroup of L consisting of all the elements P for which $P\phi_i = 1$, then we must always have either $L_i \wedge G$

contained in L_{i-1}, or else L_i/L_{i-1} of prime-power order and $L_{i-1}G$ containing L_i. This may be arranged by choosing $L_\lambda = L_0$, $L_{\lambda-1} = L_\lambda \wedge G$, and then $L_1, L_2, \ldots, L_{\lambda-2}$ may be chosen such that

$$1 < L_1 < L_2 < \ldots < L_{\lambda-1}$$

is a characteristic series of $L_{\lambda-1}$. Since $L_{\lambda-1}$ is self-conjugate in L, it then follows that the L_i are also all self-conjugate in L. And further, owing to the solubility of $L_{\lambda-1}$, each L_i/L_{i-1} is of prime-power order, for $i < \lambda$.

To prove Theorem 7.2, let H/K be any factor of L which is central for G, let $\mathfrak{S}$ be any Sylow system of G and let N be the normalizer of $\mathfrak{S}$ in L. Then we must show that N covers H/K. And the corresponding result for a factor hypercentral for G will be an immediate corollary.

Consider the homomorphism $P \to KP$ which transforms L into L/K. Denoting this homomorphism by θ, we know from Theorem 7.3 that $N\theta$ is the normalizer of $\mathfrak{S}\theta$ in $L\theta = L/K$. But $H\theta = H/K$ belongs to the centralizer of $G\theta$. $H\theta$ is therefore certainly contained in the normalizer of every Sylow system of $G\theta$, in particular $H\theta$ lies in $N\theta$. But $N\theta = NK/K$. Thus NK contains H, and so N covers H/K.

8. *Anzahl theorems.*

Comparing Theorems 6.1 and 7.1, we obtain the most complete information possible concerning the incidence relations of self-conjugate subgroups and system normalizers. In particular, for the order of the latter, the remark immediately following definition 5.3 yields

THEOREM 8.1. *The order of the absolute system normalizers of a soluble group G is equal to the product of the orders of all the central chief factors which occur in any given chief series of G.*

And for the index of the system normalizers we have correspondingly

THEOREM 8.2. *The number $\varpi(G)$ of Sylow systems of G is equal to the product of the orders of all the eccentric chief factors which occur in any given chief series of G.*

And as a corollary of this, for the index of the normalizers of a Sylow p_i-complement, we have

THEOREM 8.3. *The number $\varpi_i(G)$ of Sylow p_i-complements of G is equal to the product of the orders of all the eccentric chief factors of G which occur in a given chief series of G and have order a power of p_i.*

That these numbers must be independent of the particular chief series chosen follows not only from the results themselves, but also from the Jordan-Hölder theorem in its operator form† (with inner automorphisms of G as operators).

We may recall the fact that the order of an eccentric chief factor is always congruent to 1 modulo some prime divisor of the order of G (I, Theorem 3.7). Thus we obtain the following result which is analogous to the usual form of Sylow's Anzahl theorem:

THEOREM 8.4. *$\varpi(G)$ and the $\varpi_i(G)$ are products of prime-powers each of which is congruent to 1 modulo some prime divisor of the order of G.*

In I (2.4), we proved the corresponding result for h_m, the number of S-subgroups of order m in G. There it was shown that the prime-power factors of h_m, which are in question, each divide the order of some chief factor of G. We now know that, in the particular case of the Sylow complements, these factors of $h_m = \varpi_i$ are each equal to the order of some chief factor of G. This result cannot be extended to the general S-subgroup. For example, in the primitive soluble group of degree 8 and order 168, the normalizers of the Sylow subgroups of order 3 are of order 6, and neither cover nor avoid the unique minimal self-conjugate subgroup of order 8; in this group, therefore, $h_3 = 28$, while the orders of the eccentric chief factors are 7 and 8.

If for every chief factor H/K of the group of finite order L, the subgroup M of L either covers or avoids H/K, then M must stand in a rather special relation to L which would probably repay further investigation. Among the subgroups M with this property, we may mention (i) the self-conjugate subgroups and (ii) the system normalizers relative to L of the self-conjugate soluble subgroups. If L itself is soluble, we may add (iii) the S-subgroups of the self-conjugate subgroups of L, and (iv) the maximal subgroups of L. But, as we have just seen, the normalizers of S-subgroups (except in the case of Sylow complements) do not as a rule have the property in question.

If G is a self-conjugate subgroup of L, then L/G is obviously central for G. Hence any relative system normalizer M of G in L must cover L/G. This follows either from Theorem 7.2, or more simply from the conjugacy property of the Sylow systems of G. In any case, we have therefore

$$L = GM; \tag{8.51}$$

(8.52) the meet of M with G is an absolute system normalizer N of G;

† van der Waerden, *Moderne algebra*, 1, 2nd ed., §§ 43–46.

(8.53) the order of M is equal to the order of N multiplied by the index $(L:G) = (M:N)$.

It is also easy to obtain Theorems 4.4 and 4.5 from the covering and avoidance theorems. Thus, if H is self-conjugate in G, we deduce that $\varpi(G)/\varpi(H)$ is equal to the product of the orders of those chief factors in any given chief series of G which are eccentric for G but central for H. If K/K' is such a factor, every system normalizer M of H relative to G must cover K/K'. And further, since $G = HM$, $(M \wedge K)/(M \wedge K')$ must be an eccentric chief factor of M of the same order as K/K'. Since, on the other hand, a central chief factor L/L' of G necessarily gives rise to a central chief factor $(M \wedge L)/(M \wedge L')$ of M, while the chief factors of G which are eccentric for H are all avoided by M, it follows that the quotient $\varpi(G)/\varpi(H)$ is precisely the product of the orders of the eccentric chief factors of M in a certain chief series of M, and that is $\varpi(M)$.

Finally, for the general case of an arbitrary self-conjugate subgroup and an arbitrary relative system normalizer, we have

THEOREM 8.6. *Let G and H be self-conjugate subgroups of any group L, let M be their meet, and suppose that G is soluble and $\mathfrak{S}$ a Sylow system of G. Then, if $\mathfrak{S}$ is transformed in turn by all the elements of H, the number of distinct Sylow systems of G that will arise is equal to the product of the orders of all the eccentric chief factors of G which occur in any given chief series of G and which are covered by M.*

In applying this result, we may always use a chief series of G containing M as a member.

To prove it, we may first observe that the number of Sylow systems in question is equal to the index in H of the meet $H \wedge N_L(\mathfrak{S})$. But, by Theorem 7.2, $N_L(\mathfrak{S})$ covers H/M, since this factor is central for G. Thus we require the index in M of the meet $M \wedge N_L(\mathfrak{S}) = M \wedge N_G(\mathfrak{S})$. Every central chief factor K/K' of G is covered by $N_G(\mathfrak{S})$, by Theorem 7.1. If K/K' is also covered by M, then $(M \wedge K)/(M \wedge K')$ is also a central chief factor of G, and hence this also is covered by $N_G(\mathfrak{S})$; that is to say, $M \wedge N_G(\mathfrak{S})$ covers K/K'. On the other hand, by Theorem 6.1, $N_G(\mathfrak{S})$ avoids all eccentric chief factors of G, whether covered by M or not. Hence the order of $M \wedge N_G(\mathfrak{S})$ is equal to the product of the orders of all central chief factors of G in the given chief series which are covered by M. The order of M itself is the product of the orders of *all* the chief factors of G in the given chief series which are covered by M. Hence the result required follows.

9. *The hypercentral as a meet.*

The definition of the hypercentral given in § 5 is essentially of a commutatorial character. We should expect to find an equivalent definition in terms of "Sylow structure". This is provided by the following

THEOREM. *The hypercentral of a soluble group G is*

(9.1) *the meet of all the absolute system normalizers of G,*

(9.2) *the meet of all the normalizers of the Sylow complements of G,*

(9.3) *the meet of all the normalizers of the Sylow subgroups of G, and finally*

(9.4) *the meet of all the self-normalizing subgroups of G.*

Further, (9.3) *and* (9.4) *hold in any group of finite order.*

Let H be any subgroup of G not containing the hypercentral Z^*. If Z_i is the last term of the upper central series of G to be contained in H, then we must have $Z_i < Z_{i+1}$. The commutator of any element of Z_{i+1} with any element of G lies in Z_i and therefore in H. Thus Z_{i+1} belongs to the normalizer of H, but not to H itself. Hence the self-normalizers of G all contain Z^*. In particular, since the normalizers of the Sylow subgroups of G are self-normalizers, their meet Z' must contain Z^*, and so now Theorem 9.4 follows from Theorem 9.3.

To prove the latter, let P and Q be any two elements of Z', of coprime prime-power orders. Then their commutator must be the identity. Thus $Z' = Z_1' \times Z_2' \times \ldots \times Z_r'$ is the direct product of its Sylow subgroups Z_i'. If next P is any element of Z_1' and R any element of G of order a power of p_2, then the commutator $T = R^{-1}P^{-1}RP$ must lie in Z_1', since Z_1' is self-conjugate in G and contains P. But T also lies in every Sylow p_2-subgroup of G which contains R. Hence $T = 1$, and every element of Z_1' is permutable with every element of G whose order is prime to p_1. Thus the centralizer K_1 of Z_1' in G is of index a power of p_1. Thus G transforms the p_1-group Z_1' in accordance with a p_1-group of automorphisms G/K_1. Hence $Z_1'/1$ is a hypercentral factor of G. Similarly for the other Sylow subgroups Z_k'. Thus Z' itself must also lie in Z^*. Combining this with our previous result, we obtain $Z' = Z^*$, that is Theorem 9.3.

This result may be regarded as a generalization of the theorem of Burnside referred to in § 1.

If G is soluble, and $\mathfrak{S}$ one of its Sylow systems, then the system normalizer $N_G(\mathfrak{S})$ is the meet of the normalizers of the Sylow subgroups of $\mathfrak{S}$ and also the meet of the normalizers of the Sylow complements of $\mathfrak{S}$. Hence in this case Theorem 9.2 and Theorem 9.3 follow from Theorem 9.1. To prove Theorem 9.1, we note that by Theorem 7.1 the system normalizers of G all contain Z^*, and so their meet Z'' contains Z^*. But Z'' is self-conjugate in G, and is evidently covered by every system normalizer. Hence, by Theorem 6.1, Z'' can contain no eccentric chief factor of G. Thus Z'' is decomposable into central factors of G, so that it must be contained in Z^*. Combining these two results, we obtain $Z'' = Z^*$, that is, Theorem 9.1.

In the case of soluble G, we may also define Z^* as the meet of all subgroups M of G having the property (a) of Theorem 4.8.

These definitions suggest a comparison of the hypercentral of a group with three other interesting characteristic subgroups which are also definable in a similar way, viz., as meets of all subgroups of a given kind. These are (i) the central, which is the meet of all the centralizers; (ii) the *kern* of R. Baer, which is the meet of all the normalizers of subgroups; and (iii) the subgroup of Frattini, which is related to the systems of independent generators of the given group, and which is the meet of all the maximal subgroups. This list could easily be extended to include, *e.g.* the various types of commutator subgroup, but the latter are distinguished from the three subgroups mentioned above and from the hypercentral in that they are most naturally regarded as meets of certain self-conjugate subgroups whose *quotients* are of a specified kind.

THEOREM 9.5. *If G is a soluble self-conjugate subgroup of the group L, and M is the meet of the relative system normalizers of G in L, then the meet $M \wedge G = Z^*$, the hypercentral of G; and M/Z^* is the centralizer of G/Z^* in L_1/Z^*. M contains every self-conjugate subgroup of L whose meet with G lies in Z^*.*

The proof depends merely on Theorem 6.2 and Theorem 7.2, and may be omitted.

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AN INTRODUCTION TO THE THEORY OF OPERATORS (II): A THEOREM ON INTEGRATION

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This paper follows No. I* of the series (a knowledge of which is assumed). Here we prove the following theorem: *If $F(A)$ is a function (of a special type, called direct) of an operator A, then*

$$F(A) = \int_{-\infty}^{\infty} F(\lambda)\, dE_\lambda, \quad \textit{where} \quad A = \int_{-\infty}^{\infty} \lambda dE_\lambda.$$

Some restriction on the type of function $F(A)$ seems unavoidable, since in the general case $F(\lambda)$ is an operator and not a number; [*e.g.*

$$F(\lambda) = \cos(A\lambda + B),$$

where A and B are operators]. Since operators do not form an ordered algebra, the theory of measure has no immediate generalization. The integral that we employ corresponds to the Riemann integral and hence there is a restriction on the type of function that it integrates. If we dealt only with the particular case where $F(\lambda)$ is a number then it would be possible to extend the Lebesgue theory to integrals of the form $\int_{-\infty}^{\infty} F(\lambda)\, dE_\lambda$. This extension is not considered here since we wish to deal with the more general case when $F(\lambda)$ is not a number but an operator.

We show further that, in certain circumstances,

$$\frac{1}{F(A)} = \int_{-\infty}^{\infty} \frac{1}{F(\lambda)}\, dE_\lambda,$$

where in general the integral is of the type that we shall call improper.

It is useful to make the following remark here. If $E^2 = E$, and $E = E_1 + E_2$, $E_1 E_2 = O$, $E_1 > O$, $E_2 > O$, then we call E imprimitive, otherwise primitive ($E > O$). When $E = I$ then the cardinal number of

* Steen, *Proc. London Math. Soc.* (2), 41 (1936), 361–392. See also Freudenthal, *Konin. Akad. v. Wetensch. te Amsterdam*, 39 (1936), 641–651.

mutually orthogonal primitive projections* whose sum is I will be called the dimension number of I. No axiom about dimension number has been introduced since cases when this number is finite, $\aleph_0$ or $2^{\aleph_0}$ are known: viz. sets of permutable Hermitian matrices of order n, sets of self-adjoint operators in Hilbert space (a) when separable, (b) when non-separable. The lack of such an axiom has materially affected the enunciations and the proofs of some of the lemmas.

The numbers of the paragraphs follow those in paper No. I, and the notation is the same. The theorem $A = \int_{-\infty}^{\infty} \lambda dE_\lambda$ will be called Theorem I.

The reader is reminded that the set of real operators considered in these papers is a set of entities obeying a system of axioms similar to those satisfied by real numbers, the main difference being that the trichotomy is not assumed; on the other hand multiplication is commutative.

XV.—*Properties of limits.*

LEMMA (i). *A necessary and sufficient condition that* $A_n \to A$ *is that* (i) *the sequence is bounded,* (ii) *if* $H > O$, *then, for some* $O < K \leqslant H$,

$$AK - K^2 \leqslant A_n K \leqslant AK + K^2, \quad \textit{for} \quad n > n_0(H)\dagger.$$

Write $\bar{A}_n = \underset{p \geqslant 0}{\text{l.u.b.}} (A_{n+p})$, and $\underline{A}_n = \underset{p \geqslant 0}{\text{g.l.b.}} (A_{n+p})$.

Suppose $A_n \to A$, then $A = \text{g.l.b.}(\bar{A}_n) = \text{l.u.b.}(\underline{A}_n)$; this implies the existence of $\bar{A}_n$ and $\underline{A}_n$, and hence the inequality $\underline{A}_1 \leqslant A_n \leqslant \bar{A}_1$ for all n, and hence the boundedness of the sequence A_n. Further, given $H > O$, there is an integer n such that $A - H \not\geqslant \underline{A}_n$. Hence $(\underline{A}_n - A + H)_+ > O$. There is then a projection $E \neq O$ such that $E(\underline{A}_n - A + H) > O$. Write $K' = HE$; then $K' > O$ [for $\underline{A}_n \leqslant A$ and $E(\underline{A}_n - A) + K' > O$], and $K' A_{n+p} \geqslant K' \underline{A}_n \geqslant K' A - K'^2$, for $p \geqslant o$. Again $A + K' \not\leqslant \bar{A}_m$ for some

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* *I.e.* if there are any such projections.

† As explained in the introduction to the first paper of this series real operators can be represented by real measurable functions defined in $(-\infty, \infty)$. $A_n \to A$ is then represented by $F_n(\lambda) \to F(\lambda)$ for all λ. But this does not imply that, if $K(\lambda) \geqslant o$ for all λ, the

$$F(\lambda) - K(\lambda) \leqslant F_n(\lambda) \leqslant F(\lambda) + K(\lambda), \quad \text{for} \quad n > n_0,$$

for all λ. Thus the theorem that, if $A_n \to A$ and if $K > O$, then

$$A - K \leqslant A_n \leqslant A + K, \quad \text{for} \quad n < n_0,$$

is false.

integer m; as before, there is a projection $E' \neq O$ such that

$$E'(A+K'-\bar{A}_m) > O.$$

Write $K = K'E'$; then $K > O$, and $AK+K^2 \geqslant \bar{A}_m K \geqslant A_{m+p} K$, for $p \geqslant o$. The result now follows on multiplying by E' the inequality which was first obtained.

Now suppose that A_n is a bounded sequence and that condition (ii) of the lemma is satisfied. Then, if $\underline{A} = \underline{\lim}\, A_n$ and $\bar{A} = \overline{\lim}\, A_n$, we have

$$AK-K^2 \leqslant \underline{A}_n K \leqslant \underline{A}K \leqslant \bar{A}K \leqslant \bar{A}_n K \leqslant AK+K^2 \quad [n > n_0(H)].$$

Hence $(\bar{A}-\underline{A})K \leqslant 2K^2$. If $\bar{A} > \underline{A}$, take $H = \frac{1}{3}(\bar{A}-\underline{A})$; then

$$O < 2K^2 \leqslant \tfrac{2}{3}(\bar{A}-\underline{A})K < (\bar{A}-\underline{A})K,$$

which is a contradiction. Thus $\bar{A} = \underline{A}$, and so $\lim A_n$ exists. If

$$\lim A_n \neq A,$$

then either $(\lim A_n - A)_+$ or $(\lim A_n - A)_-$ is not zero. Suppose that

$$(\lim A_n - A)_+ > O;$$

then there is a projection $E \neq O$ such that $E \lim A_n > EA+\delta E$ for some $\delta > o$ (for $\lim A_n - A - \delta I \not\leqslant O$ for all $\delta > O$). If we take $2H = \delta E$, then for any $O < K \leqslant H$ we have $K \lim A_n \geqslant KA+\delta K > KA+K^2$, which is a contradiction Hence $(\lim A_n - A)_+ = O$; similarly $(\lim A_n - A)_- = O$, and so $\lim A_n = A$.

LEMMA (ii). *A necessary and sufficient condition that $A_n \to A$ is that,* (i) *the sequence is bounded,* (ii) *if $B \not\leqslant A \not\leqslant C$, then $B \not\leqslant \bar{A}_n$ and $\underline{A}_n \not\leqslant C$, for $n > n_0(B, C)$. If A_n and A are complete and positive we can take $B \geqslant O$ and $C \geqslant O$.*

If $A_n \to A$, then $\underline{A}_n \leqslant A \leqslant \bar{A}_n$. Let $B \not\leqslant A$; then there is an integer n such that $\bar{A}_n \not\geqslant B$, and so for $p \geqslant o$, $\bar{A}_{n+p} \not\geqslant B$. Similarly $\underline{A}_{m+p} \not\leqslant C$ for some integer m and $p \geqslant o$.

If $B \not\leqslant \bar{A}_n$ and $\underline{A}_n \not\leqslant C$ for $n > n_0(B, C)$, then $B \not\leqslant \bar{A}$ and $\underline{A} \not\leqslant C$. If $A \not\geqslant \bar{A}$, then there is a projection E such that $E\bar{A} > EA$, and hence an operator $B \not\leqslant A$ such that $EA < EB < E\bar{A}$, $B < \bar{A}$; this is a contradiction. Hence $A \geqslant \bar{A}$. If $A > \bar{A}$, then we can find an operator $C > \bar{A}$ and $C \not\geqslant A$, whence $\underline{A} \not\leqslant C$; but these results imply that $\underline{A} \not\leqslant \bar{A}$, which is false; hence $A = \bar{A}$. Similarly $A = \underline{A}$.

LEMMA (iii). *A necessary and sufficient condition that A_n should tend to a limit is that,* (i) *the sequence is bounded,* (ii) $|A_{n+p}-A_n|\to O$ *uniformly in* p; *i.e. given* $H>O$, *then, for some* $O<K\leqslant H$, *we have*

$$|A_{n+p}-A_n|K\leqslant K^2, \quad \textit{for} \quad n>n_0(K), \quad p\geqslant o.$$

If $A_n\to A$, then by Lemma (i), given $H>O$, there is a K, $O<K\leqslant H$, such that

$$AK-K^2\leqslant A_{n+p}K\leqslant AK+K^2$$

for $n>n_0(H)$ and $p\geqslant o$. Hence

$$-2K^2\leqslant(A_{n+p}-A_n)K\leqslant 2K^2.$$

If we write $2K=K'$, we obtain the required result with $2H$ for H. If

$$-K^2\leqslant(A_{n+p}-A_n)K\leqslant K^2$$

for some $O<K\leqslant H$, $\quad n>n_0(H)$, $\quad p\geqslant o$, then

$$-K^2+A_nK\leqslant A_{n+p}K\leqslant A_nK+K^2.$$

Thus $$-K^2+A_nK\leqslant \underline{A}_nK\leqslant \bar{A}_nK\leqslant A_nK+K^2,$$

so that $(\bar{A}-\underline{A})K\leqslant 2K^2$, and the result follows as in the proof of Lemma (i).

LEMMA (iv). *If* $A_n=A$, *then* $\lim A_n=A$.

LEMMA (v). *If* $A_n\to A$, *then* $-A_n\to -A$.

These two results follow at once from the definition of a limit.

LEMMA (vi). *If* $A_n\to A$ *and* $B_n\to B$, *then* $A_n+B_n\to A+B$.

It is clear that the sequence A_n+B_n is bounded if the two sequences A_n and B_n are both bounded. It remains to show that, given $H>O$, there is a K, $O<K\leqslant H$, such that

$$(A+B)K-K^2\leqslant(A_n+B_n)K\leqslant(A+B)K+K^2, \quad \text{for} \quad n>n_0(H).$$

Now we have

$$AK'-K'^2\leqslant A_nK'\leqslant AK'+K'^2 \quad \text{for} \quad O<K'\leqslant H, \quad n>n_0(H);$$

also for some K, $O<K\leqslant K'$, and $n>n_0(K')$, we have

$$BK-K^2\leqslant B_nK\leqslant BK+K^2.$$

Further [*vide* proof of Lemma (i)] we may take $K = EK'$, and thus obtain

$$AK - K^2 \leqslant A_n K \leqslant AK + K^2, \quad \text{for} \quad n > n_0(H).$$

By adding the last two sets of inequalities the result follows with $2H$ for H.

LEMMA (vii). *If* $A_n \to A$, *then* $BA_n \to BA$.

Clearly the sequence BA_n is bounded if the sequence A_n is bounded. If $B > O$ and B is complete, then $\overline{BA}_n = B.\overline{A}_n$ and $\underline{BA}_n = B.\underline{A}_n$. If $V \nleqslant BA$, then $V/B \nleqslant A$, and so $V/B \nleqslant \bar{A}_n$, *i.e.* $V \nleqslant \overline{BA}_n$. Similarly, if $W \ngeqslant BA$, then $W \ngeqslant \underline{BA}_n$; the result now follows from Lemma (ii). Secondly, if $B > O$, then $B + I > O$ and is complete. Hence, using the result just proved and Lemmas (v) and (vi), we have

$$BA_n = (B+I)A_n - A_n \to (B+I)A - A = BA.$$

Lastly, if $B = B_+ - B_-$, then $BA_n = B_+ A_n - B_- A_n \to B_+ A - B_- A = BA$.

LEMMA (viii). *If* $A_n \to O$, *then* $|A_n| \to O$ *and conversely.*

Suppose that $A_n \to O$; then $\underline{A}_n \leqslant O \leqslant \bar{A}_n$, and $\underline{A}_n \leqslant A_{n+p} \leqslant \bar{A}_n$, for $p \geqslant O$; thus $|A_{n+p}| \leqslant \bar{A}_n - \underline{A}_n$, and so $\overline{|A_n|} \leqslant \bar{A}_n - \underline{A}_n$. Thus

$$O \leqslant \text{g.l.b.}\,\overline{|A_n|} \leqslant \text{g.l.b.}\,(\bar{A}_n - \underline{A}_n) = O$$

(for $\underline{A}_n - \bar{A}_n$ is a decreasing sequence and its g.l.b. is its limit), thus $|A_n| \to O$.

If $|A_n| \to O$, then $A_{(n+p)+} \leqslant \overline{|A_n|}$, and so $\overline{A_{n+}} \leqslant \overline{|A_n|}$, hence $A_{n+} \to O$. Similarly $A_{n-} \to O$, and the result follows.

LEMMA (ix). *If* $A_n \to O$ *and* $B_n \to O$, *then* $A_n B_n \to O$.

It is sufficient to prove that $|A_n B_n| \to O$ when $|A_n| \to O$ and $|B_n| \to O$. We have $|A_{n+p} B_{n+p}| \leqslant \overline{|A_n|}\,\overline{|B_n|}$, and so

$$\overline{|A_n B_n|} \leqslant \overline{|A_n|}\,\overline{|B_n|} \leqslant \overline{|A_n|}\,\overline{|B_{n_0}|}, \quad \text{if} \quad n > n_0.$$

The result now follows since, by Lemma (vii), $\lim A_n \overline{|B_{n_0}|} = O$.

LEMMA (x). *If* $A_n \to A$ *and* $B_n \to B$, *then* $A_n B_n \to AB$.

It is sufficient to prove that $A_n B_n - AB \to O$. We have

$$A_n B_n - AB = (A_n - A)(B_n - B) + (A_n - A)B + (B_n - B)A;$$

the result now follows from Lemmas (ix) and (vi).

LEMMA (xi). *If* $A_n \to A$, *then* $|A_n| \to |A|$.

Now $|A| = \sqrt{A^2}$; further $A_n^2 \to A^2$; hence, if $H > O$, there is an operator H_1 such that $O < H_1 \leqslant H^2$ and $H_1 < A^2$, if $HA \neq O$. Suppose first that $HA \neq O$; then there is an operator K, $O < K \leqslant H_1$, such that $A^2K - K^2 \leqslant A_n^2 K \leqslant A^2K + K^2$, for $n > n_0(H_1)$. Write $K = G^2$, $G > O$, then $A^2K + K^2 \leqslant (|A|G + G^2)^2$. Further, $A^2K - K^2 \geqslant (|A|G - G^2)^2$, for this is $2|A|G^3 \geqslant 2G^4$, and this is true since $O < G^2 \leqslant H_1 < A^2$, whence $G < |A|$. Also $|A|G - G^2 \geqslant 0$. Hence, for $n > n_0(H_1)$, we have

$$|A|G - G^2 \leqslant |A_n|G \leqslant |A|G + G^2,$$

and $O < G^2 \leqslant H^2$, *i.e.* $O < G \leqslant H$.

If $HA = O$, then $H^2A^2 = O$, and so $K^2A^2 = O$, whence $GA = O$; thus $-K^2 \leqslant A_n^2 K \leqslant K^2$, or $-G^2 \leqslant |A_n|G \leqslant G^2$; this is the same as

$$|A|G - G^2 \leqslant |A_n|G \leqslant |A|G + G^2.$$

Thus in each case the desired result follows.

LEMMA (xii). *If* $B_n \to B$, *and if* B_n *and* B *are complete, then*

$$\frac{1}{B_n} \to \frac{1}{B}.$$

If $B_n > O$, then $B > O$. Thus, if $V > O$ and $W > O$ and if $V \nleqslant B \nleqslant W$, then $V \nleqslant \overline{B}_n$ and $\underline{B}_n \nleqslant W$; but, from these inequalities it follows that, if

$$\frac{1}{W} \nleqslant \frac{1}{B} \nleqslant \frac{1}{V},$$

then $1/W \nleqslant \overline{1/B_n}$, and $\underline{1/B_n} \nleqslant 1/V$, and the result follows. If $B_n \not> O$ for an ∞ of n it is sufficient to prove that $|(B - B_n)/BB_n| \to O$; to do this it is sufficient to show that $1/|B_n| \to 1/|B|$, and this follows by the result just proved and Lemma (xi).

Similar lemmas may be proved for sets of operators A_x when $x \to \infty$ or $x \to a$. The proofs are similar to those given and so are omitted.

XVI. *Continuity of functions.*

Def. (i). If $F(\lambda) = F(\lambda I)$ is an operator defined for $\lambda_1 \leqslant \lambda \leqslant \lambda_2$, then we say that $F(\lambda)$ is continuous at λ if $F(\mu) \to F(\lambda)$ as $\mu \to \lambda$.

LEMMA (i). *If* $F(\lambda)$ *and* $G(\lambda)$ *are continuous at* λ, *then so are* $F(\lambda) \pm G(\lambda)$, $AF(\lambda)$, $F(\lambda).G(\lambda)$ *and, if* $G(\mu)$ *is complete for all* μ *in* $\lambda - \delta \leqslant \mu \leqslant \lambda + \delta$ *for some* $\delta > o$, *then* $F(\lambda)/G(\lambda)$ *is continuous at* λ.

Def (ii). If $F(\lambda)$ is continuous at all λ in $\lambda_1 \leqslant \lambda \leqslant \lambda_2$, then it is said to be continuous in the interval $\lambda_1 \leqslant \lambda \leqslant \lambda_2$.

Def. (iii). If $F_n(\lambda) \to F(\lambda)$ as $n \to \infty$ for each λ in the interval $\lambda_1 \leqslant \lambda \leqslant \lambda_2$, then we say that $F_n(\lambda) \to F(\lambda)$ uniformly in $\lambda_1 \leqslant \lambda \leqslant \lambda_2$ if, given $H > O$, there is an operator K, $O < K \leqslant H$, such that

$$F(\lambda)\,K - K^2 \leqslant F_n(\lambda)\,K \leqslant F(\lambda)\,K + K^2,$$

for $n > n_0(H)$, where $n_0(H)$ is independent of λ and where λ is in the interval $\lambda_1 \leqslant \lambda \leqslant \lambda_2$.

LEMMA (ii). *If $F_n(\lambda)$ is, for each integer n, continuous in the interval $\lambda_1 \leqslant \lambda \leqslant \lambda_2$, and if the sequence $F_n(\lambda)$ is uniformly convergent to $F(\lambda)$ in this interval, then $F(\lambda)$ is continuous in the interval $\lambda_1 \leqslant \lambda \leqslant \lambda_2$.*

We have to show that $|F(\mu) - F(\lambda)| \to O$ as $\mu \to \lambda$. Given λ, $\lambda_1 \leqslant \lambda \leqslant \lambda_2$, we have

$$|F(\mu) - F(\lambda)| \leqslant |F_n(\mu) - F_n(\lambda)| + |F_n(\lambda) - F(\lambda)| + |F_n(\mu) - F(\mu)|.$$

If $H > O$, there is a K, $O < K \leqslant H$, such that $|F_n(\mu) - F(\mu)|\,K \leqslant K^2$, for $n > n_0(H)$ and all μ, $\lambda_1 \leqslant \mu \leqslant \lambda_2$. Fix such an n; then there is a $G = EK > O$, where E is a projection, such that $|F_n(\mu) - F_n(\lambda)|\,G \leqslant G^2$, for $|\mu - \lambda| < \delta(K)$. Thus $|F(\mu) - F(\lambda)|\,G \leqslant 3G^2$, and the result follows.

Def. (iv). We say that $F(\lambda)$ is uniformly continuous in $\lambda_1 \leqslant \lambda \leqslant \lambda_2$ if $F(\mu) \to F(\lambda)$ uniformly in λ; *i.e.* if for $H > O$ there is a G, $O < G \leqslant H$, such that $|F(\mu) - F(\lambda)|\,G \leqslant G^2$, whenever $|\mu - \lambda| < \delta(H)$ and when μ and λ lie in the closed interval (λ_1, λ_2).

LEMMA (iii). *If $F(\lambda)$ and $G(\lambda)$ are uniformly continuous in (λ_1, λ_2), then so are $F(\lambda) \pm G(\lambda)$, $AF(\lambda)$, $F(\lambda)\,.\,G(\lambda)$, and if $G(\mu)$ is complete for $|\mu - \lambda| \leqslant \delta$, then $F(\lambda)/G(\lambda)$ is uniformly continuous.*

LEMMA (iv). *λI is uniformly continuous.*

LEMMA (v). *If $F_n(\lambda)$ is, for each n, uniformly continuous in λ for $\lambda_1 \leqslant \lambda \leqslant \lambda_2$, and if the sequence $F_n(\lambda)$ is uniformly convergent to $F(\lambda)$ in $\lambda_1 \leqslant \lambda \leqslant \lambda_2$, then $F(\lambda)$ is uniformly continuous in $\lambda_1 \leqslant \lambda \leqslant \lambda_2$.*

In Definition (iv) we may take $G = EH$. For if $|A|G \leqslant G^2$, and $O < G \leqslant H$, let $E = \text{g.l.b.}\,(E')$, $GE' = G$; then $GE \leqslant HE$. If $GE < HE$, let $B \geqslant I$ be such that $BG = HE$; then $|A|BG \leqslant BG^2 \leqslant (BG)^2$, which is the required result. The proof of Lemma (v) follows on the same lines

as the proof of Lemma (ii), for when n is fixed $G = HE$ is independent of λ, if

$$|\mu - \lambda| < \delta(n, H).$$

XVII. *Properties of integrals.*

LEMMA (i). *If* $\int_{-\infty}^{\infty} F(\lambda)\,dA_\lambda$ *exists, then it is the limit as* $\delta \to o$ *of* $\sum_{-\infty}^{\infty} F(\lambda'_n)\Delta_n$, *where* $\lambda_n \leqslant \lambda_n' \leqslant \lambda_{n+1}$, $\lambda_{n+1} - \lambda_n \leqslant \delta$, *and where for shortness we write* $\Delta_n = A_{\lambda_{n+1}} - A_{\lambda_n}$.

By the definition of an integral, $\Delta_n \geqslant O$; hence

$$\sum_{-\infty}^{\infty} F_{\underline{\Delta}_n}\Delta_n \leqslant \sum_{-\infty}^{\infty} F(\lambda_n')\Delta_n \leqslant \sum_{-\infty}^{\infty} F_{\overline{\Delta}_n}\Delta_n,$$

where*

$$F_{\underline{\Delta}_n} = \underset{\lambda_n \leqslant \lambda \leqslant \lambda_{n+1}}{\text{g.l.b.}} \big(F(\lambda)\big), \quad F_{\overline{\Delta}_n} = \underset{\lambda_n \leqslant \lambda \leqslant \lambda_{n+1}}{\text{l.u.b.}} \big(F(\lambda)\big).$$

The result now follows.

This is the Riemann-Stieltjes definition; the definition of §VIII was in terms of the upper and lower Darboux-Stieltjes integrals.

LEMMA (ii). *If* $\int_{-\infty}^{\infty} F(\lambda)\,dA_\lambda$ *and* $\int_{-\infty}^{\infty} G(\lambda)\,dA_\lambda$ *both exist, then so does*

$$\int_{-\infty}^{\infty} [F(\lambda) + G(\lambda)]\,dA_\lambda,$$

and

$$\int_{-\infty}^{\infty} F(\lambda)\,dA_\lambda + \int_{-\infty}^{\infty} G(\lambda)\,dA_\lambda = \int_{-\infty}^{\infty} [F(\lambda) + G(\lambda)]\,dA_\lambda.$$

This follows since

$$(F+G)_{\overline{\Delta}_n} \leqslant F_{\overline{\Delta}_n} + G_{\overline{\Delta}_n}, \quad F_{\underline{\Delta}_n} + G_{\underline{\Delta}_n} \leqslant (F+G)_{\underline{\Delta}_n}.$$

LEMMA (iii). $$A \int_{-\infty}^{\infty} F(\lambda)\,dA_\lambda = \int_{-\infty}^{\infty} AF(\lambda)\,dA_\lambda.$$

Write $A = A_+ - A_-$. By Lemma (ii) it is sufficient to prove the result for $A = A_+$. Now $(A_+F)_{\overline{\Delta}_n} = A_+ . F_{\overline{\Delta}_n}$, and $(A_+F)_{\underline{\Delta}_n} = A_+ . F_{\underline{\Delta}_n}$. Further

$$-\int_{-\infty}^{\infty} F(\lambda)\,dA_\lambda = \int_{-\infty}^{\infty} [-F(\lambda)]\,dA_\lambda.$$

For $(-F)_{\overline{\Delta}_n} = -F_{\underline{\Delta}_n}$, etc.

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* Note that the end points are included.

LEMMA (iv).

$$\int_{-\infty}^{\infty} F(\lambda)\,dA_\lambda = \int_{-\infty}^{\lambda} F(\lambda)\,dA_\lambda + \int_{\lambda}^{\infty} F(\lambda)\,dA_\lambda.$$

It is clear that any sum of the form $\sum_{-\infty}^{\infty} F_{\bar{\Delta}_n}\Delta_n'$ is greater than or equal to any sum of the form $\sum_{-\infty}^{\infty} F_{\underline{\Delta}_n'}\Delta_n'$, where $\Delta_n' = A_{\lambda_{n+1}'} - A_{\lambda_n'}$, so that Δ_n' refers to a different subdivision of the real axis. If $\delta = \max(\lambda_{n+1} - \lambda_n)$, then $\lim_{\delta\to 0} \sum_{-\infty}^{\infty} (F_{\bar{\Delta}_n} - F_{\underline{\Delta}_n})\Delta_n = O$, since we are supposing that $\int_{-\infty}^{\infty} F(\lambda)\,dA_\lambda$ exists. Now $(F_{\bar{\Delta}_n} - F_{\underline{\Delta}_n})\Delta_n \geqslant O$; hence $\sum_{-\infty}^{\lambda} (F_{\bar{\Delta}_n} - F_{\underline{\Delta}_n})\Delta_n \leqslant \sum_{-\infty}^{\infty} (F_{\bar{\Delta}_n} - F_{\underline{\Delta}_n})\Delta_n$, where we take (as we clearly may) λ as a point of subdivision, and $\sum_{-\infty}^{\lambda}$ means the sum over all Δ_n for which $\lambda_{n+1} \leqslant \lambda$. Thus $\lim_{\delta\to 0} \sum_{-\infty}^{\lambda} (F_{\bar{\Delta}_n} - F_{\underline{\Delta}_n})\Delta_n = O$. Now $\sum_{-\infty}^{\lambda} F_{\bar{\Delta}_n}\Delta_n$ decreases as we add new points of subdivision, and since it is bounded below $\left(\text{it is greater than } \sum_{-\infty}^{\lambda} F_{\underline{\Delta}_n'}\Delta_n', \text{ for a fixed subdivision } (\lambda_n')\right)$ it then tends to a limit as $\delta \to o$, and this limit is the same as the limit of $\sum_{-\infty}^{\lambda} F_{\underline{\Delta}_n}\Delta_n$. The common value of these two limits is then $\int_{-\infty}^{\lambda} F(\lambda)\,dA_\lambda$. In a similar manner $\sum_{\lambda}^{\infty} F_{\bar{\Delta}_n}\Delta_n \to \int_{\lambda}^{\infty} F(\lambda)\,dA_\lambda$. We now apply Lemma (vi) of § XV to the equation

$$\sum_{-\infty}^{\lambda} F_{\bar{\Delta}_n}\Delta_n + \sum_{\lambda}^{\infty} F_{\bar{\Delta}_n}\Delta_n = \sum_{-\infty}^{\infty} F_{\bar{\Delta}_n}\Delta_n$$

and obtain the required result.

LEMMA (v). *If* $F(\lambda) \geqslant O$, *then* $\int_{-\infty}^{\infty} F(\lambda)\,dA_\lambda \geqslant O$, *if it exists.*

For $F_{\underline{\Delta}_n} \geqslant O$.

LEMMA (vi). $\left|\int_{-\infty}^{\infty} F(\lambda)\,dA_\lambda\right| \leqslant \int_{-\infty}^{\infty} |F(\lambda)|\,dA_\lambda$, *if both integrals exist.*

For $F_{\bar{\Delta}_n} \leqslant |F|_{\bar{\Delta}_n}$ and $-(F_{\underline{\Delta}_n}) = (-F)_{\bar{\Delta}_n} \leqslant |F|_{\bar{\Delta}_n}$, and the result follows.

LEMMA (vii). *If* E_λ *is a resolution of the identity, and if the integrals* $\int_{-\infty}^{\infty} F(\lambda)\,dE_\lambda$ *and* $\int_{-\infty}^{\infty} G(\lambda)\,dE_\lambda$ *exist, then so does* $\int_{-\infty}^{\infty} F(\lambda)\,G(\lambda)\,dE_\lambda$, *and*

$$\int_{-\infty}^{\infty} F(\lambda)\,dE_\lambda \int_{-\infty}^{\infty} G(\lambda)\,dE_\lambda = \int_{-\infty}^{\infty} F(\lambda)\,G(\lambda)\,dE_\lambda.$$

If $\Delta_n = E_{\lambda_{n+1}} - E_{\lambda_n}$, then, if $F(\lambda) \geqslant O$, $G(\lambda) \geqslant O$ for all λ, we have

$$\sum_{-\infty}^{\infty} F_{\underline{\Delta}_n} \Delta_n \sum_{-\infty}^{\infty} G_{\underline{\Delta}_n} \Delta_n = \sum_{-\infty}^{\infty} F_{\underline{\Delta}_n} G_{\underline{\Delta}_n} \Delta_n \leqslant \sum_{-\infty}^{\infty} (FG)_{\underline{\Delta}_n} \Delta_n \leqslant \sum_{-\infty}^{\infty} (FG)_{\underline{\Delta}_n} \Delta_n$$

$$\leqslant \sum_{-\infty}^{\infty} F_{\overline{\Delta}_n} G_{\overline{\Delta}_n} \Delta_n = \sum_{-\infty}^{\infty} F_{\overline{\Delta}_n} \Delta_n \sum_{-\infty}^{\infty} G_{\overline{\Delta}_n} \Delta_n,$$

and the result follows. If $F(\lambda) \geqslant A$, $G(\lambda) \geqslant B$ for all λ, we have, by what has just been proved,

$$\int_{-\infty}^{\infty} \big(F(\lambda) - A\big)\, dE_\lambda \int_{-\infty}^{\infty} \big(G(\lambda) - B\big)\, dE_\lambda = \int_{-\infty}^{\infty} \big(F(\lambda) - A\big)\big(G(\lambda) - B\big)\, dE_\lambda\big),$$

and the result follows.

In the remaining case $\underset{\lambda_1 \leqslant \lambda \leqslant \lambda_2}{\text{g.l.b.}} \big(F(\lambda)\big)$ exists. Hence

$$\int_{\lambda_1}^{\lambda_2} F(\lambda)\, dE_\lambda \int_{\lambda_1}^{\lambda_2} G(\lambda)\, dE_\lambda = \int_{\lambda_1}^{\lambda_2} F(\lambda)\, G(\lambda)\, dE_\lambda.$$

We now let $\lambda_1 \to -\infty$ and $\lambda_2 \to +\infty$, then $\int_{\lambda_1}^{\lambda_2} F(\lambda)\, dE_\lambda \to \int_{-\infty}^{\infty} F(\lambda)\, dE_\lambda$, and the result follows.

LEMMA (viii). *If* $\int_{-\infty}^{\infty} G(\lambda)\, dE_\lambda$ *exists and if* $\underset{\lambda_1 \leqslant \lambda \leqslant \lambda_2}{\text{g.l.b.}} |G(\lambda)|$ *is complete for all* λ_1, λ_2, *then*

$$1 \Big/ \int_{-\infty}^{\infty} G(\lambda)\, dE_\lambda = \int_{-\infty}^{\infty} \frac{1}{G(\lambda)}\, dE_\lambda.$$

First suppose that $G(\lambda) > O$ for all λ. Then $G_{\underline{\Delta}_n}$ and $G_{\overline{\Delta}_n}$ are complete. Further, $\sum_{-\infty}^{\infty} G_{\underline{\Delta}_n} \Delta_n$ $(\Delta_n = E_{\lambda_{n+1}} - E_{\lambda_n})$ is complete; for, if $B \sum_{-\infty}^{\infty} G_{\underline{\Delta}_n} \Delta_n = O$, then $B G_{\underline{\Delta}_n} \Delta_n = O$, and so $B \Delta_n = O$, whence $B \sum_{-N}^{M} \Delta_n = O$, and so $BI = O$, *i.e.* $B = O$. Similarly, $\sum_{-\infty}^{\infty} G_{\overline{\Delta}_n} \Delta_n$ is complete. Now

$$\sum_{-N}^{M} \frac{1}{G_{\overline{\Delta}_n}} \Delta_n = \sum_{-N}^{M} \Delta_n \Big/ \sum_{-\infty}^{\infty} G_{\overline{\Delta}_n} \Delta_n \to 1 \Big/ \sum_{-\infty}^{\infty} G_{\overline{\Delta}_n} \Delta_n$$

as N, $M \to \infty$; thus $\sum_{-\infty}^{\infty} \frac{1}{G_{\underline{\Delta}_n}} \Delta_n$ exists and is equal to $\Big[\sum_{-\infty}^{\infty} G_{\overline{\Delta}_n} \Delta_n\Big]^{-1}$. Similarly,

$$\sum_{-\infty}^{\infty} \frac{1}{G_{\underline{\Delta}_n}} \Delta_n = \Big[\sum_{-\infty}^{\infty} G_{\underline{\Delta}_n} \Delta_n\Big]^{-1}.$$

Also $(1/G)_{\underline{\Delta}_n} = 1/G_{\overline{\Delta}_n}$, and $(1/G)_{\overline{\Delta}_n} = 1/G_{\underline{\Delta}_n}$. For we have $G(\lambda) \leqslant G_{\overline{\Delta}_n}$ for

$$\lambda_n \leqslant \lambda \leqslant \lambda_{n+1},$$

and if $G(\lambda) > O$ and complete, then $1/G_{\overline{\Delta}_n} \leqslant 1/G(\lambda)$. Further, if $H \not\geqslant G_{\overline{\Delta}_n}$, then there is a λ such that $G(\lambda) \not\leqslant H$; these are equivalent (if H is complete) to $1/G_{\overline{\Delta}_n} \not\geqslant 1/H$, $1/H \not\leqslant G(\lambda)$; thus $1/G_{\overline{\Delta}_n} = (1/G)_{\underline{\Delta}_n}$. We thus have, finally,

$$\sum_{-\infty}^{\infty} \left(\frac{1}{G}\right)_{\underline{\Delta}_n} \Delta_n = 1 \Big/ \sum_{-\infty}^{\infty} G_{\overline{\Delta}_n} \Delta_n, \quad \sum_{-\infty}^{\infty} \left(\frac{1}{G}\right)_{\overline{\Delta}_n} \Delta_n = 1 \Big/ \sum_{-\infty}^{\infty} G_{\underline{\Delta}_n} \Delta_n ;$$

on taking l.u.b. of the first sum and g.l.b. of the second sum the result follows. $\left(\int_{-\infty}^{\infty} G(\lambda)\, dE_\lambda \geqslant \sum_{-\infty}^{\infty} G_{\underline{\Delta}_n} \Delta_n \text{ and so is complete.}\right)$

Secondly, if $G(\lambda) \not> O$ for all λ, then $\underset{\lambda_1 \leqslant \lambda \leqslant \lambda_2}{\text{g.l.b.}} \left(G^2(\lambda)\right)$ is complete. Using the result just proved, we obtain

$$1 \Big/ \int_{-\infty}^{\infty} G^2(\lambda)\, dE_\lambda = \int_{-\infty}^{\infty} \frac{1}{G^2(\lambda)}\, dE_\lambda ;$$

the desired result now follows if we multiply both sides by $\int_{-\infty}^{\infty} G(\lambda)\, dE_\lambda$ and use Lemma (vii).

Def. (i). If l.u.b. $F(\lambda)$ or g.l.b. $F(\lambda)$ taken for any interval enclosing λ_0 does not exist, and if A_λ is continuous in a neighbourhood of λ_0, then we write

$$\int_{-\infty}^{\infty} F(\lambda)\, dA_\lambda = \lim_{\delta,\, \delta' \to o} \left[\left(\int_{-\infty}^{\lambda_0 + \delta} + \int_{\lambda_0 - \delta'}^{\infty} \right) F(\lambda)\, dA_\lambda \right] \quad (\delta > o,\ \delta' > o)$$

if this limit exists. The resulting limit is called the improper integral. A similar definition applies when the set of such points λ_0 is an isolated set.

LEMMA (ix). *If the integrals in question are improper, then Lemmas* (ii) *to* (vii) *inclusive are still true. Lemma* (viii) *remains true if the integrals* $\int_{-\infty}^{\infty} G(\lambda)\, dE_\lambda$ *and* $\int_{-\infty}^{\infty} \frac{1}{G(\lambda)}\, dE_\lambda$ *exist properly or improperly.*

We omit the proofs of the various parts of the first statement since they are straightforward. To prove the second statement, replace $G(\lambda)$ by I in the open intervals $(\lambda_0 - \delta,\ \lambda_0 + \delta')$, where $G(\lambda)$ or $1/G(\lambda)$ are not properly integrable. Call the resulting operator $G_\delta(\lambda)$. Clearly, $G_\delta(\lambda)$ satisfies the

conditions of Lemma (viii), and so we have*

$$1\Big/\int_{-\infty}^{\infty} G_\delta(\lambda)\,dE_\lambda = \int_{-\infty}^{\infty} \frac{1}{G_\delta(\lambda)}\,dE_\lambda.$$

Now

$$\int_{-\infty}^{\infty} G_\delta(\lambda)\,dE_\lambda = \left(\int_{-\infty}^{\lambda_0-\delta} + \int_{\lambda_0+\delta'}^{\infty}\right) G(\lambda)\,dE_\lambda + E_{\lambda_0+\delta'} - E_{-\lambda_0-\delta}$$

$$\to \int_{-\infty}^{\infty} G(\lambda)\,dE_\lambda \quad \text{as} \quad \delta,\ \delta' \to o,$$

since by supposition† $E_{\lambda_0+\delta'} - E_{\lambda_0-\delta} \to O$, and $\int_{-\infty}^{\infty} G(\lambda)\,dE_\lambda$ exists either properly or improperly. Similarly

$$\int_{-\infty} \frac{1}{G_\delta(\lambda)}\,dE_\lambda \to \int_{-\infty}^{\infty} \frac{1}{G(\lambda)}\,dE_\lambda.$$

LEMMA (x). *If $F(\lambda)$ is uniformly continuous in every finite interval of λ, then $\int_{-\infty}^{\infty} F(\lambda)\,dE_\lambda$ exists.*

For, if $H > O$, there is a G, $O < G \leqslant H$, such that

$$-G^2 \leqslant G\Big(F(\mu) - F(\lambda)\Big) \leqslant G^2,$$

for all λ, $\lambda_0 \leqslant \lambda \leqslant \lambda_N$ and all μ, $|\mu - \lambda| \leqslant \delta$. Hence

$$-G^2 \leqslant G\Big(F_{\bar{\Delta}} - F(\lambda)\Big) \leqslant G^2, \quad -G^2 \leqslant G\Big(F_{\underline{\Delta}} - F(\lambda)\Big) \leqslant G^2.$$

Thus $-2G^2 \leqslant G(F_{\bar{\Delta}} - F_{\underline{\Delta}}) \leqslant 2G^2$ for all intervals Δ of length less than or equal to 2δ. Hence

$$O \leqslant (F_{\bar{\Delta}_n} - F_{\underline{\Delta}_n})\,G\Delta_n \leqslant 2G^2\Delta_n, \quad \Delta_n = E_{\lambda_{n+1}} - E_{\lambda_n};$$

* $G_\delta(\lambda)$ is integrable, for we may take $\lambda_0 - \delta$ and $\lambda_0 + \delta'$ as points of subdivision, and choose $\delta > o$, $\delta > o'$ so small that E_λ is continuous in $(\lambda_0 - \delta,\ \lambda_0 + \delta')$ [see Def. (i) of this section]. $G_\delta(\lambda)$ is then integrable in $(\lambda_0 - \delta,\ \lambda_0 + \delta')$. $G_\delta(\lambda_0 - \delta) = G(\lambda_0 - \delta)$, so that, in general, $\underset{\lambda_0-\delta \leqslant \lambda \leqslant \lambda'}{\text{l.u.b.}}\ G_\delta(\lambda) \neq I$, but the discontinuities of $G_\delta(\lambda)$ at the ends of the interval $(\lambda_0 - \delta,\ \mu_0 + \delta')$ do not affect the value of the integral of $G_\delta(\lambda)$ over this interval since E_λ is continuous in this interval. The restriction in Def. (i) of this section of taking A_λ continuous in a neighbourhood of λ_0 is essential when $A_\lambda = E_\lambda$, since discontinuity of E_λ cannot be "arbitrarily small" in bound. $\|E_{\lambda+0} - E_\lambda\| = 1$ or 0, and is equal to 0 if and only if $E_{\lambda+0} = E_\lambda$, *i.e.* if E_λ is continuous.

† Since either $G(\lambda)$ or $1/G(\lambda)$ is improperly integrable in the neighbourhood of λ_0, so that E_λ is continuous at $\lambda = \lambda_0$.

thus $$\sum_n (F_{\bar{\Delta}_n} - F_{\underline{\Delta}_n}) \Delta_n G \leqslant 2G^2.$$

This implies that $\lim_{\delta \to 0} \sum_n (F_{\bar{\Delta}_n} - F_{\underline{\Delta}_n}) \Delta_n = O$ and so it implies the existence of $\int_{\lambda_0}^{\lambda_N} F(\lambda)\, dE_\lambda$. If the intervals (λ_1, λ_2) and (λ_3, λ_4) do not overlap, then $\int_{\lambda_1}^{\lambda_2} F(\lambda)\, dE_\lambda$ and $\int_{\lambda_3}^{\lambda_4} F(\lambda)\, dE_\lambda$ have zero for their product. The existence of $\int_{-\infty}^{\infty} F(\lambda)\, dE_\lambda$ now follows from

LEMMA (xi). *If A_n exists for all n and if $A_n A_m = O$ when $n \neq m$, then $\sum_{n=0}^{\infty} A_n$ exists.*

If we write $A_n = A_{n+} - A_{n-}$, then it is sufficient to prove the result for A_{n+}. We therefore take $A_n > O$; let $E_n = \text{g.l.b.}\,(E')$, $E' A_n = A_n$; then $B = E_n/(I + A_n)$ exists and $B_n \leqslant E_n$, $B_n B_m = O$ when $n \neq m$ (for $E_n E_m = O$ when $n \neq m$). Thus $\sum_0^N B_n \leqslant I$ for all N, and so $B = \sum_0^\infty B_n$ exists. $\sum_0^\infty E_n$ exists for $\sum_1^N E_n \leqslant I$. Write $E = I - \sum_0^\infty E_n$; then $B + E$ is complete. Thus

$$\sum_{n\ 0}^{N} A_n (B+E) = \sum_{n=0}^{N} \frac{A_n}{A_n + I} \leqslant I,$$

and so $$\sum_{n\ 0}^{N} A_n \leqslant \frac{1}{B+E}.$$

The sequence $C_N = \sum_{n=0}^{N} A_n$ is therefore bounded; hence $\sum_{n=0}^{\infty} A_n$ exists.

LEMMA (xii). *If $F_n(\lambda)$ is uniformly continuous for each n, and if also $F_n(\lambda) \to F(\lambda)$ uniformly in every finite interval of λ, then*

$$\lim_{n \to \infty} \int_{-\infty}^{\infty} F_n(\lambda)\, dE_\lambda = \int_{-\infty}^{\infty} F(\lambda)\, dE_\lambda.$$

By Lemma (x) of this paragraph and Lemma (v) of paragraph XVI, both integrals exist. It is sufficient to show that, if $F_n(\lambda) \to O$ uniformly in every finite interval of λ and if $F_n(\lambda)$ is uniformly continuous, then $\int_{-\infty}^{\infty} F_n(\lambda)\, dE_\lambda \to O$. To do this it is sufficient to take $F_n(\lambda) \geqslant O$, for, if

$F_n(\lambda) \to O$ uniformly, then so does $|F_n(\lambda)|$, and if $F_n(\lambda)$ is uniformly continuous, then so is $|F_n(\lambda)|$ [see the proof of Lemma (xi), § XV]; the desired result will then follow from Lemma (vi) of this paragraph.

Given $H > O$, there is a K, $O < K \leqslant H$, such that $K F_n(\lambda) \leqslant K^2$ for $n > n_0(H)$ and all λ in $\lambda_1 \leqslant \lambda \leqslant \lambda_2$, $F_n(\lambda) \geqslant O$. Thus

$$O \leqslant K \int_{\lambda_1}^{\lambda_2} F_n(\lambda)\, dE_\lambda \leqslant K^2, \quad i.e. \quad \int_{\lambda_1}^{\lambda_2} F_n(\lambda)\, dE_\lambda \to O \quad \text{for all } (\lambda_1, \lambda_2).$$

Now we can find λ_1 and λ_2 so that $(E_{\lambda_2} - E_{\lambda_1}) H > O$ (for $E_{\lambda_2} - E_{\lambda_1} \to I$ as $\lambda_2 \to \infty, \lambda_1 \to -\infty$). Then, if $\Delta = E_{\lambda_2} - E_{\lambda_1}$, we have $\Delta \int_{-\infty}^{\infty} F_n(\lambda)\, dE_\lambda \not> \Delta H$. Thus there exists a projection E, $E < \Delta$, such that $E \int_{-\infty}^{\infty} F_n(\lambda)\, dE_\lambda \leqslant EH$. Thus $\int_{-\infty}^{\infty} F_n(\lambda)\, dE_\lambda \not> H$ for $n > n_0(H)$; this completes the proof.

XVIII. *Direct functions of an operator.*

Def. (i). By a direct function of an operator A we mean an operator $F(A)$ which is obtained from A and other operators B, C, ... by the use of addition, subtraction, multiplication, and limits of convergent sequences each applied a finite number of times, and where the sequences, $F_n(A)$, are such that $F_n(\lambda I)$ is uniformly continuous and uniformly convergent.

THEOREM II. *If $F(A)$ and $G(A)$ are direct functions and if*

$$A = \int_{-\infty}^{\infty} \lambda\, dE_\lambda, \quad \textit{then} \quad F(A) = \int_{-\infty}^{\infty} F(\lambda)\, dE_\lambda.$$

If $R(A) = F(A)/G(A)$ and if $1/G(\lambda)$ is improperly integrable, then

$$R(A) = \int_{-\infty}^{\infty} R(\lambda)\, dE_\lambda.$$

If $P(A)$ is a polynomial function of $A = \int_{-\infty}^{\infty} \lambda\, dE_\lambda$, then from Lemmas (ii), (iii), and (vii) of § XVII we see that

$$P(A) = \int_{-\infty}^{\infty} P(\lambda)\, dE_\lambda.$$

Further, from Lemmas (iii) and (iv) of § XVI, $P(\lambda)$ is uniformly continuous

in every finite interval of λ. From Lemma (xii) of § XVII we see that, if a sequence of polynomials $P_n(\lambda)$ is uniformly convergent to $F(\lambda)$, then

$$F(A) = \lim_{n\to\infty} P_n(A) = \lim_{n\to\infty} \int_{-\infty}^{\infty} P_n(\lambda)\, dE_\lambda = \int_{-\infty}^{\infty} F(\lambda)\, dE_\lambda.$$

Further, from Lemma (v) of § XVI, $F(\lambda)$ is uniformly continuous in every finite interval of λ. By applying this argument a finite number of times to uniformly convergent sequences $F_n(\lambda)$, we obtain the first part of the theorem. The second part follows from Lemma (ix) of § XVII.

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ON COMPUTABLE NUMBERS, WITH AN APPLICATION TO THE ENTSCHEIDUNGSPROBLEM. A CORRECTION

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In a paper entitled "On computable numbers, with an application to the Entscheidungsproblem"* the author gave a proof of the insolubility of the Entscheidungsproblem of the "engere Funktionenkalkül". This proof contained some formal errors† which will be corrected here: there are also some other statements in the same paper which should be modified, although they are not actually false as they stand.

The expression for Inst $\{q_i S_j S_k L q_l\}$ on p. 260 of the paper quoted should read

$$(x, y, x', y') \left\{ \left(R_{S_j}(x, y) \,\&\, I(x, y) \,\&\, K_{q_i}(x) \,\&\, F(x, x') \,\&\, F(y', y) \right) \right.$$

$$\rightarrow \Big(I(x', y') \,\&\, R_{S_k}(x', y) \,\&\, K_{q_l}(x') \,\&\, F(y', z) \vee \Big[\Big(R_{S_0}(x, z) \rightarrow R_{S_0}(x', z) \Big)$$

$$\left. \& \Big(R_{S_1}(x, z) \rightarrow R_{S_1}(x', z) \Big) \,\&\, \ldots \,\&\, \Big(R_{S_M}(x, z) \rightarrow R_{S_M}(x', z) \Big) \Big] \Big) \right\},$$

$S_0, S_1, \ldots, S_M$ being the symbols which $\mathcal{M}$ can print. The statement on p. 261, line 33, viz.

$$\text{"Inst } \{q_a S_b S_d L q_c\} \,\&\, F^{(n+1)} \rightarrow (CC_n \rightarrow CC_{n+1})$$

is provable" is false (even with the new expression for Inst $\{q_a S_b S_d L q_c\}$): we are unable for example to deduce $F^{(n+1)} \rightarrow \big(- F(u, u'') \big)$ and therefore can never use the term

$$F(y', z) \vee \Big[\Big(R_{S_0}(x, z) \rightarrow R_{S_0}(x', z) \Big) \,\&\, \ldots \,\&\, \Big(R_{S_M}(x, z) \rightarrow R_{S_M}(x', z) \Big) \Big]$$

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† The author is indebted to P. Bernays for pointing out these errors.

in Inst $\{q_a S_b S_d L q_c\}$. To correct this we introduce a new functional variable G [$G(x, y)$ to have the interpretation "x precedes y"]. Then, if Q is an abbreviation for

$$(x)(\exists w)(y, z)\{F(x, w) \& \big(F(x, y) \to G(x, y)\big) \& \big(F(x, z) \& G(z, y) \to G(x, y)\big) \\ \& \Big[G(z, x) \vee \big(G(x, y) \& F(y, z)\big) \vee \big(F(x, y) \& F(z, y)\big) \to \big(-F(x, z)\big)\Big]\},$$

the corrected formula Un($\mathfrak{U}$) is to be

$$(\exists u) A(\mathfrak{U}) \to (\exists s)(\exists t) R_{S_1}(s, t),$$

where $A(\mathfrak{U})$ is an abbreviation for

$$Q \& (y) R_{S_0}(u, y) \& I(u, u) \& K_{q_1}(u) \& \text{Des}(\mathfrak{U}).$$

The statement on page 261 (line 33) must then read

$$\text{Inst}\{q_a S_b S_d L q_c\} \& Q \& F^{(n+1)} \to (CC_n \to CC_{n+1}),$$

and line 29 should read

$$r\big(n, i(n)\big) = b, \quad r\big(n+1, i(n)\big) = d, \quad k(n) = a, \quad k(n+1) = c.$$

For the words "logical sum" on p. 260, line 15, read "conjunction". With these modifications the proof is correct. Un($\mathfrak{U}$) may be put in the form (I) (p. 263) with $n = 4$.

Some difficulty arises from the particular manner in which "computable number" was defined (p. 233). If the computable numbers are to satisfy intuitive requirements we should have:

If we can give a rule which associates with each positive integer n two rationals a_n, b_n satisfying $a_n \leqslant a_{n+1} < b_{n+1} \leqslant b_n$, $b_n - a_n < 2^{-n}$, then there is a computable number α for which $a_n \leqslant \alpha \leqslant b_n$ each n. (A)

A proof of this may be given, valid by ordinary mathematical standards, but involving an application of the principle of excluded middle. On the other hand the following is false:

There is a rule whereby, given the rule of formation of the sequences a_n, b_n in (A) *we can obtain a* D.N. *for a machine to compute α.* (B)

That (B) is false, at least if we adopt the convention that the decimals of numbers of the form $m/2^n$ shall always terminate with zeros, can be seen in this way. Let $\mathfrak{U}$ be some machine, and define c_n as follows: $c_n = \frac{1}{2}$ if $\mathfrak{U}$ has not printed a figure 0 by the time the n-th complete configuration is reached $c_n = \frac{1}{2} - 2^{-m-3}$ if 0 had first been printed at the m-th

complete configuration $(m \leqslant n)$. Put $a_n = c_n - 2^{-n-2}$, $b_n = c_n + 2^{-n-2}$. Then the inequalities of (A) are satisfied, and the first figure of α is 0 if $\mathfrak{M}$ ever prints 0 and is 1 otherwise. If (B) were true we should have a means of finding the first figure of α given the D.N. of $\mathfrak{M}$: *i.e.* we should be able to determine whether $\mathfrak{M}$ ever prints 0, contrary to the results of § 8 of the paper quoted. Thus although (A) shows that there must be machines which compute the Euler constant (for example) we cannot at present describe any such machine, for we do not yet know whether the Euler constant is of the form $m/2^n$.

This disagreeable situation can be avoided by modifying the manner in which computable numbers are associated with computable sequences, the totality of computable numbers being left unaltered. It may be done in many ways* of which this is an example. Suppose that the first figure of a computable sequence γ is i and that this is followed by 1 repeated n times, then by 0 and finally by the sequence whose r-th figure is c_r; then the sequence γ is to correspond to the real number

$$(2i-1)\,n + \sum_{r=1}^{\infty} (2c_r - 1)(\tfrac{2}{3})^r.$$

If the machine which computes γ is regarded as computing also this real number then (B) holds. The uniqueness of representation of real numbers by sequences of figures is now lost, but this is of little theoretical importance, since the D.N.'s are not unique in any case.

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* This use of overlapping intervals for the definition of real numbers is due originally to Brouwer.